

A disk embedding starter kit

- 0) Intro
- 1) Motivating problem for disc embedding
- 2) Starting Top. Input
- 3) Disc embedding thm statement
- 4) Towers & Skyscrapers
- 5) Kirby diagrams of Towers

 future  present  past

} mostly used
of 3 & earlier

DET = "disc embed. thm" book.

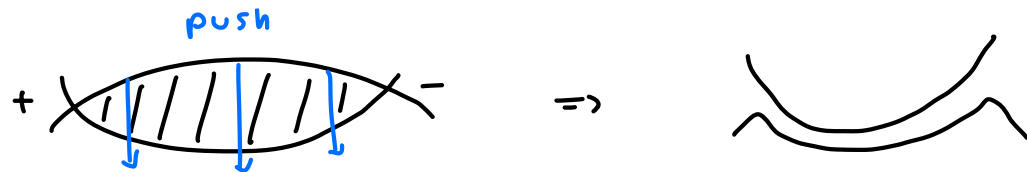
0) Intro

Σ mins

Dim $n \geq 5$ is easy, because

1) I can embed disks

2) I can use Whitney disks to resolve opposite $\cap$ s



So I can geometrically realize alg $\cap \neq$.

What abt $n=4$?

1) Motivating problem for disc embedding

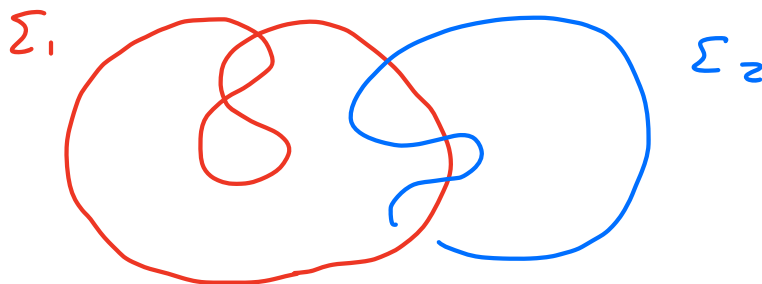
of DET ch 1

Here's a good toy example. Consider X^4 sm, $\pi_1 = 0$

- $Q_{K_3} \cong 2E_8 \oplus 3H$
- $Q_X \cong E_8$ impos., Rokhlin ($e = 8 \neq 0 \pmod{16}$)
- Is $Q_X \cong 2E_8$ possible? No Rokhlin obstruction.

Want to kill H factors in Q_X .

- $\pi_1 X = 0$, so $\pi_2 X = \overset{\text{Hurewicz}}{H_2(X; \mathbb{Z})}$. So $\exists$ immersed spheres Σ_1, Σ_2 s.t. on $\text{span}([\Sigma_1], [\Sigma_2])$, $Q_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



Say Σ_1, Σ_2 embedded, & "geom dual": $\# |\Sigma_1 \cap \Sigma_2| = 1$.

Surger on WLOG Σ_2 ; kill $S^2 \times D^2$, add $D^3 \times S^1$.

$$\begin{array}{c}
 \overline{X} = X / S^2 \times D^2 \\
 \begin{array}{c}
 \cancel{H_2 S^2 \times S^1} \xrightarrow{\cong} \cancel{H_2 S^2 \times D^2} \oplus H_2 \overline{X} \xrightarrow{\text{kills } \Sigma_1} H_2 X \\
 \uparrow \scriptstyle \mathbb{Z} \quad \quad \quad \uparrow \scriptstyle \mathbb{Z} \\
 \cancel{H_1 S^2 \times S^1} \xrightarrow{\cong} \cancel{H_1 S^2 \times D^2} \oplus H_1 \overline{X} \xrightarrow{\text{kills } \Sigma_1} H_1 X
 \end{array} \\
 \Rightarrow \quad 0 \rightarrow H_2 \overline{X} \xrightarrow{\text{kills } \Sigma_1} H_2 X \xrightarrow{\mathbb{Z}} 0
 \end{array}$$

$$X' = \bar{X} \cup S' \times D^3$$

$[\Sigma_2]$
nontrivial
in X

$$\begin{aligned} H_2 S' \times S^1 &\rightarrow H_2 S' \times D^3 \oplus H_2 \bar{X} \rightarrow H_2 X' \\ H_1 S' \times S^1 &\rightarrow H_1 S' \times D^3 \oplus H_1 \bar{X} \rightarrow H_1 X' \end{aligned}$$

$\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$

$$\Rightarrow 0 \rightarrow \mathbb{Z} \cdot [\Sigma_2] \rightarrow H_2 \bar{X} \rightarrow H_2 X' \rightarrow 0$$

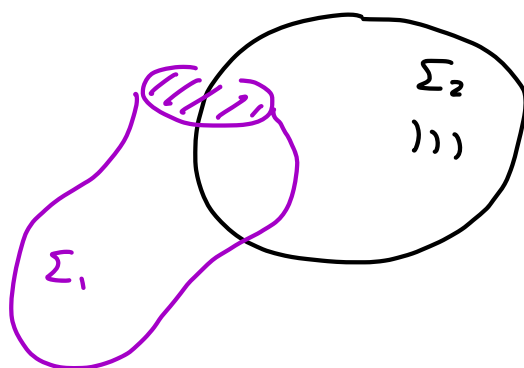
So X' loses both $[\Sigma_1]$, $[\Sigma_2]$, kills H .

