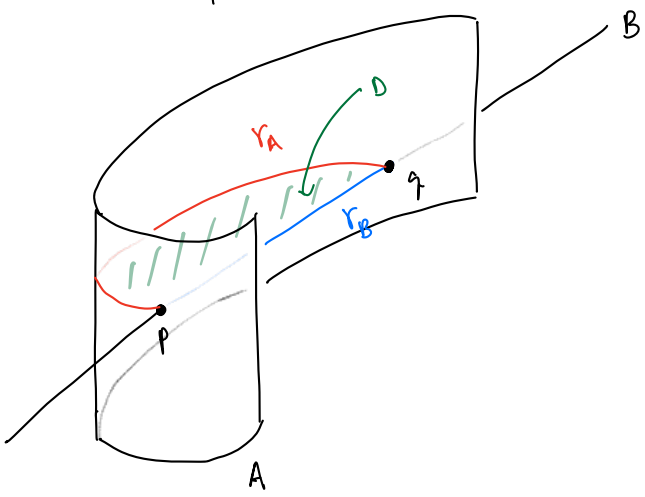
§1. Casson handles

Dream: 4d Whitney trick

To make this dream a reality, we need:

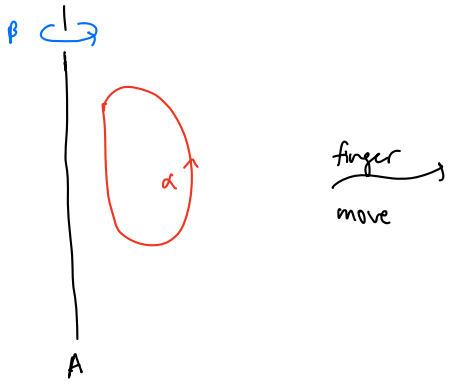


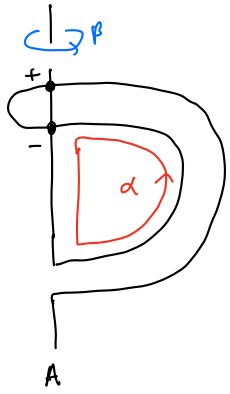
γ_A, γ_B exist (easy)

$\gamma_A \cup \gamma_B$ nullhomotopic (we'll be in settings with, e.g., $\pi_1(X \setminus (A \cup B)) = 1$)

D embedded (HARD!)

- To get these settings, we will often need finger moves which increase the (geom.) self-int. of A (or B), but kill certain commutators in π_1 of the complement



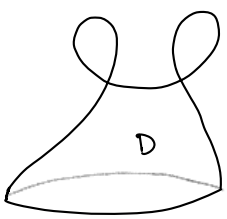


kills $[\beta, \alpha^{-1}\beta\alpha]$ in $\pi_1(X \setminus A)$

(Proof. Near an intersection, A is modeled on $D^2 \times \{0\} \cup \{0\} \times D^2$. In $X \setminus A$, the torus $S^1 \times S^1$ lives w/ $S^1 \times \{0\} = \beta$ and $\{0\} \times S^1$ = meridian around fingered branch = conj. of β by α. Since $\pi_1(\text{torus})$ = commutative, know β, $\alpha^{-1}\beta\alpha$ commute.)

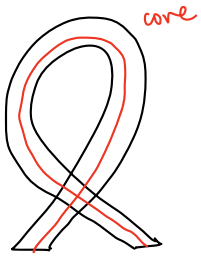
In general, 4d Whitney must fail: non-slice knots exist, so not every $D^2 \looparrowright B^4$ can be made embedded (rel ∂).

What if we allow D to be immersed?



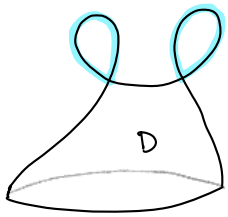
Obs. The difference between ν(embedded D^2) and ν(immersed D^2) is double point loops generating π_1.

