To experts: $\widetilde{Kh}(L)$ means $\widetilde{Kh}(m(L))$
also need to choose pt to reduce along (suppressed from notation)

① Survey of Khovanov & Heegaard Floer homology

	Definition	Topological info
$L \subset S^3$ link $\rightsquigarrow \widetilde{Kh}(L)$:	combinatorial (unoriented cube of resol)	???.
$Y^3 \rightsquigarrow \widehat{HF}(Y)$; $L \subset Y^3$ link $\rightsquigarrow \widehat{HFL}(Y,L)$ (denoted $\widehat{HFK}$ for knots)	Floer theoretic	detects some (detects Seifert genus: Ozsváth–Szabó '03; detects fiberedness: Ni '06)

surprisingly $\exists$ lots of similarities

$\exists$ Floer theoretic def^n for Kh: Seidel–Smith '04, Abouzaid–Smith '15, Aganagic '21, LePage–Stende '25

$\exists$ combinatorial def^n for HFL: Manolescu–Ozsváth–Sarkar '06

Manolescu, Rasmussen: $L \subset S^3$ noticed similarities btw $\widetilde{Kh}(L)$ & $\widehat{HF}(\Sigma(L))$

Thm (Ozsváth–Szabó '03): $\exists$ spec. seq^n $\widetilde{Kh}(L) \Rightarrow \widehat{HF}(\Sigma(L))$ over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

Reason why $\exists$ spec seq^n $\widetilde{Kh} \Rightarrow$ some link homology thy $H(L)$:

$\exists$ one sentence summary: if $H(L)$ admits an unoriented skein exact Δ, then (quite often) $\exists$ spec seq^n from Kh

crossing $\times$ of $L \rightsquigarrow$ unoriented resol^n: $)($ L_0, $\asymp$ L_1

Ex

L L_0 L_1

If $\exists$ unoriented skein exact Δ $\left(\exists \text{ exact } \begin{array}{c} H(L_0) \to H(L_1) \\ \nwarrow \;\; \swarrow \\ H(L) \end{array} \right)$,

and can iterate it

$\rightsquigarrow$ unoriented cube of resol^n (like $\widetilde{Kh}$ but w/ extra ∂)

$\rightsquigarrow$ get spec. seq^n $\Rightarrow H(L)$

show $H = \widetilde{Kh}$ for very simple links & maps btw them (planar links, planar cobordisms)

$\rightsquigarrow E_1$ page $= \widetilde{CKh}$

Rmk Carried out many times: eg. Li '24 for HMR

② Survey of $\widetilde{Kh}$ and $\widehat{HFK}$:

<u>Q.</u> Rasmussen '05: Is rk $\widetilde{Kh}(K) \geq$ rk $\widehat{HFK}(S^3,K)$ for ~~knots~~ $K \subset S^3$?

<u>Rmk</u> "Stronger" than other statements: implies $\widetilde{Kh}$ detects the unknot ○ sutured inv...

(proved by Kronheimer-Mrowka '10)

<u>Thm</u> (Dowlin '18) Yes, over $\mathbb{Q}$.

oriented cube of resol^n

(Building on Ozsváth-Szabó '07, Manolescu '11, Beliakova-Putyra-Robert-Wagner '22)

<u>Rmk</u> • Need to divide by 2 (doesn't work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$)

• Need technical condition on the singular braid one starts with

∃ "similar" spectral seq^n but in the spirit of Ozsváth-Szabó's double branched cover spec seq^n:

unreduced

Kronheimer-Mrowka '10: $L \subset Y^3$ link ⇝ defined $I^\natural(Y,L)$, $I^\#(Y,L)$ / $\mathbb{Z}$

(instantons)

"right Floer theoretic <u>link</u> inv" s.t.

experts: $I^\natural$ also needs distinguished pt to reduce along

① Can iterate unoriented Δ & $\widetilde{Kh}(L) \Rightarrow I^\natural(S^3,L)$

$Kh(L) \Rightarrow I^\#(S^3,L)$

② behaves like $\widehat{HFK}$ for knots ($I^\natural(S^3,K)$ is a "sutured inv" like $\widehat{HFK}$)

↳ can show $\widetilde{Kh}$ detects ○

<u>Conj</u> (KM '08, '10) $K \subset S^3$ knot: $I^\natural(S^3,K) \cong \widehat{HFK}(S^3,K)$ / $\mathbb{Q}$

⚠ Tricky in Heegaard Floer thy: Need the right <u>link</u> inv $H(L)$ s.t.