Also, this surgery adds a meridian loop around Σ_2 to π_1 . [But if Σ_1 geom dual to Σ_2 ,



then meridian = 0 in $\pi_1(X \setminus \nu \Sigma_2)$!]

Why g. dual
impt!



Here, only need Σ_1 to be immersed.

(but Σ_2 should be embedded).

So surgery still has $\pi_1 = 0$, kills H , yay!

How to realize g. duality?

- Want to cancel alg $\cap$ s btwn Σ_1, Σ_2 .

$\Rightarrow$ Add Whitney disks.

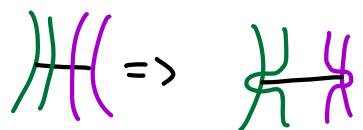
- ① Need $\pi_1(\text{Whitney } S') = 0$, else no immersed disk.

① Use $\pi_1(X)$ coeffs in the $\cap$ form.

② Need framing cond'n to use Whitney disk W to slide over.

① We'll have tools to deal w/ framing (e.g. "boundary twist"), but we'll need to care abt it.

③ What if $W \cap \Sigma_1, \Sigma_2$ nontrivially?

① Use finger move! 

④ What abt self $\cap$ s of W ?

① Add more disks!

$\Rightarrow$ Add ∞ tower of fixing disc: Casson

(Works, but hard)

② Add ∞ mix of surface & disc: skyscraper.

$\Rightarrow$ better by Bing-Whitehead thy.

This is the hard part!

Thm (Casson)

Casson handles are prop. h Top equiv (rel. attaching region) to open 2 handle

$$(D^2 \times \text{int } D^2, \partial' \times \text{int } D^2)$$

Thm (Freedman)

Casson handles are homeomorphic (rel. attaching region) to open 2 handle

$$(D^2 \times \text{int } D^2, \partial' \times \text{int } D^2)$$

§2 Starting Topological Input (Skippable)

Let's elaborate on a few moves we did earlier.

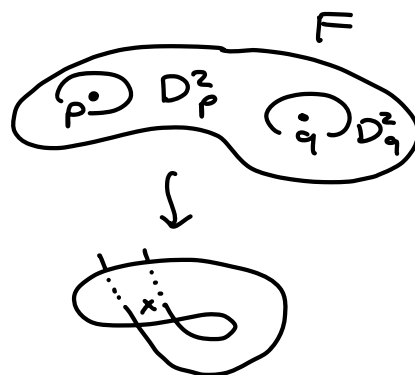
§2.1 Framings (DEF 11.1, 11.2.1)

Def Given a surface F , a **plumbed model** of F is the result of:

1) picking a few pairs of pts on F
(all distinct)

2) "Plumbing" $F \times \mathbb{R}$ at each pair, via the identification

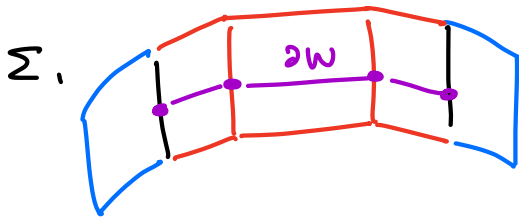
$$\begin{array}{ccc} F & & F \\ \cup & & \cup \\ D_p^2 \times \mathbb{R}^2 & \xrightarrow{\cong} & D_q^2 \times \mathbb{R}^2 \\ (x, y) & \longrightarrow & (y, x). \end{array}$$



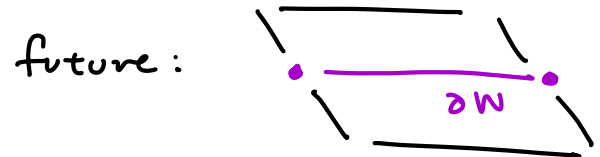
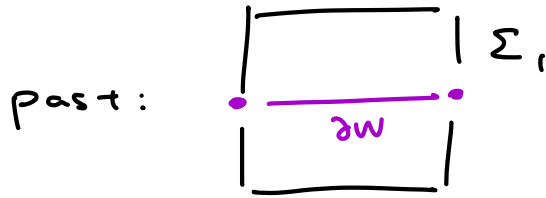
Def A **framed immersion** is an embed.
of some plumbed model of F .

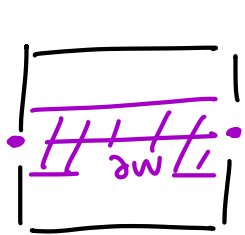
Equivalently, it's an immersion of F w/
trivial normal bundle.