Half-dim'l picture:

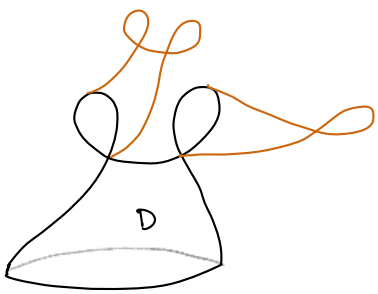


Thickening up, ν(immersed D^2 w/ n double pts) is $\natural n(S^1 \times B^3)$, w/ π_1 generated by DPLs.

n-times self-plumbed 2-h.



Can ask $\pi_1(X \setminus D) = 1$ by finger moves. So DPLs bound imm. disks:



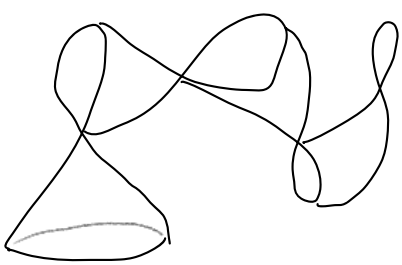
Suppose one of these disks were embedded. We'd kill a double point!



Keep going. At each stage D_n, kill all $\pi_1(\nu D_{n-1})$, possibly introducing more π_1. Do ∞-ly.

$\rightsquigarrow$ the thickened nbhd is $T_n = \nu D_n$; T_∞ is a Casson handle.

Rmk. The immersed disks at each stage may be made disjoint from all previous stages by finger moves ($\pi_1(X \setminus D_{n-1}) = 1$), and from each other (again by FMs).



... = simplest Casson handle

Thm (Casson 1974). Let $\pi_1(X) = 1$ and $D_1, \dots, D_n$ be immersed transverse 2-disks w/ $\partial D_i \hookrightarrow \partial X^4$ embedded disjointly. Assume $\exists\ \beta_1, \dots, \beta_n \in H_2(X; \mathbb{Z})$ st $\beta_i \cdot \beta_i$ even and $D_i \cdot \beta_j = \delta_{ij}$. Then the D_i's may be regularly homotoped rel ∂ to be disjoint, and we may build disjoint, embedded CHs $CH_1, \dots, CH_n$

with the first stage of CH_i being D_i such that each CH_i is proper htpy equiv. rel ∂_- ($= S^1 \times D^2$) to $D^2 \times \mathbb{R}^2$.

From the picture, one can see $\pi_1(CH) = 1$. One can check it has the correct relative homology.

Casson does some more to get proper htpy equiv.

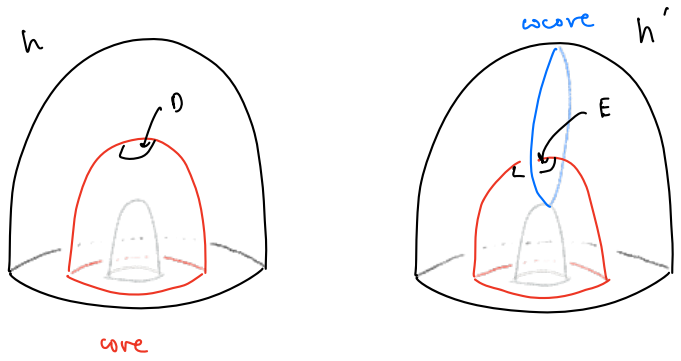
Thm (Freedman 1982). Each CH_i is in fact homeomorphic rel ∂ to $D^2 \times \mathbb{R}^2$.

("open 2-h.")

So can replace D_i with the core of this open 2-h., and perform Whitney tricks!

§2. Kirby diagrams

① Kirby diagrams of plumbings / self-plumbings



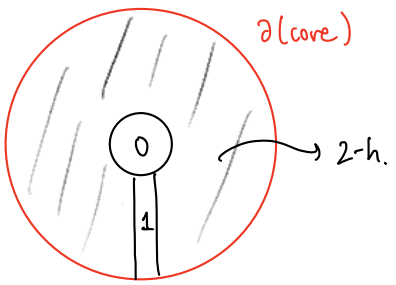
positive plumbing:

$(x,y) \in D \times D^2 \sim (y,x) \in E \times D^2$

(negative plumbing: $(x,y) \sim (-y,-x)$)

core of h is $D^2 \times 0$:

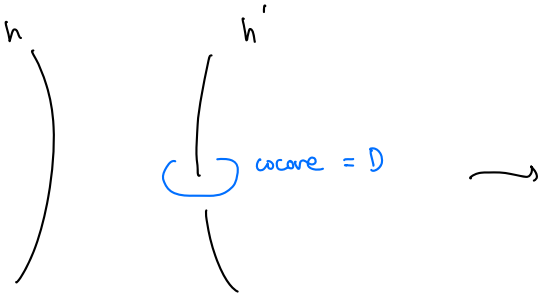
(1)



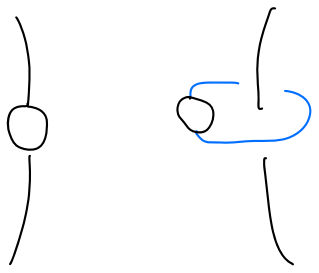
thicken this picture up to get relative handle decomp. of h (wrt attaching region) w/ one 0-h., one 1-h., one 2-h.

↳ this is D!

plumbing identifies D w/ cocore of h':

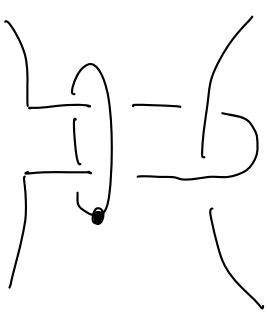


where does the 1-h.'s feet go?

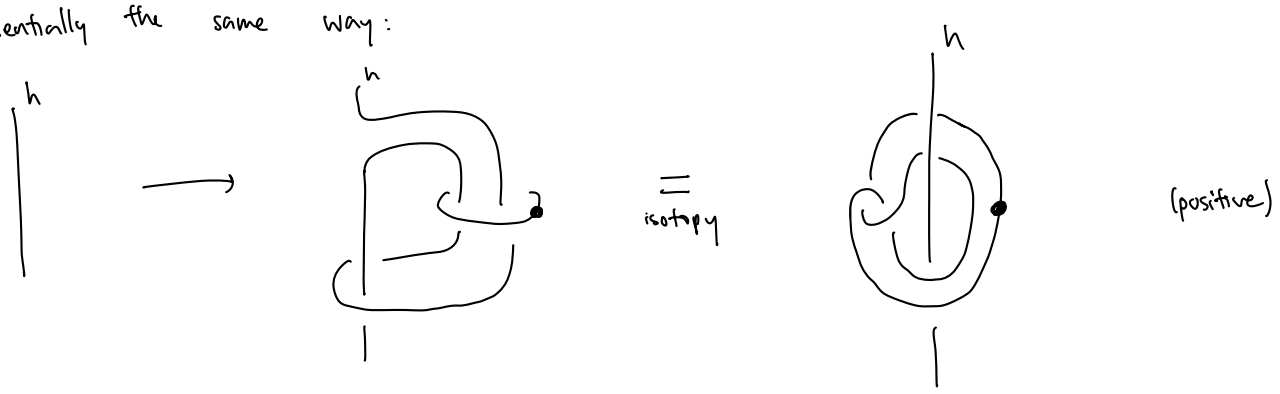


verify that this 2-h. att. circle is exactly what's drawn in (1)

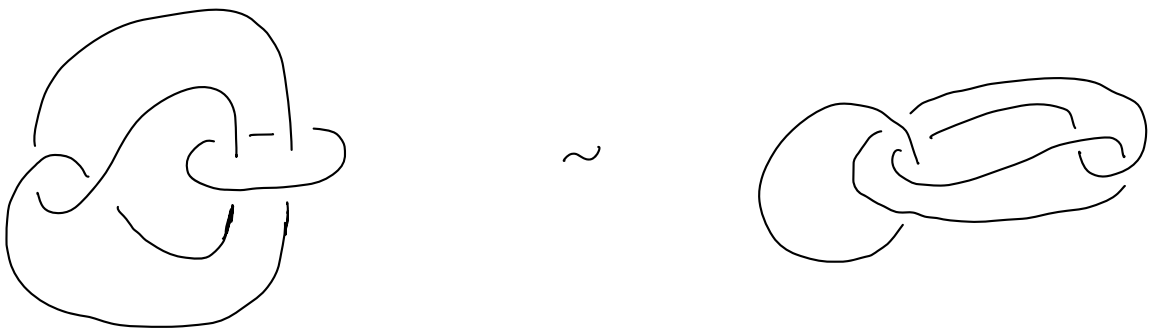
in dotted circle notation, a plumbing looks like (sign of the clasp depends on sign of plumbing)



self-plumb in essentially the same way:



Rmk. Here, we use a symmetry of a Whitehead link. By untwisting the left side + isotoping,



Whitehead double of a knot K is $Wh(K)$:



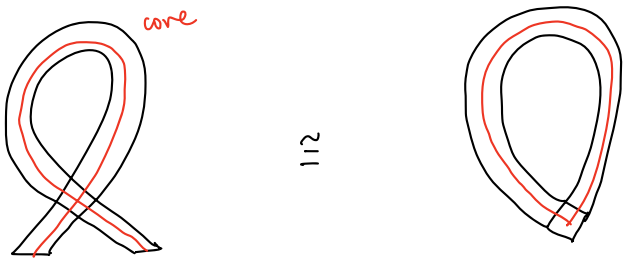
If $Wh_{i,j}(H)$ denotes Wh^i of left component of Hopf and Wh^j of right, then $Wh_{i,j}(H) = Wh_{i',j'}(ff')$ iff $i+j = i'+j'$.

Example. self-plumbed 2-h., i.e, immersed 2-h.



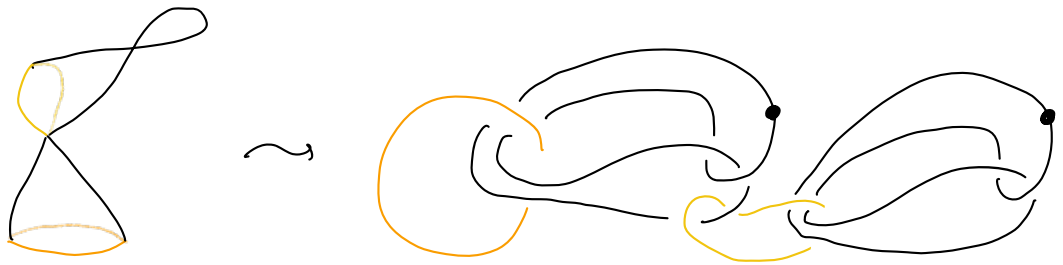
Sanity check! Ignoring the attaching circle, h is just $\bigcirc = S^1 \times B^3$.

Half-dim'l picture:



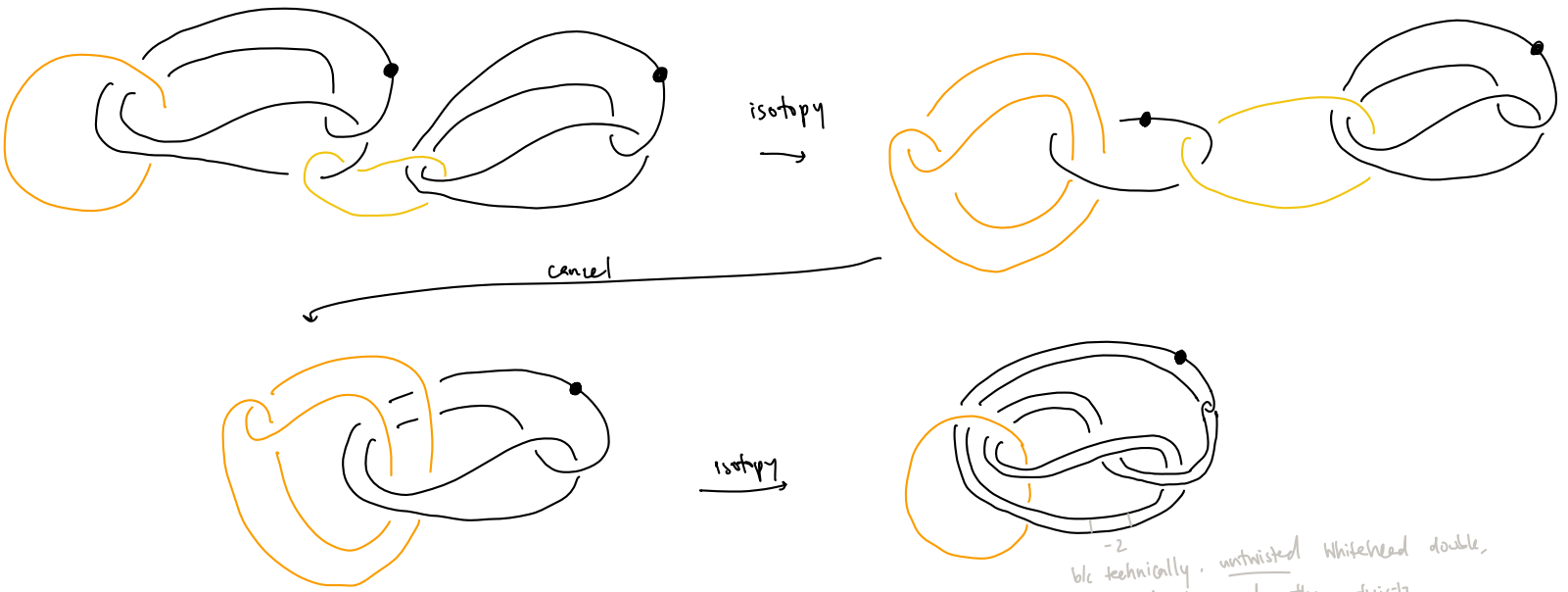
Thickening up, n-times self-plumbed 2-h. is $\natural\, n(S^1 \times B^3)$.