① ∃ unoriented Δ ② $H(K) = \widehat{HFK}(S^3,K)$ for knots.

$\widehat{HFL}$ does <u>NOT</u> work:

rk $\widehat{HFL}$:	4	1	1
	$\not\leq$		
rk $\widetilde{Kh}$:	2	1	1

∃ "Ax": Manolescu '06: ∃ unoriented Δ for $\widehat{HFL}$ for "links w/ extra basepts"

but does not agree w/ $\widetilde{Kh}$

Baldwin-Levine '11: Iterated, but E_2 page not even a link inv

Baldwin-Levine-Sarkar '15: defined Kh for "links w/ extra basepts"

suggested new approach

But $\exists$ hope: Maybe we can find a HF inv like $I^{\flat}$

③ Statement of main result

Thm (N. '25) $L \subset S^3$ link, work over $\mathbb{F}_2$

$\exists$ Heegaard Floer theoretic invariants $HF^{\flat}(S^3,L)$, $HF^{\#}(S^3,L)$, $HF^{\times}(S^3,L)$

① can iterate unoriented Δ

$$\rightsquigarrow \quad \begin{aligned} \widetilde{Kh}(m(L)) &\Rightarrow HF^{\flat}(S^3,L) \\ Kh(m(L)) &\Rightarrow HF^{\#}(S^3,L) \\ Kh^{-}(m(L)) &\Rightarrow HF^{\times}(S^3,L) \end{aligned}$$

② $HF^{\flat}(S^3,K) = \widetilde{HFK}(S^3,K)$ for knots.

Rmk 1) $HF^{\times}, HF^{\#}, HF^{\flat}$ are variants of the unoriented link Floer homology

defined by Ozsváth-Stipsicz-Szabó '15

2) rank ineq / $\mathbb{F}$ new, but conjectured to be weaker

3) Main contribution: noticing that unoriented link Floer homology is the right inv to look at,

constructing unoriented Δ, and iterating it

④ Khovanov homology

$L \subset S^3$ ℓ-component link $\rightsquigarrow (CKh(L), \partial)$ free chain cpx / $\mathbb{Z}$

"minus version" Lee '02, Rasmussen '04: $(CKh^-(L), \partial)$ free chain cpx / $\mathbb{Z}[u]$

A point $x \in L \rightsquigarrow$ "basepoint action" (Hedden-Ni '12) $x : CKh^-(L) \hookrightarrow CKh^-(L)$, $x^2 = u$.

($x, y \in L$ on same component $\rightsquigarrow$ $x \sim y$ htpic)

So choose $x_i \in L_i$ i^{th} component,

$CKh^-(L)$ is a $\underbrace{\mathbb{Z}[x_1, \cdots, x_\ell]/(x_1^2 = \cdots = x_\ell^2)}_{= Kh^-(\ell\text{-comp unlink})}$ -mod

Unreduced: $CKh(L) = CKh^-(L)/u$

Reduced: $\widetilde{CKh}(L,p) = CKh^-(L)/x_i$ if $p \in L_i$

⚠ over $\mathbb{F}_2$, $Kh^- \cong Kh \otimes_{\mathbb{F}} \mathbb{F}[u]$ as $\mathbb{F}[u]$-mods

Feature & not a bug?

