A PROOF OF [RW24, THEOREM 5.5]

The proof of Theorem 5.5 in [RW24] has three steps, and utilizes [RW24, Lemma 5.8], which is proven in [RW24, Section A]. We sketch a proof of [RW24, Theorem 5.5] assuming [RW24, Lemma A.3 (1), (2), (3)].

Let us first state the main technical result we need for Steps 2 and 3.

We usually denote elements in $\mathbb{Z}^m$ with a boldface letter such as $\boldsymbol{n}$, and its coordinates as n_i. The ith unit vector is $\boldsymbol{e_i}$.

Claim 1. Assume assumption (ii) of [RW24, Theorem 5.5]. Let

$$H := \max_{\boldsymbol{n}\in\boldsymbol{\alpha}+(2\mathbb{Z})^2} h(\boldsymbol{n}).$$

Then, we have

(1) $\widetilde{KhR}^h(K^{\boldsymbol{p}}(\boldsymbol{n})) = 0$ whenever $h > H/2$.
(2) $\widetilde{KhR}^h(K^{\boldsymbol{p}}(\boldsymbol{n})) \simeq \mathrm{gr}\widetilde{KhR}^h_{Lee}(K^{\boldsymbol{p}}(\boldsymbol{n}))$ for $h = H/2$, as quantum graded vector spaces.
(3) For i such that $n_i \geq 2$, the annular creation map (band map)

$$\widetilde{KhR}^{H/2}(K^{\boldsymbol{p}}(\boldsymbol{n}-2\boldsymbol{e_i}) \sqcup U) \to \widetilde{KhR}^{H/2}(K^{\boldsymbol{p}}(\boldsymbol{n}))$$

is injective.

Let's recall how [RW24, Theorem 5.5] follows. Assume both assumptions (i) and (ii). Assumption (i) says that $h(\boldsymbol{\alpha}) = H$.

(1) follows directly from Claim 1 (1) since $w(K^{\boldsymbol{p}}(\boldsymbol{\alpha}+\boldsymbol{r},\boldsymbol{r})) = h(\boldsymbol{\alpha})$.
(2) follows as in Step 2. The first part of (2) (the dimension of $\mathcal{S}^{Lee}_{0,0}$) follows directly from assumptions (i) and (ii) (using Claim 2). The second part of (2) (upper bound of the dimension of $\mathcal{S}_{0,0}$) follows from Claim 1 (2), as in Step 2.
(3) follows directly from Claim 1 (2) and (3): recall from [RW24, Proposition 5.4] that

$$s(X;\boldsymbol{\alpha}) = \lim_{\boldsymbol{r}\to\infty^m}(s_{\mathfrak{gl}_2}(K^{\boldsymbol{p}}(\boldsymbol{\alpha}+\boldsymbol{r},\boldsymbol{r})) - 2|\boldsymbol{r}|) - |\alpha| + 1.$$

Let $x_{\boldsymbol{r}} \in KhR_{Lee}(K^{\boldsymbol{p}}(\boldsymbol{\alpha}+\boldsymbol{r},\boldsymbol{r}))$ be the canonical Lee generator of $K^{\boldsymbol{p}}(\boldsymbol{\alpha}+\boldsymbol{r},\boldsymbol{r})$. By definition,

$$s_{\mathfrak{gl}_2}(K^{\boldsymbol{p}}(\boldsymbol{\alpha}+\boldsymbol{r},\boldsymbol{r})) := q(x_{\boldsymbol{r}}) - 1.$$