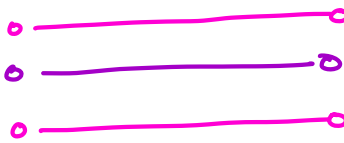
we'll need the right framing when we actually
do Whitney disk moves. We want
the local picture (in $v(\text{Whitney disk})$)
to look as follows.



W is whitney disk

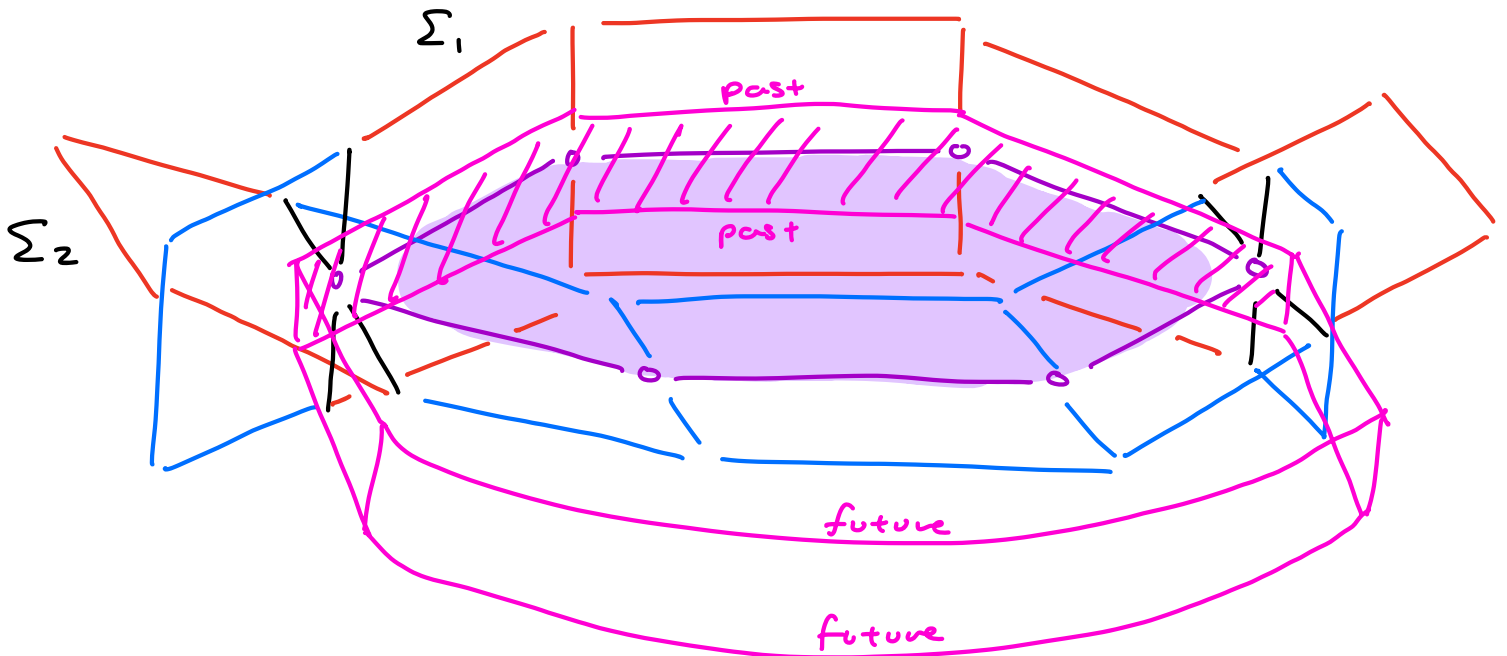


Want to cut out , and glue

 W to Σ_1 , cap off pink in far future.