⑤ Unoriented link Floer homology

Def Link Floer homology (Ozsváth–Szabó, Rasmussen)

$L \subset S^3$ ℓ-component link

Put two basepts $w_i, z_i \in L_i$ (the i^{th} component)

Use a Heegaard diagram $\rightsquigarrow$ Symp M, Lagrangians $\alpha, \beta \subset M$

link Floer chain cpx $CFL(L) =$ "$CF(\alpha, \beta)$"

free chain cpx over $\mathbb{F}[z_1, \cdots, z_\ell, w_1, \cdots, w_\ell] = \mathcal{R}$

$= \bigoplus_{x \in \alpha \cap \beta} \mathcal{R} x$

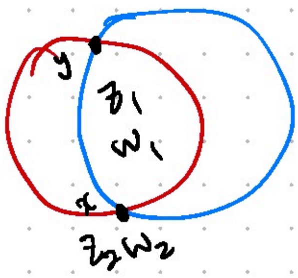
∂: count index 1 curves, count z_i w/ weight z_i
w_i w/ weight w_i

The hat version is $\widehat{CFL}(L) = CFL(L) \otimes_{\mathcal{R}} \mathcal{R}/(z_1, \cdots, z_\ell, w_1, \cdots, w_\ell)$

⚠ $\exists$ another convention: $\mathcal{R} = \mathbb{F}[U_1, \cdots, U_\ell, V_1, \cdots, V_\ell]$, $w_i \longleftrightarrow U_i$, $z_i \longleftrightarrow V_i$,

Example 

$M = S^2$



$$\mathcal{R}x \xrightarrow{z_1 w_1 + z_2 w_2} \mathcal{R}y$$

Borodzik–Liu–Zemke '21

Rmk. on CFL, $\underbrace{z_i w_i}_{= U_i}$ & $\underbrace{z_j w_j}_{= U_j}$ htpic $\rightsquigarrow$ "U" action on homology

Def Unoriented link Floer homology

"minus version": $CF^{\times}(L) = \mathcal{CFL}(L) \underset{\mathcal{R}}{\otimes} \mathcal{R}/(z_1-w_1, \cdots, z_\ell - w_\ell)$ ← double sharp

Convention: $z_i = w_i = U_i^{\frac{1}{2}}$. $U_i^{\frac{1}{2}}$: basept action

⚠ Usually basept action on HF refers to sth else (Φ, Ψ closely related to Sarkar map"s")

i.e. work over $\mathbb{F}[U_1^{\frac{1}{2}}, \cdots, U_\ell^{\frac{1}{2}}]$, and z_i, w_i have weight $U_i^{\frac{1}{2}}$

$\rightsquigarrow HF^{\times}(L)$ is an $\underbrace{\mathbb{F}[U_1^{\frac{1}{2}}, \cdots, U_\ell^{\frac{1}{2}}]/(U_1 = \cdots = U_\ell)}_{= HF^{\times}(\ell\text{-comp unlink})}$-module. so $\mathbb{F}[U]$-module

Rmk Original defn: $CFL(L) \underset{\mathcal{R}}{\otimes} \mathcal{R}/(z_1 = \cdots = z_\ell = w_1 = \cdots = w_\ell)$

$\rightsquigarrow$ Frøyshov type inv gives lower bound of unoriented four-ball genus

Def "Unreduced hat version": $CF^{\#}(L) \overset{\text{def}}{=} CF^{\times}(L)/U_1$

$\rightsquigarrow HF^{\#}(L)$ is an $\mathbb{F}[U_1^{\frac{1}{2}}, \cdots, U_\ell^{\frac{1}{2}}]/(U_1, \cdots, U_\ell)$-module

"reduced hat version": Choose a distinguished basept $p \in L$

$CF^{\natural}(L, p) \overset{\text{def}}{=} CF^{\times}(L)/U_i^{\frac{1}{2}}$ where $p \in L_i$

Rmk If K is a knot, $CF^{\natural}(K, p) = CF^{\times}(K)/U_1^{\frac{1}{2}} = \widehat{CFK}(K)$

Rmk Ye suggested similar counterpart of $I^{\#}$ for knots in HF theory:

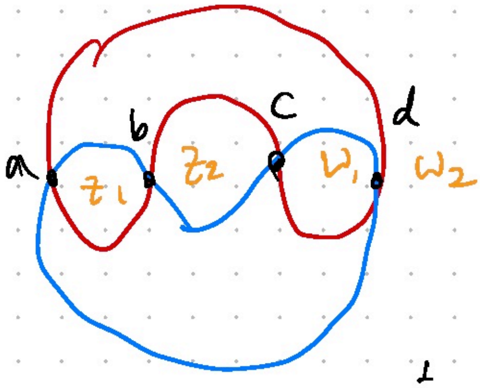
work over $\mathbb{Z}[U]/U^2$ $\quad$ z, w have weight $2U$

Lemma If L has ℓ components, then

$$2^{\ell-1} \operatorname{rk} HF^{\natural}(L, p) \geq \operatorname{rk} \widehat{HFL}(L)$$

Cor $2^{\ell-1} \operatorname{rk} \widetilde{Kh}(L) \geq \operatorname{rk} \widehat{HFL}(L)$

Ex



$$\begin{tikzcd} a \arrow[r, "z_1"] \arrow[dr, "w_2"'] & b \\ c \arrow[ur, "z_2"'] \arrow[r, "w_1"'] & d \end{tikzcd}$$

$$CF^x = \begin{tikzcd} Ra \arrow[r, "u_1^{\frac12}"] \arrow[dr, "u_2^{\frac12}"'] & Rb \\ Rc \arrow[ur, "u_2^{\frac12}"'] \arrow[r, "u_1^{\frac12}"'] & Rd \end{tikzcd} \qquad R = \mathbb{F}[u_1^{\frac12}, u_2^{\frac12}]$$

$CF^g($  $) =$

$$\mathbb{F}[u_1^{\frac12}] \xrightarrow{u_1^{\frac12}} \mathbb{F}[u_1^{\frac12}]$$

$$\mathbb{F}[u_1^{\frac12}] \xrightarrow{u_1^{\frac12}} \mathbb{F}[u_1^{\frac12}]$$

$rk = 2$

⑥ Why $HF^{b,\#,\lambda}$?

Conj (Kronheimer-Mrowka) $I^{\#}(Y) \cong \widehat{HF}(Y)$ over $\mathbb{C}$ ($L=\phi$)

Thm (Bhat '24) let $K \subset Y^3$ knot, λ: framing, μ: meridian

$\exists$ exact Δ

$$\begin{tikzcd} I^{\#}(Y_\lambda(K)) \arrow[rr, "F"] & & I^{\#}(Y_{\lambda+2\mu}(K)) \arrow[dl] \\ & I^{\#}(Y,K) \arrow[ul] & \end{tikzcd} \quad /\mathbb{Z}$$

So reduces to finding analogue of F in Heegaard Floer

Turns out that $\exists$ two "natural" choices (Spin^c structure of $\mathbb{RP}^3$)

One was studied by Ozsváth-Szabó '01

The other one gives analogue of Bhat's Δ

Thm (N. '25): $\exists$ exact Δ

$$\begin{tikzcd} \widehat{HF}(Y_\lambda(K)) \arrow[rr, "F"] & & \widehat{HF}(Y_{\lambda+2\mu}(K)) \arrow[dl] \\ & HF^{\#}(Y,K) \arrow[ul] & \end{tikzcd} \quad /\mathbb{F}$$

(in fact, $\exists$ exact Δ
$$\begin{tikzcd} HF^- \arrow[r] & HF^- \arrow[dl] \\ HF^{\lambda} \arrow[u] & \end{tikzcd}$$
in completed versions)

Rmk $\exists$ different 2-surgery exact Δ:

Thm (Ozsváth-Szabó '01):

$$\begin{tikzcd} \widehat{HF}(Y_\lambda(K)) \arrow[rr, "G"] & & \widehat{HF}(Y_{\lambda+2\mu}(K)) \arrow[dl] \\ & \widehat{HF}(Y, \mathbb{F}[\mathbb{Z}/2\mathbb{Z}]) \arrow[ul] & \end{tikzcd}$$

twisted by $[K] \in H_1(Y)$

$\widehat{HF}(Y, \mathbb{F}[\mathbb{Z}/2\mathbb{Z}]) \simeq \widehat{HF}(Y) \oplus \widehat{HF}(Y)$ if K null-homologous