Then, Claim 1 (2) and (3) imply that the annular creation map

$$KhR_{Lee}(K^{\boldsymbol{p}}(\boldsymbol{\alpha}+\boldsymbol{r},\boldsymbol{r}) \sqcup U) \to KhR_{Lee}(K^{\boldsymbol{p}}(\boldsymbol{\alpha}+\boldsymbol{r}+\boldsymbol{e_j},\boldsymbol{r}+\boldsymbol{e_j}))$$

has quantum filtration degree exactly 1. This map sends $x_{\boldsymbol{r}} \otimes \mathbf{a}$ or $x_{\boldsymbol{r}} \otimes \mathbf{b}$ to $x_{\boldsymbol{r}+\boldsymbol{e_j}}$, depending on the convention. (It will always map the other one to $-\sigma x_{\boldsymbol{r}+\boldsymbol{e_j}}$, where σ is the map on KhR_{Lee} given by swapping the two strands that the dotted annular creation map creates). Hence

$$q(x_{\boldsymbol{r}+\boldsymbol{e_j}}) = q(x_{\boldsymbol{r}} \otimes \mathbf{a}) + 1 = q(x_{\boldsymbol{r}}) + 2,$$

which is what we wanted to show.

Note that we are assuming [RW24, Proposition 5.4] and a slightly stronger statement for Claim 1 (3), so we don't have to consider the irreducible $S(r)$-subrepresentations again.

Claim 2. Let A, B be real $m \times m$-matrices, where $A_{ij} \geq B_{ij} \geq 0$ for $i \neq j$ and $B_{ii} \geq 0$. Then for any $\boldsymbol{x} \in \mathbb{R}^m$,

$$\boldsymbol{x}^T(A - 2B)\boldsymbol{x} \leq |\boldsymbol{x}|^T A |\boldsymbol{x}|,$$

where $|\boldsymbol{x}| \in \mathbb{R}^m$ such that $|x|_i = |x_i|$. Note that if $\boldsymbol{x} \in \boldsymbol{\alpha} + (2\mathbb{Z})^m$ for some $\boldsymbol{\alpha} \in \mathbb{Z}^m$, then $|\boldsymbol{x}| \in \boldsymbol{\alpha} + (2\mathbb{Z})^m$ as well.

Proof. This follows since $(a_{ij} - 2b_{ij})x_i x_j \leq a_{ij}|x|_i|x|_j$ for all i, j. □

Now, let us show Claim 1. We will show (1) and (2) of Claim 1 first, and deal with (3) at the end. Consider the full induction argument (the proof of [RW24, Lemma 5.8]), and consider all the links B that come up in the induction argument.

Notation 3. Let's say in this particular induction step, we are going from framing $\boldsymbol{f} + \boldsymbol{e_j}$ to $\boldsymbol{f}$. Recall that B is the same as an $\boldsymbol{f}$-framed $\boldsymbol{n'}$ cable of K for some $\boldsymbol{n'} \in \mathbb{Z}^m_{\geq 0}$, $2|\boldsymbol{n'} - \boldsymbol{n}$, for all but one link component, K_j of K. Instead of taking the n'_j cable of K_j, we insert the braid closure $D^i_{a,b}$ ($a = n'_j$) to an f_j-framed diagram, where $D^i_{a,b}$ is as in [RW24, Appendix A], but where the σ_i's correspond to negative crossings. Also, B might have an additional unlinked unlink (which might be empty). We will write such B as $B = D^i_{a,b} \sqcup^k U$ (omit $j, \boldsymbol{f}, \boldsymbol{n'}$). The link that corresponds to $D^i_{a,0}$ is an $(\boldsymbol{f} + \boldsymbol{e_j})$-framed cable (but the blackboard framing of the components that correspond to K_j (in the diagram B) have framing f_j) and $D^{a-1}_{a,a-1}$ is an $\boldsymbol{f}$-framed cable. (Also note that we are treating the $D^i_{a,b}$'s for different values of $a, b, i, j, \boldsymbol{f}, \boldsymbol{n'}$ as different objects.)