② Kirby diagram of (simplest) CH



and so on!

Rmk. Deleting orange, get cancellation $\Longrightarrow$ ignoring attaching data, CH is trivial (i.e., diffeo. to $D^2 \times D^2$)



In general, the n-th stage T_n is a 2-h. and a 1-h. (i.e., deleted slice disk)

↳ attaching region for T_n

- T_1 = 2-h. attached, then cut out a slice disk SD_1
- T_2 : can make it so that this slice disk lies on $\nu(SD_1)$; same story

⋮

- $T_\infty = D^2 \times D^2 \setminus \bigcap \nu(SD_i)$ is the simplest Casson handle!

In principle, it'd be nice to work with an endpoint compactified CH and show that that's standard (homeo. to $D^2 \times D^2$ rel ∂). But this is false!

Pf that compactified CH is not $D^2 \times D^2$.

① $\partial(CH) = S^3 - Wh$ where Wh is the Whitehead continuum ($Wh \subset S^3$)

② $\partial(\overline{CH}) = S^3 / Wh$

③ $S^3/Wh \neq S^3$ (in fact, S^3/Wh isn't a mfld.) □

Rmk. The diagrams above were for the simplest CH. For more double pts, allow replication:



§3. The Whitehead continuum

DET §7

Kirby §12.4

Construction.

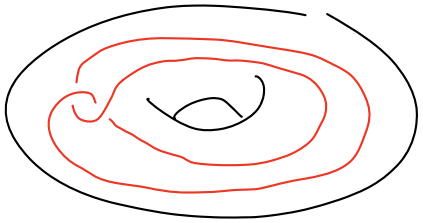
$Y_0 = S^1 \times D^2 \subset S^3$

Clasp lives inside, and has $Y_1 := \nu(\text{clasp}) = S^1 \times D^2$

⋮

keep going

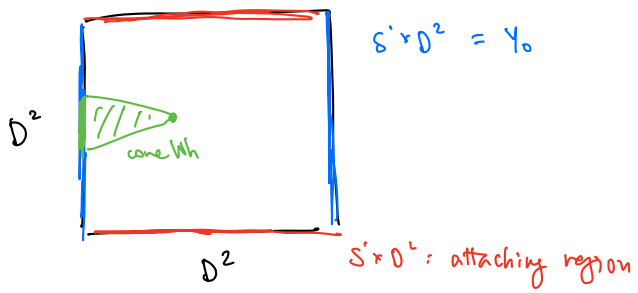
$Wh := \bigcap Y_n \subset S^3$



In the language of decompositions, the Whitehead pattern defines a decomposition of X as $\mathcal{D} = \pi_0(Wh)$.