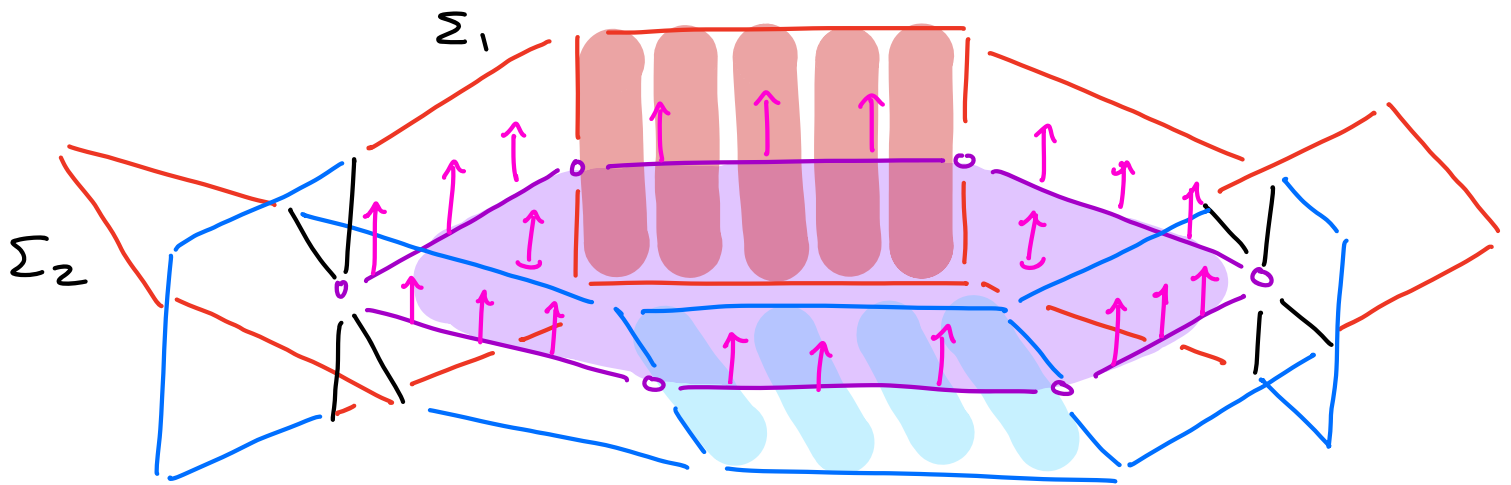
Pink is disjoint from B in this case.

In our model, this will look as follows:

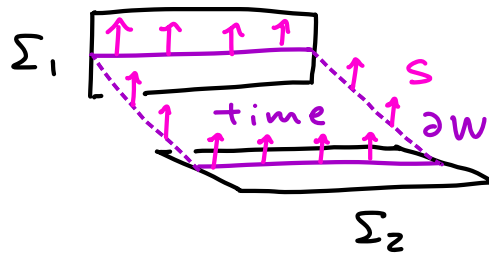


Cut out shaded pink from Σ_1 , add in rest of pink (goes into future).

So the pink "disk surgery" is determined by the following pink framing on W :



So look at the VF s on ∂W given by being $\parallel$ to Σ_1 on $\Sigma_1 \cap \partial W$, $\perp$ to Σ_2 on $\Sigma_2 \cap \partial W$, $\nparallel \perp$ to W . Then want s to actually be con't, & want s to extend to W (in which case, say W is "framed").



Possible issues:

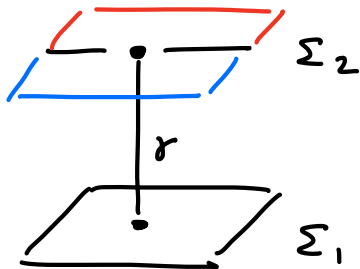
- 1) If $\cap$ pts of Σ_1, Σ_2 same sign, then s gives mobius strip on ∂W , bad.
- 2) If framing s doesn't extend, then the pink framing $s|_{\Sigma_1 \cap \partial W}$, when moved across W , will hit Σ_2 :



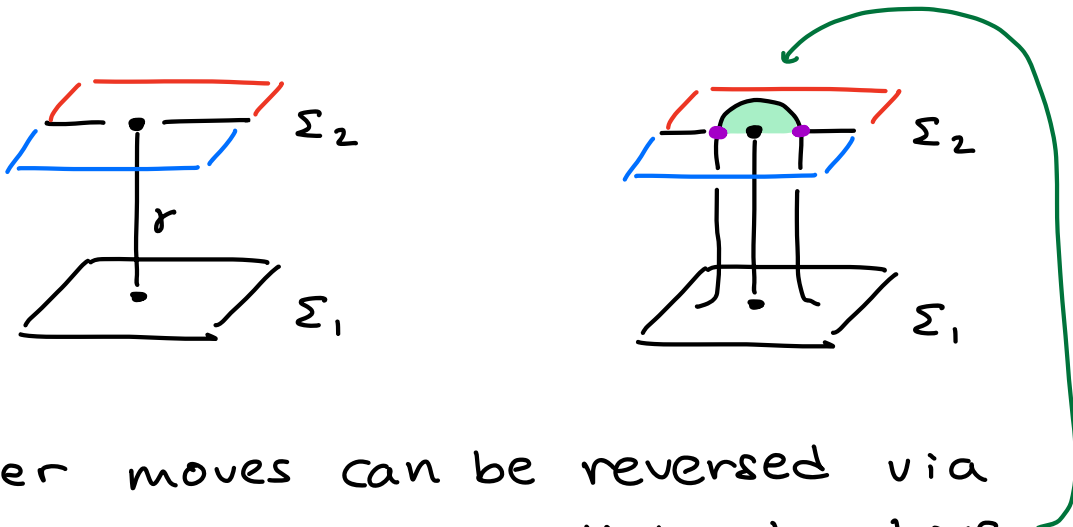
So can't perform Whitney move.