50'

⑦ Proof of unoriented △

Exact △'s in Floer theory:

① Represent the three chain cpxes as

$CF(\alpha,\beta)$, $CF(\alpha,\gamma)$, $CF(\alpha,\delta)$ ($\alpha,\beta,\gamma,\delta$ in some A_∞ cat)

② define maps: for

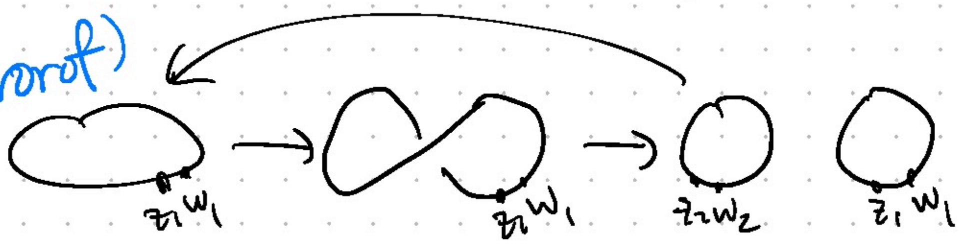
choose cycle $\theta \in CF(\beta,\gamma)$

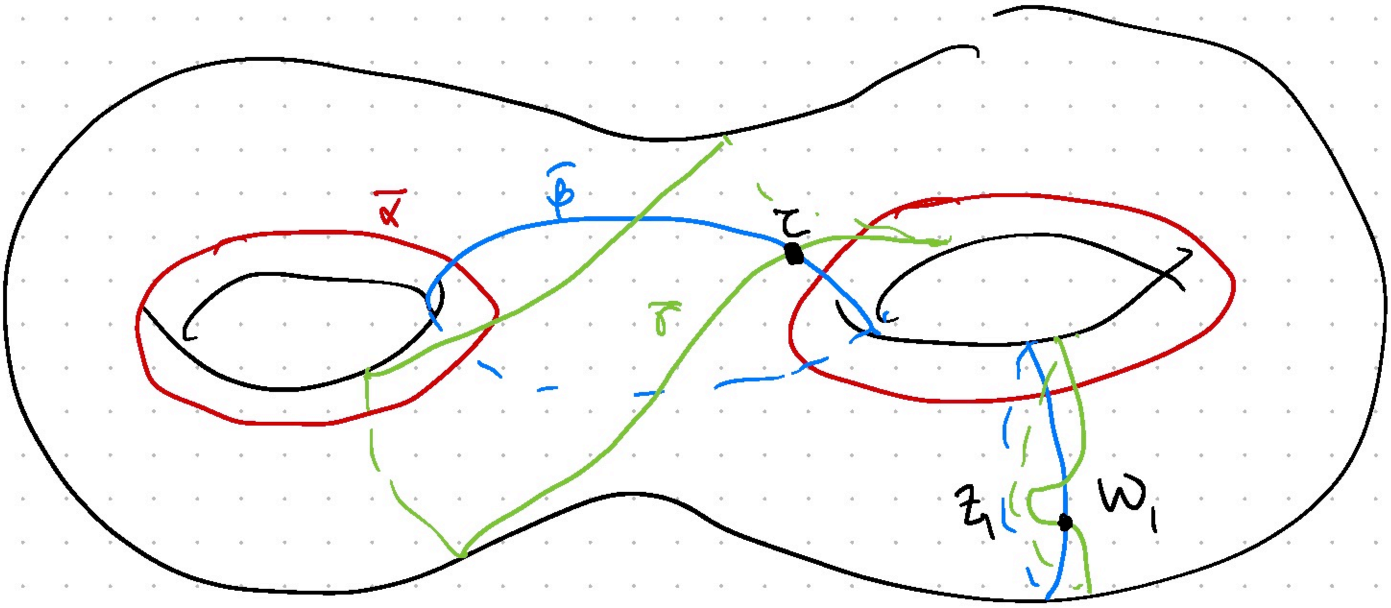
and define $CF(\alpha,\beta) \xrightarrow{-\otimes\theta} CF(\alpha,\beta)\otimes CF(\beta,\gamma) \xrightarrow{\mu_2} CF(\alpha,\gamma)$