① Why is $\partial(CH) = S^3 - Wh$?

- Think of $Y_0 = S^1 \times D^2 \subset \partial(D^2 \times D^2)$. Then each Y_n is a thickening $(\times D^2)$ of $\partial(SD_n)$. Of course $T_n = D^2 \times D^2 \setminus SD_n \Longrightarrow CH = T_\infty = D^2 \times D^2 - \text{cone}(Wh)$.
- Note Wh lives in the $S^1 \times D^2$ boundary, and the cone is into int:



- So $\partial(CH) = S^3 - Wh$.

② Why is $\partial(\overline{CH}) = S^3 / Wh$?

- Not sure, the compactification just crushes each connected component of Wh (which is connected anyways) to a point

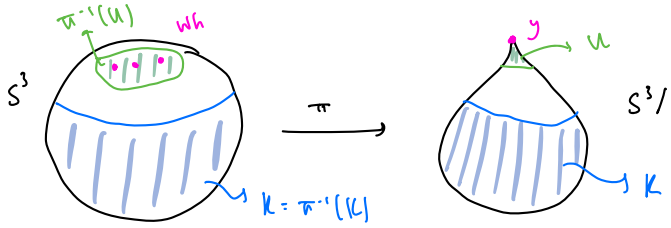
③ Why is $S^3/Wh \ncong S^3$? (i.e., why doesn't the Whitehead decomposition shrink?)

Def. X is simply connected at ∞ if $\forall$ cpt $K \subset X$, $\exists$ cpt $L \supset K$ st loops outside L can be shrunk outside K, i.e., st $\pi_1(X-L) \xrightarrow{i_*} \pi_1(X-K)$ is the zero map.

Prop. $S^3 - Wh$ is not simply conn. at ∞.

Prop $\Longrightarrow$ ③. • $\pi: S^3 \longrightarrow \underbrace{S^3/Wh}_{\text{supp. mfd}}$, let $y := \pi(Wh)$ be a mfd pt.

- $K \subseteq S^3 - Wh$ cpt, then $\exists$ coord. nbhd U of $y \in S^3/Wh$ st $U \cap K = \emptyset$



Rmk. U exists b/c S^3/Wh Hausdorff. Requires "upper semi-continuity."

- $L := \pi^{-1}(S^3/Wh - U)$ is preimage of closed, hence closed in S^3, hence cpt
- WTS $\pi_1((S^3 - Wh) - L) \longrightarrow \pi_1((S^3 - Wh) - K)$ is trivial
- Enough to note $(S^3 - Wh) - L = \pi^{-1}(U - \{y\}) \cong U - \{y\}$ since π is homeo away from Wh and y, so $\pi_1(\downarrow) = 1$.
  - $U - \{y\} \simeq S^2$

□

Pf of prop. $K_i := S^3 \setminus \text{int}(Y_i)$. Claim $\forall$ cpt $K' \supset K_0$, $\exists$ loop outside K' which hits K_0 during its contraction, i.e.,

$$\pi_1((S^3 - Wh) - K') \longrightarrow \pi_1((S^3 - Wh) - K_0) \quad \text{nontrivial.}$$
$$\underbrace{\quad}_{Y}$$

① Supp. o/w.

$K' \subset K_i$ for some i ($S^3 - Wh = \bigcup K_i$ is an exhaustion by cpt sets)

$$\pi_1(Y - K_i) \longrightarrow \pi_1(Y - K_0)$$
$$\searrow \quad \nearrow 0$$
$$\pi_1(Y - K')$$

$$\Longrightarrow \quad \pi_1(Y - K_i) \longrightarrow \pi_1(Y - K_0) \quad \text{trivial!}$$
$$\| \qquad\qquad \|$$
$$\pi_1(\text{int } T_i - Wh) \qquad \pi_1(\text{int } T_0 - Wh)$$