§2.2 Finger moves. (OET 11.2.2)

These look locally around an arc connecting surfaces.



They push Σ_1 as follows:



Finger moves can be reversed via the whitney disk highlighted above.

§3 Stating The Disk Embed Thm

§3.1 Upgraded $\cap$ form (DET 11.3)

I want to keep track of π_1 data when looking at intersections.

Def $\omega: \pi_1(M) \rightarrow \{\pm 1\}$ ori character
(if π_1 action preserves ori).

Def Equivariant $\cap$ form on M : Def'd for M cpt

$$\lambda: H_2(M; \mathbb{Z}\pi) \times H_2(M, \partial M; \omega) \longrightarrow \mathbb{Z}\pi$$
$$(x, y) \longrightarrow \langle PD^{-1}(y), x \rangle$$

image is $\mathbb{Z}\pi$, so keeps track of π_1 data

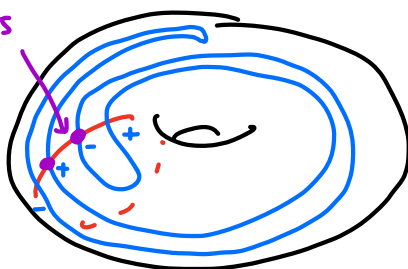
- Here $\mathbb{Z}\pi^\omega$ is a left $\mathbb{Z}\pi$ mod via

$$g \cdot r = \omega(y) gr \quad \begin{array}{l} g \in \mathbb{Z}\pi \\ r \in \mathbb{Z}\pi^\omega \end{array}$$

- PD gives

$$H_2(M; \mathbb{Z}\pi) \cong H^2(M, \partial M; \mathbb{Z}\pi^\omega).$$

don't cancel,
live in diff'elts
of $\mathbb{Z}\pi$



Ok, we'll now want to upgrade this to work for noncpt M . Here's the gist of the idea.

DET Def 11.2

sry handwriting
got worse

Consider f, g sm immersed, $\uparrow$ based ori
discs/spheres in M , s.t. $\partial f, \partial g$ are
disjointly embedded.

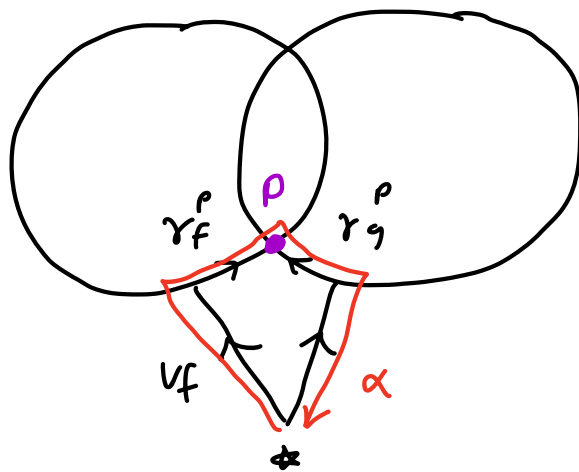
- Pick paths v_f, v_g in M joining
base pt of M to basepts of f, g ("whiskers")

Set

$$\lambda(f, g) = \sum_{p \in f \not\cap g} \varepsilon(p) \alpha(p) \in \mathcal{K}[\pi_1(M)]$$

Here:

- γ_f^p is a simple path in f from $\star_f$ to p .
Def γ_g^p similarly. Pick γ_f^p, γ_g^p to
not $\cap$ other pts in $f \not\cap g$
- $\varepsilon(p) \in \{\pm 1\}$ is 1 if local ori at p
induced by ori's of f, g matches local
ori induced by transporting $\tau_p M$ local
ori along $v_g \gamma_g^p$, and is -1 else.
- $\alpha(p) = [v_f \gamma_f^p (\gamma_g^p)^{-1} v_g^{-1}] \in \pi_1(M)$.



Some definitional properties: (DET Prop 11.4 i), iii)

1) $\lambda(f, g)$ is inv. under reg. hTopies in the interiors of f, g .
hTopies thru immersions

- hTopies pushing $\cap$ pts off the ∂ of an immersed disc can change λ .

2) Replacing $\nu_f \rightarrow \nu'_f$ changes

$$\lambda(f, g) \rightarrow \nu'_f \nu_f^{-1} \lambda(f, g)$$

Similarly $\nu_g \rightarrow \nu'_g$ changes

$$\lambda(f, g) \rightarrow \lambda(f, g) \nu_g (\nu'_g)^{-1}$$

They also have an upgraded version of the self- $\cap \#$ (which is slightly diff't than $\lambda(f, f)$; see Prop 11.8). You can find a def' in DET Def 11.5.

§ 3.2 DET Statement (DET 11.4)

Given $f, \bar{f} : (D^2, S') \rightarrow (M, \partial M)$, say $f, \bar{f}$ have the same framed ∂ if $f(S') = \bar{f}(S')$.
and we may pick fibers of the normal bundles of $f, \bar{f}$ which induce a Topic framings on $f(S') = \bar{f}(S')$.