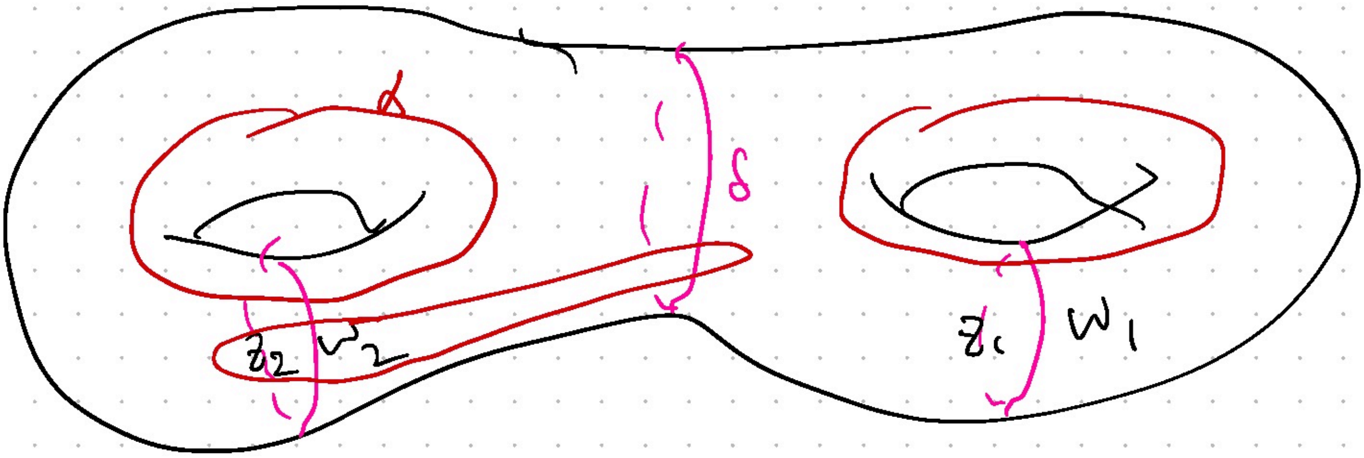
③ use the "triangle detection lemma" (often reduces to a local computation)

in other words, show $\beta \xrightarrow{\theta} \gamma \simeq \delta$

Ex (mirror of)







Challenge: Heegaard diagrams are different

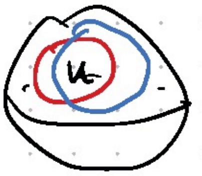
$CF(\bar{\alpha},\bar{\beta})$, $CF(\bar{\alpha},\bar{\gamma})$: over $\mathbb{F}[U^{\frac{1}{2}}]$ —— Ⓐ

$CF(\alpha,\delta)$: over $\mathbb{F}[U_1^{\frac{1}{2}}, U_2^{\frac{1}{2}}]$ —— Ⓑ

Sol^n: meet in the middle   work over $\mathbb{F}[U_1^{\frac{1}{2}}, U_2]$

Ⓐ: Ozsváth-Szabó, Manolescu-Ozsváth, Zemke:

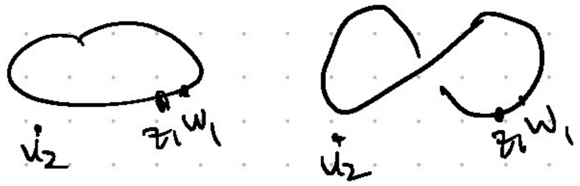
Given HD $\mathcal{H}$ & work over R.