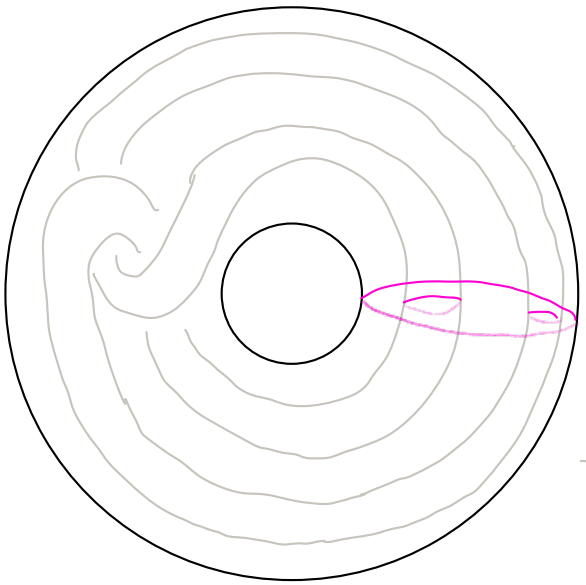
② $M_j \subseteq \partial T_j$ a meridian pt $\times \partial D^2$ is essential in $T_j - Wh$:

- If not, it bds immersed (Dehn's lemma $\Longrightarrow$ embedded) disk in $T_j - Wh$.
- Cptness $\Longrightarrow$ disk lives in $T_j - T_k$ for some $k \gg j$.
- Delete thickened disk from T_j $\rightsquigarrow$ get a ball containing T_k
- M_j and core of T_k form an iterated Whitehead link which is split, hence unlink!
- Can compute invariants (e.g., Milnor invariant) to show not so.

③ Push M_i into $\text{int}(T_i)$ so M_i essential in $\pi_1(\text{int}(T_i) - Wh)$.

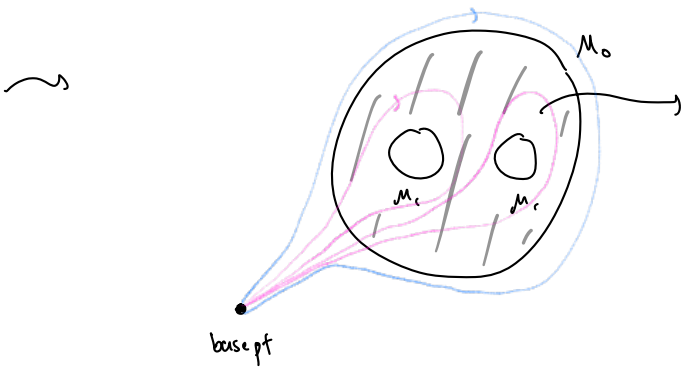
Claim. It's essential in $\pi_1(\text{int}(T_0) - Wh)$ too (so contradiction)!

Why?



M_0's disk intersects T_k in 2^k meridians

$T_i \subset T_0$



disk living in $T_0 - T_k \subset T_0 - Wh$

blue and pink loops are clearly htpic!

Upshot. M_0 is product of conjugates of M_i. But $M_0 \in \pi_1(\text{int } T_0 - Wh)$ nontriv., so M_i is also nontriv.

□

So compactified CH is not a std 2-h. $D^2 \times D^2$, since $\partial \overline{CH} \neq S^3$!

Rmk. The fact that $S^3/Wh \not\cong_h S^3$ can be reworded as saying "the Whitehead decomposition doesn't shrink."

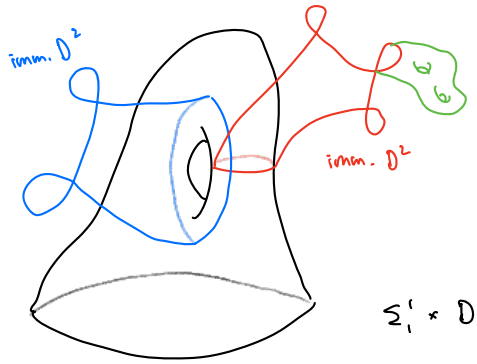
Rmk. In fact, $\mathbb{R}^3/Wh \times \mathbb{R} \cong_h \mathbb{R}^4$. That's how Freedman used open Casson handles to prove disk embedding.