We renormalize $KhR(B)$ and $KhR_{Lee}(B)$ (in the homological degree) such that the canonical Lee generator of B where all the strands of B are oriented the same as K is in homological grading $g(\boldsymbol{n'})/2$ where

$$g(\boldsymbol{n'}) := \boldsymbol{n'}^T(P - W + N_+ - N_-)\boldsymbol{n'}.$$

Write the renormalized groups as $\overline{KhR}(B)$ and $\overline{KhR}_{Lee}(B)$, and call the above canonical Lee generator the *special Lee generator* and denote it as s_B. (We are being ambiguous about what the orientations are on the additional unlink $\sqcup^k U$, but this doesn't matter.) Note that $\overline{KhR}$ is insensitive of the framing and the orientation of the link components of B.

Note that $\widetilde{KhR}(K^{\boldsymbol{p}}(\boldsymbol{n})) = \overline{KhR}(K^{\boldsymbol{p}}(\boldsymbol{n}))$ (ignore the quantum gradings).

We show the following by inducting on B.

Claim 4. We have the following:

1. $\overline{KhR}^h(B) = 0$ whenever $h > H/2$.
2. $\overline{KhR}^h(B) \simeq \mathrm{gr}\overline{KhR}^h_{Lee}(B)$ for $h = H/2$ (and so this holds for all $h \geq H/2$).

The base cases are $B = K^{\boldsymbol{w}}(\boldsymbol{n'}) \sqcup^k U$ for $0 \leq \boldsymbol{n'} \leq \boldsymbol{n}$, $2|\boldsymbol{n} - \boldsymbol{n'}$. The maximum homological grading of $\overline{KhR}^h(B)$ is at most

$$\frac{1}{2}g(\boldsymbol{n'}) + \boldsymbol{n'}^T N_- \boldsymbol{n'} = \frac{1}{2}h(\boldsymbol{n'}).$$

Condition (1) is satisfied since $h(\boldsymbol{n'}) \leq H$. Condition (2) is trivially satisfied if $h(\boldsymbol{n'}) < H$, and if $h(\boldsymbol{n'}) = H$, then Condition (2) is satisfied since B has a positive diagram.

Now let us do the induction step. Recall that if $B = D^i_{a,b} \sqcup^k U$ (here, $a, b, i \in \mathbb{Z}_{\geq 0}$ and $i \geq 1$, $a > b$), then there is an unoriented skein exact triangle involving B_1, B, B_2 where $B_1 = E^{i-1}_{a,b} \sqcup^k U$ and $B_2 = D^{i-1}_{a,b} \sqcup^k U$. Assume both conditions (1) and (2) for B_1 and B_2.

Let us figure out the grading shifts of the exact triangle. The special Lee generator of B maps to the special Lee generator of B_2, so the map from B to B_2 has grading 0. Hence the exact triangle is

$$\cdots \to t^d\overline{KhR}(B_1) \to \overline{KhR}(B) \to \overline{KhR}(B_2) \to t^{d-1}\overline{KhR}(B_1) \to \cdots \tag{1}$$

for some d. Here, by t^dA, we mean $A[d]$ if we use the notation of [RW24], i.e. $(t^dA)_k = A_{k-d}$. Note that we also have the same exact sequence in Lee homology. The following claim is the main technical lemma.

Claim 5 ([RW24, Lemma A.3 (4)]). We have $d \le 0$. Furthermore, if $b < a - 1$, then $d < 0$.

Proof. This is [RW24, Lemma A.3 (4)]. We repeat this proof (phrased differently) at the end. □

Let's continue proving the induction step. Condition (1) holds for B: let $h > H/2$. Equation 1 in the relevant degrees read

$$\cdots \to \overline{KhR}^{h-d}(B_1) \to \overline{KhR}^{h}(B) \to \overline{KhR}^{h}(B_2) \to \cdots$$

but $h - d > H/2$, and so $\overline{KhR}^{h-d}(B_1)$ and $\overline{KhR}^{h}(B_2)$ vanish. Hence $\overline{KhR}^{h}(B)$ also vanishes.