We'll need the notion of a "good group". Any f.g. gp w/ subexponentially growing cayley gps, e.g. finite or abelian gps, are good.

Thm (Disc Embedding)

Say M^4 is sm, conn, $\partial M \neq \emptyset$, & $\pi_1 M$ is good.

Consider

$$F = (f_1, \dots, f_n) : (D^2 \cup \dots \cup D^2, S' \cup \dots \cup S') \rightarrow (M, \partial M)$$

an immersed collection of discs in M , w/

disjoint emb ∂S . Say F has an im. collection of ALG dual, framed immersed S^2 s

$$G = (g_1, \dots, g_n) : (S^2 \cup \dots \cup S^2) \rightarrow (M, \partial M)$$

Here ALG duality gives
$$\begin{cases} \lambda(f_i, g_j) = \delta_{ij} \\ \lambda(g_i, g_j) = 0 \\ \mu(g_i) = 0 \end{cases}$$

Then $\exists$ a collection of disj, flat, top. emb. discs

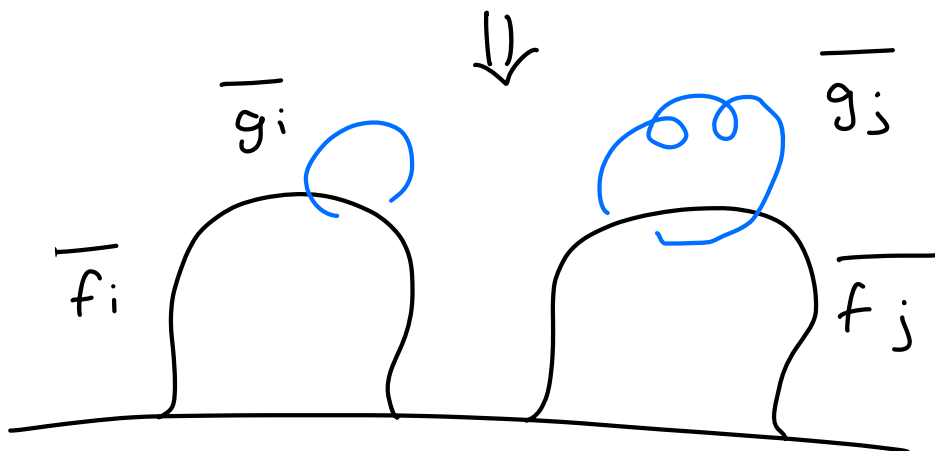
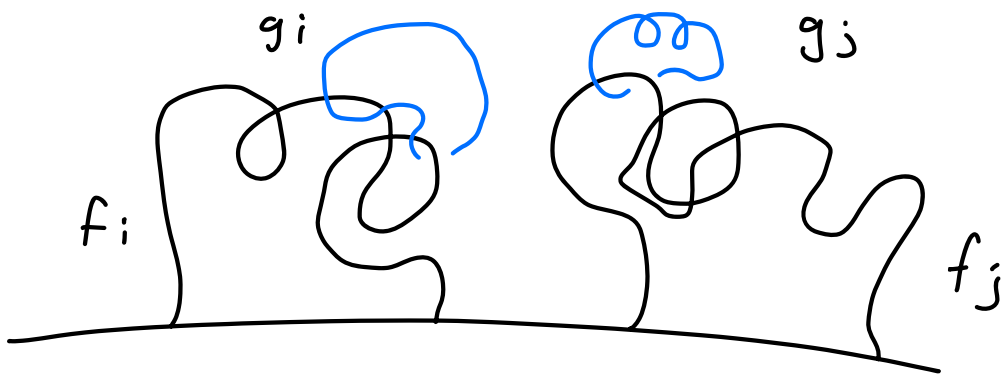
$$\overline{F} = (\overline{f}_1, \dots, \overline{f}_n) : (D^2 \cup \dots \cup D^2, S' \cup \dots \cup S') \hookrightarrow (M, \partial M)$$

$\&$ GEODual, framed, immersed S^2_s

$$\overline{G} = (\overline{g}_1, \dots, \overline{g}_n) : S^2 \cup \dots \cup S^2 \hookrightarrow M$$

s.t. $\forall i,$

- $\overline{f}_i, f_i$ have same framed ∂
- $\overline{g}_i$ is hTopic to g_i .



(skip!)

(OE7 Prop 22.6)

Aside: How do you even use above to prove classification of top. manif, if you only work in sm category?

Here's an imp't starting pt:

Prop 22.6

Every $\mathbb{Z}_2 H S^3 \Sigma$ bounds a cpt contractible top 4-manif.

pf 1) start w/ $\Sigma \times I$. This will have some π_1 , generated by loops α_i , which bound disjoint surfaces F_i (cuz $H_1 = 0$).
Surger along α_i (framed via F_i), replacing
$$\sqcup S^1 \times D^3 \rightarrow \sqcup D^2 \times S^2$$

2) These $D^2 \times S^2$'s cap off the F_i w/ discs D_i .