Rmk. For CH (not the simplest), use replication in the Whitehead doubling procedure



§4. Skyscrapers and Bing decompositions

skyscrapers allow attaching (thickened) surfaces, not just self-plumbed thickened disks



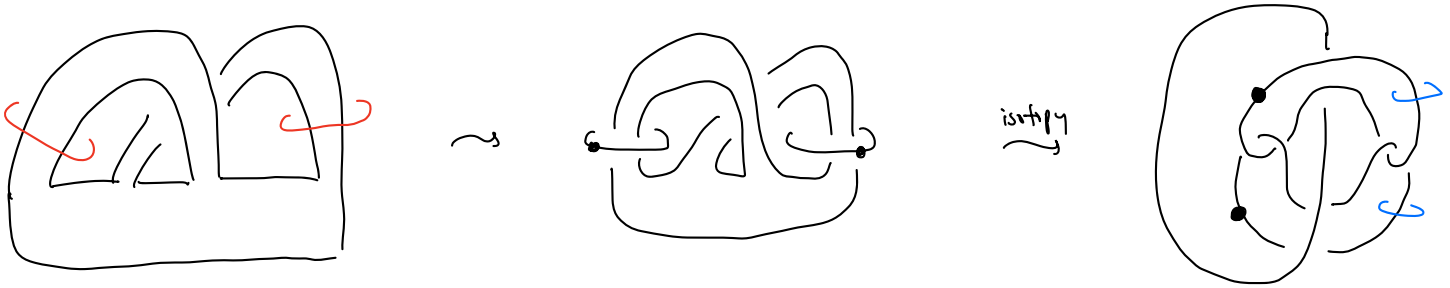
Recall. A skyscraper is a kind of infinite tower, where a tower is made of stages. Each stage is a bunch of thickened surfaces or thickened immersed disks (blocks). Stages attach to the previous stage along generators of π_1.

Technically, a skyscraper requires $N_j \geq 4$ surface stages for each j-th disk stage st $\sum N_j/2^j \to \infty$.

Also, a skyscraper is the unapt compactified infinite tower.

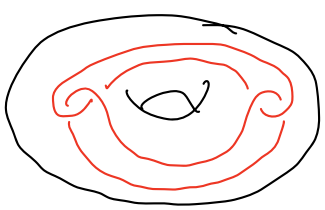
Why does this simplify the pf of DET?

- Kirby diagrams of skyscrapers now involve diagrams of $\Sigma_g \times D^2$ (technically Σ_g')



(and connect sums)

- thing will get attached along meridians of 1-h. (blue)
- so skyscraper has lots of Bing doubling!
  - ↳ CH is Wh. doubling only, since Kirby diagram is Whitehead link
  - ↳ Now Bing:



(ν(red) = $S^1 \times D^2$, put this torus in, trace out another red curve. keep going.)

• Why do we like Bing doubling?

Thm. Consider a solid torus Y_0. Consider $Y_1 \subset Y_0$ formed by either Bing or Whitehead doubling, and so on, where we choose Bing / Wh. based on a sequence

$$B^{b_1} W B^{b_2} W \dots$$

(so $Y_1, \dots, Y_{b_1}$ are formed by Bing doubling; Y_{b_1+1} is by Wh., etc.).

Rmk. Allow for ramification

Let $\mathcal{D} := \bigcap Y_i$. This is the decomposition assoc. to this sequence. Define the decomp space $S^3/\mathcal{D}$ as the quotient where each conn. component is crushed to a point. Then $\mathcal{D}$ shrinks, i.e., the quotient $\pi: S^3 \to S^3/\mathcal{D}$ is approximable by homeomorphisms, iff $\sum b_i/2^i \to \infty$.

Rmk. $Wh = \bigcap Y_n$ has only one conn. comp.

Such decomp. are called mixed Bing-Whitehead decompositions.

- $b_i = 0 \implies$ Whitehead decomp. (assoc. to CH), which doesn't shrink ($S^3 \to S^3/Wh$ isn't ABH, since S^3/Wh isn't even a mfd!)
- Only Bing $\implies$ shrinks (Bing 1952, or DET §5.2)

Upshot. Skyscrapers simplify the pf b/c $\partial(\text{skyscraper}) \cong_h S^3$ and in fact skyscraper $\cong_h D^2 \times D^2$. (Also, recall Casson's thm requires $\pi_1 = 1$; this pf works for a bigger class of π_1.)