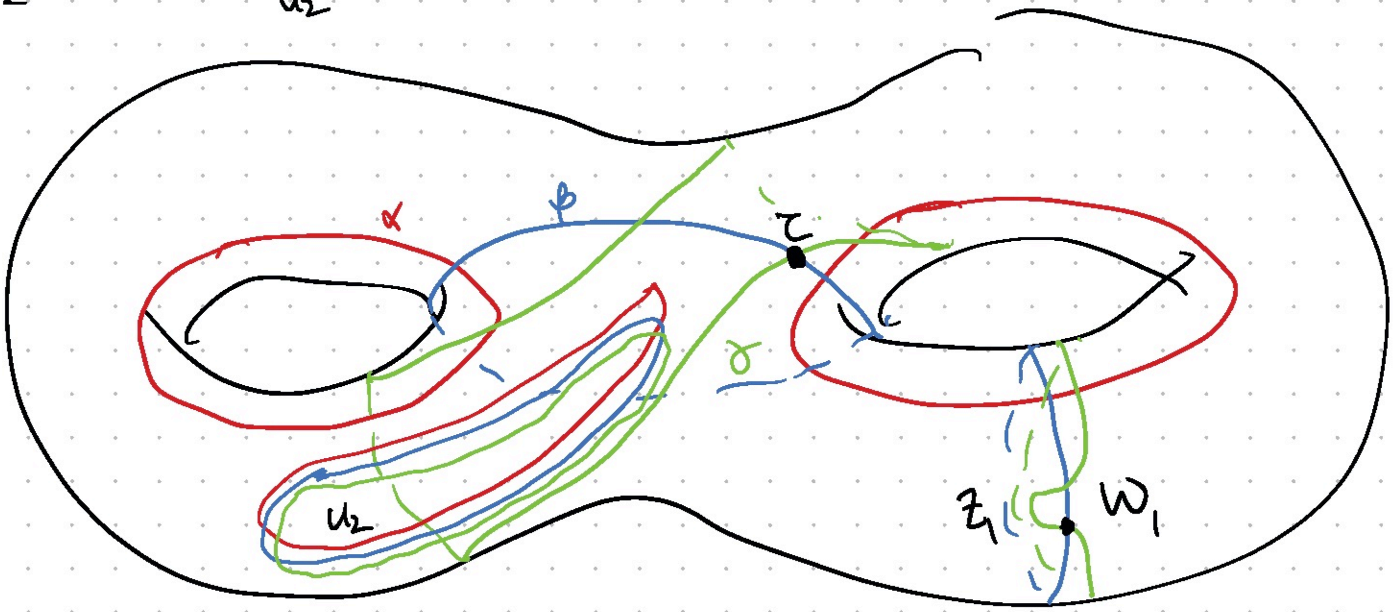
$\mathcal{H}^{stab} = \mathcal{H} \#$ 

work over $R[U]$, u has weight U

$\Rightarrow CF_{\mathcal{H}}(\alpha,\beta) \simeq CF_{\mathcal{H}^{stab}}(\alpha,\beta)$

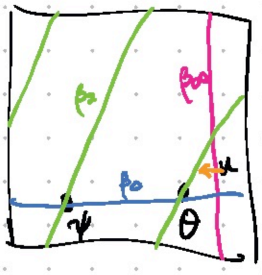
Interpretation: "add a basept in the ambient 3 mfd"





(B): Similar "challenge" for the 2-surgery exact Δ's:

Use local systems!



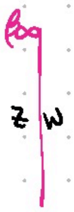
Ozsváth-Szabó:

$$\beta_0 \xrightarrow{\psi} \beta_2 \simeq \beta_\infty^{E} \quad \leftarrow \text{rank 2 local system}$$

$$E = e_0\mathbb{F}[U] \oplus e_1\mathbb{F}[U]$$

$$\phi = e_1 e_0^* + e_0 e_1^* \qquad \text{get } \phi^{\#(\leftarrow\, \cap\, \partial_{\beta_\infty} D)}$$