The surgered manifold has $\pi_1 = 0$, so you can contract an H_1 -generating set of curves on $F_i \cup D_i$ to get a collection of immersed spheres $\{f_i\}$, w/

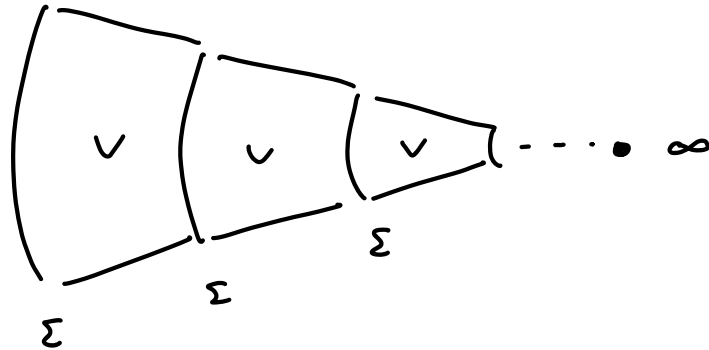
$$\lambda(f_i, f_j) = \mu(f_i) = 0.$$

I have a collection of alg. dual spheres $\{g_i\}$ given by $\{*\} \times S^2 \subset D^2 \times S^2$.

Apply sphere embed thm $\Rightarrow$ replace f_i w/ top. emb. spheres $\overline{f_i}$, w/ geo. transv. spheres $\overline{g_i}$.

3) Surger (as in $\S 1$) on emb. spheres $\overline{f_i}$ to get

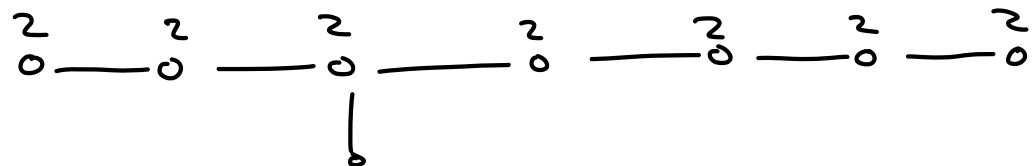
top manif V^4 . Then $\pi_1 V = H_2 V = 0$. Note $\partial V = \Sigma \cup -\Sigma$ still. Now glue ∞ copies of V , 1 pt compactify. Then this is a manif, and in fact satisfies the desired. $\square$



This shows e.g.

Prop 22.7 $\exists E_8$ manif.

pf observe:



This plumbing diag def's an M^4 w/ $\partial M = \Sigma(2,3,5)$, which is a $\mathbb{Z}H^5$? Apply Prop 22.6 to cap M off by a top. contractible W^4 . Then $M \cup W$ works. $\Sigma(2,3,5)$

Goal of next section: skyscrapers are supposed to be "generalized ∞ Casson handles". This section will formalize that notion.

4) Towers & Skyscrapers (DET 12)

We'll also set $D^n = n$ -ball, no corners.

In this section, we'll deal w/ manifolds w/ corners; i.e. manifolds always homeo to $(\mathbb{R}_{\geq 0})^k \times \mathbb{R}^{n-k}$ in sm charts. You can uniquely smooth these corners.

4.1 Blocks (DET 12.1)

Def A block is a quadruple $\mathcal{N} = (N, \Sigma, \phi, \psi)$,

where:

1) N^{int} is cpt, conn, w/ corners, s.t. ∂N is conn
"thickened spine" $\leftarrow$ gives framing of spine.

2) $\Sigma \hookrightarrow N^{\text{int}}$ is prop immersed, ori, w/ $\partial \Sigma \neq \emptyset$,
 $\pi_0 \partial \Sigma = 1$. "spine"

3) $\phi: S' \times D^2 \hookrightarrow \partial N$ sm embed,
 $\phi(S' \times 0) = \partial \Sigma$ "attaching region"

4) $\psi: \bigsqcup S' \times D^2 \hookrightarrow \partial N$ sm embed,

$\text{fin. disj union} \rightarrow \text{Im } \psi \cap \text{Im } \phi = \emptyset$ "tip region" $\leftarrow$

5) The corners of N are

$$\phi(S' \times S') \cup \psi(\bigsqcup S' \times S').$$

This is where we'll attach the next layer.

Here are the 3 standard block types we'll use.