Let us check Condition (2) for B. If $b < a-1$, then $H/2 - d > H/2$, and so Equation 1 implies that

$$0 \to \overline{KhR}^{H/2}(B) \to \overline{KhR}^{H/2}(B_2) \to 0$$

is exact, and the same holds in the Lee version. Hence Condition (2) follows.

If $b = a-1$, then B has one more component than B_2 (and so also one more than B_1). Hence, the map

$$\overline{KhR}_{Lee}(B_2) \to t^{d-1}\overline{KhR}_{Lee}(B_1)$$

is a "nonorientable band map" which is zero, and so Equation 1 implies that the following sequences are exact:

$$\overline{KhR}^{H/2-1}(B_2) \to \overline{KhR}^{H/2-d}(B_1) \to \overline{KhR}^{H/2}(B) \to \overline{KhR}^{H/2}(B_2) \to 0,$$

$$0 \to \overline{KhR}^{H/2-d}_{Lee}(B_1) \to \overline{KhR}^{H/2}_{Lee}(B) \to \overline{KhR}^{H/2}_{Lee}(B_2) \to 0.$$

Hence,

$$\begin{aligned}\dim \overline{KhR}^{H/2}_{Lee}(B) &\le \dim \overline{KhR}^{H/2}(B)\\ &\le \dim \overline{KhR}^{H/2-d}(B_1) + \dim \overline{KhR}^{H/2}(B_2)\\ &= \dim \overline{KhR}^{H/2-d}_{Lee}(B_1) + \dim \overline{KhR}^{H/2}_{Lee}(B_2)\\ &= \dim \overline{KhR}^{H/2}_{Lee}(B),\end{aligned}$$

(the first equality follows from the induction hypothesis, Condition (2) for B_1 and B_2) and so Condition (2) holds for B.

In particular, the map

$$\overline{KhR}^{H/2-d}(B_1) \to \overline{KhR}^{H/2}(B)$$

is injective. If $B = K^{\boldsymbol{p}}(\boldsymbol{n})$, then $B_1 = K^{\boldsymbol{p}}(\boldsymbol{n} - 2\boldsymbol{e_j}) \sqcup U$, and so this is exactly Claim 1 (3). (Note that $d = 0$ in this case.)

Proof of Claim 5. Use the same notations $a, b, i, j, \boldsymbol{f}, \boldsymbol{n'}$ as Notation 3. Recall that $B = D^i_{a,b} \sqcup^k U$ and we are going from framing $\boldsymbol{f} + \boldsymbol{e_j}$ to $\boldsymbol{f}$.

By the *ith strand*, we mean the ith input strand of the *braid* (not the braid closure) $(\sigma_1 \cdots \sigma_{a-1})^b \sigma_1 \cdots \sigma_i$ of which $D^i_{a,b}$ corresponds to the braid closure (here, σ_l's are negative crossings). In particular, the leftmost σ_i swaps the first strand and the $i+1$th strand. Note that the ith strand might not close up to a link component (but it will be a part of a link component) in the braid closure (and so in B). Also note that it does close up in $D^{a-1}_{a,a-1}$.

Let $I \subset [a] := \{1, \cdots, a\}$ be a set of strands such that $1 \leq |I| \leq a/2$, and say that the ith strands for $i \in I$ form a sublink of $D^i_{a,b}$. Let B^I be this sublink, and let s^I_B be the canonical Lee generator of B where everything is oriented as K but this sublink's orientation is reversed. Let $C = D^{a-1}_{a,a-1}$. The main idea is to compare the number of crossings between B^I and $B^{[a]\setminus I}$, and the number of crossings between C^I and $C^{[a]\setminus I}$, which gives

$$\mathrm{gr}_h s^I_B - \mathrm{gr}_h s_B \leq \mathrm{gr}_h s^I_C - \mathrm{gr}_h s_C. \tag{2}$$

Indeed, the crossings that differ come from the braid, and these are all positive in $C^{[a]\setminus I} \cup (r(C^I))$, where r means the orientation is reversed. If $b < a-1$, then inequality (2) is strict.