Rmk



$$\widehat{CF}(Y;\mathbb{F}[\mathbb{Z}/2\mathbb{Z}]) \cong CFK(Y,K) \otimes_{\mathbb{F}[z,w]} E$$

where z acts as ϕ

w acts as $U\phi^{-1}$

HF analogue of Bhat:

$$\beta_0 \xrightarrow{\theta} \beta_2 \simeq \beta_\infty^{E_u} \quad \leftarrow \text{rank 2}$$

gives unoriented link Floer

$$E_u = e_0\mathbb{F}[U] \oplus e_1\mathbb{F}[U]$$

$$\phi_u = e_1 e_0^* + U e_0 e_1^*$$

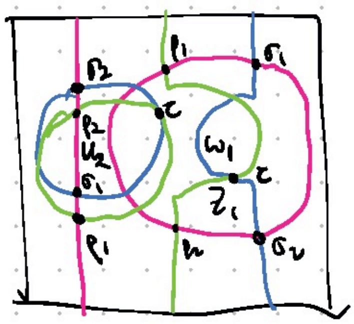
$$E_u \cong \mathbb{F}[U^{\frac{1}{2}}] = \mathbb{F}[Z,W]/(Z=W)$$

⚠ $\exists$ subtlety since $\phi_u^{-1} = U^{-1} e_1 e_0^* + e_0 e_1^*$ involves negative powers of U

⇝ Need to check "positivity" (cf. Zemke '23)

good if ≤ 2 objs w/ nontrivial local system

Now can run traditional strategy



$$\beta \xrightarrow{\tau} \gamma \simeq \delta^{E_u}$$

$$\begin{tikzcd}
\beta \arrow[r, "\tau"] & \gamma \arrow[d, dashed, "e_0\rho_1 + e_0\rho_2"] & \\
& \delta^{E_u} \arrow[d, dashed, "e_1^*\sigma_1 + e_1^*\sigma_2"] & \\
& \beta \arrow[r, "\tau"] & \gamma
\end{tikzcd}$$

⑧ Pf of spectral seq^n

Step 1. construct A_∞ category

obj : α, β_ε for each resolution $\varepsilon \in \{)(, \asymp, X\}^{\#\text{crossings}}$

$CF^X(L_\varepsilon) = CF(\alpha, \beta_\varepsilon)$

⚠ Positivity : for each U_i, want ≤ 2 β_ε's where U_i' can arise

$\rightsquigarrow$ don't reuse basepts

New basepts for each β_ε

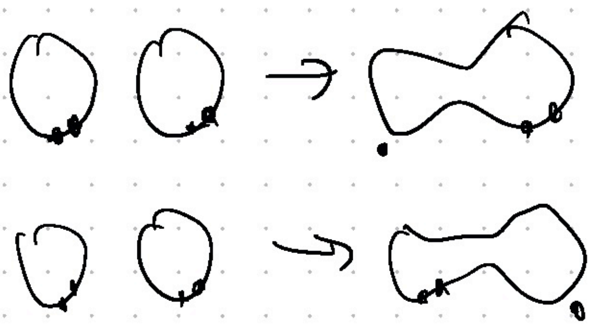
1 crossing : $\beta_{)(} \to \beta_{\asymp} \simeq \beta_{X}$

2 crossings :

$$\begin{tikzcd} \beta_{)(,)(} \arrow[r] \arrow[d] \arrow[dr] & \beta_{)(,\asymp} \arrow[d] \\ \beta_{\asymp,)(} \arrow[r] & \beta_{\asymp,\asymp} \end{tikzcd} \simeq \beta_{X,X}$$

Step 2 : Check band maps commute up to htpy

⚠ The most obvious choice doesn't work.



$\exists$ 2 choices

$\rightsquigarrow$ modify band maps so that it works

Step 3 . Check higher commutativity : grading argument

the A_∞ subcategory of β_ε's has a homological $\mathbb{Z}$-grading

Say top h-gr of $HF(\beta_\varepsilon, \beta_{\varepsilon'})$ is 0 for $|\varepsilon - \varepsilon'| = 1$ gr 0

Then : $HF(\beta_\varepsilon, \beta_{\varepsilon'})$ is supported on h-gr ≤ 0