We also have the following, where F is the diagonal matrix with entries f_i.

$$\begin{aligned} \mathrm{gr}_h s^I_C - \mathrm{gr}_h s_C &= \frac{1}{2}(\boldsymbol{n'} - 2|I|\boldsymbol{e_j})^T (F - W + N_+ - N_-)(\boldsymbol{n'} - 2|I|\boldsymbol{e_j}) \\ &\qquad - \frac{1}{2}\boldsymbol{n'}^T (F - W + N_+ - N_-)\boldsymbol{n'} \\ &\leq \frac{1}{2} g(\boldsymbol{n'} - 2|I|\boldsymbol{e_j}) - \frac{1}{2} g(\boldsymbol{n'}). \end{aligned} \tag{3}$$

(Recall that the difference of the homological gradings is the same as the difference of the writhes divided by two.)

Let's now verify the claim. We consider two cases: the map $\overline{KhR}_{Lee}(B) \to \overline{KhR}_{Lee}(B_2)$ is either a merge map or a split map.

If it is a merge map, then $t^d \overline{KhR}_{Lee}(B_1) \to \overline{KhR}_{Lee}(B)$ is a split map. The image of the special Lee generator s_{B_1} of B_1 is s^I_B for some subset $I \subset \{1, \cdots, a\}$ such that $1 \leq |I| \leq a/2$. Then, the above inequalities give

$$\mathrm{gr}_h s^I_B = (\mathrm{gr}_h s^I_B - \mathrm{gr}_h s_B) + \mathrm{gr}_h s_B \leq \frac{1}{2} g(\boldsymbol{n'} - 2|I|\boldsymbol{e_j}) = \mathrm{gr}_h s_{B_1},$$

and so $d \leq 0$. If $b < a-1$, then the inequality is strict, and so $d < 0$.

If it is a split map, then $\overline{KhR}_{Lee}(D_2) \to t^{d-1}\overline{KhR}_{Lee}(D_1)$ is a merge map. The canonical Lee generator of B_2 that maps to s_{B_1} is again $s^I_{B_2}$ for some subset $I \subset \{1, \cdots, a\}$ such that $1 \leq |I| \leq a/2$. Here, one of the two strands that the rightmost σ_i of $(\sigma_1 \cdots \sigma_{a-1})^b \sigma_1 \cdots \sigma_i$ swaps is in B^I_2, and the other one is in $B^{[a]\setminus I}_2$. The crossing corresponding to this rightmost σ_i is not in B_2 (as B_2 corresponds to $(\sigma_1 \cdots \sigma_{a-1})^b \sigma_1 \cdots \sigma_{i-1}$). Hence, we have

$$\mathrm{gr}_h s^I_{B_2} - \mathrm{gr}_h s_{B_2} \leq \mathrm{gr}_h s^I_C - \mathrm{gr}_h s_C - 1,$$

and this inequality is strict if $b < a-1$. Thus, we similarly get

$$\mathrm{gr}_h s^I_{B_2} \leq \frac{1}{2} g(\boldsymbol{n'} - 2|I|\boldsymbol{e_j}) - 1 = \mathrm{gr}_h s_{B_1} - 1$$

and so $d \leq 0$. If $b < a-1$, then $d < 0$. □

Remark 6. In fact, $d < 0$ if the framing of the component K_j of B that we are considering is not minimal, because inequality (3) would be strict. This is not weird: when we iterate [RW24, Lemma A.2] to get [RW24, Lemma A.1], the highest t-degree of $R_{\boldsymbol{n},\boldsymbol{n'},\boldsymbol{p}-\boldsymbol{w}}$ comes from letting n_i decrease to n'_i when the framing is maximal, i.e. when we go from p_i to $p_i - 1$.

References

[RW24] Qiuyu Ren and Michael Willis. Khovanov homology and exotic 4-manifolds, 2024.