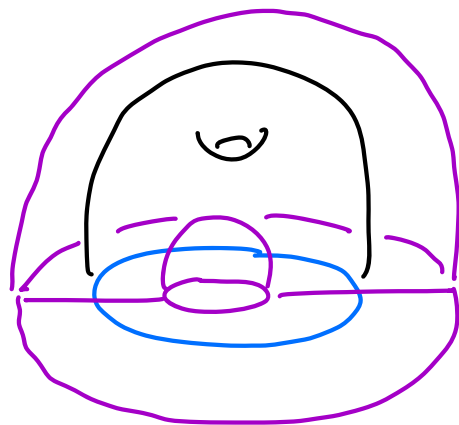
1) Surface blocks

Start w/

$$\Sigma_g \hookrightarrow D^2 \times [0, 1]$$

$$\partial \Sigma_g = S^1 \times \{1/2\}$$

this initial D^2 doesn't really matter except for parameterizing the nbhd of S^1 .



Then

$$V(\Sigma_g) = \Sigma_g \times I \subset D^2 \times I$$

Thus get

$$\Sigma_g \times I \times I \subset D^2 \times I \times I$$

Smooth the corners of $I \times I$, to get

$$N := \Sigma_g \times D^2 \subset D^2 \times D^2$$

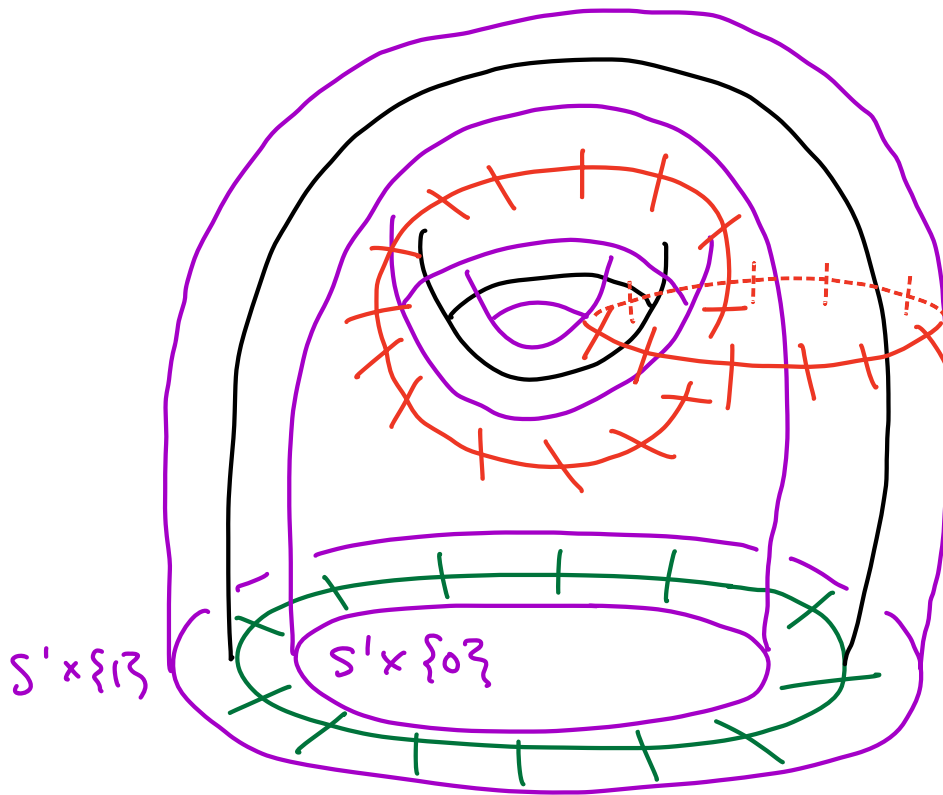
The spine is $\Sigma := \Sigma_g \times \{0\} \subset N$.

For the attaching map ϕ , let core of $S^1 \times D^2$ be $\partial \Sigma_g = \partial \Sigma$. Determine the D^2

factor via the two thickened tangent vectors (i.e. $\phi(S^1 \times D^2) = S^1 \times \text{small } D^2 \hookrightarrow \Sigma_g \times D^2$)

For the tip map ψ , pick pairs of sac's of Σ_g forming a symp. basis of $H_1(\Sigma_g; \mathbb{Z})$.

- For each pair α_i, β_i where $\alpha_i \cap \beta_i = \emptyset$,
push α_i into $\Sigma_g \times \{0\} \subset D^2 \times I$,
 β_i into $\Sigma_g \times \{1\} \subset D^2 \times I$
- These give the cores of the tip;
thicken into tori via tangent
vectors in $\Sigma_g \times S^0$ and extra I
directions.



$$N = \text{[purple box with diagonal lines]}$$

$$\Sigma = \text{[black box with diagonal lines]}$$

$$\phi = \text{[green box with diagonal lines]}$$

$$\psi = \text{[red box with diagonal lines]}$$



These also have a suppressed
 $4+1$ (time) dimension.

2) Disc blocks

Start w/ thickened disc

$$(D^2 \times D^2, D^2 \times \{0\}).$$

We can plumb in two (signed) ways

(corr' to signed self $\cap$ s):

$$1) \quad \begin{array}{ccc} D^2 \times D^2 & \longrightarrow & D^2 \times D^2 \\ x, y & & y, x \end{array}$$

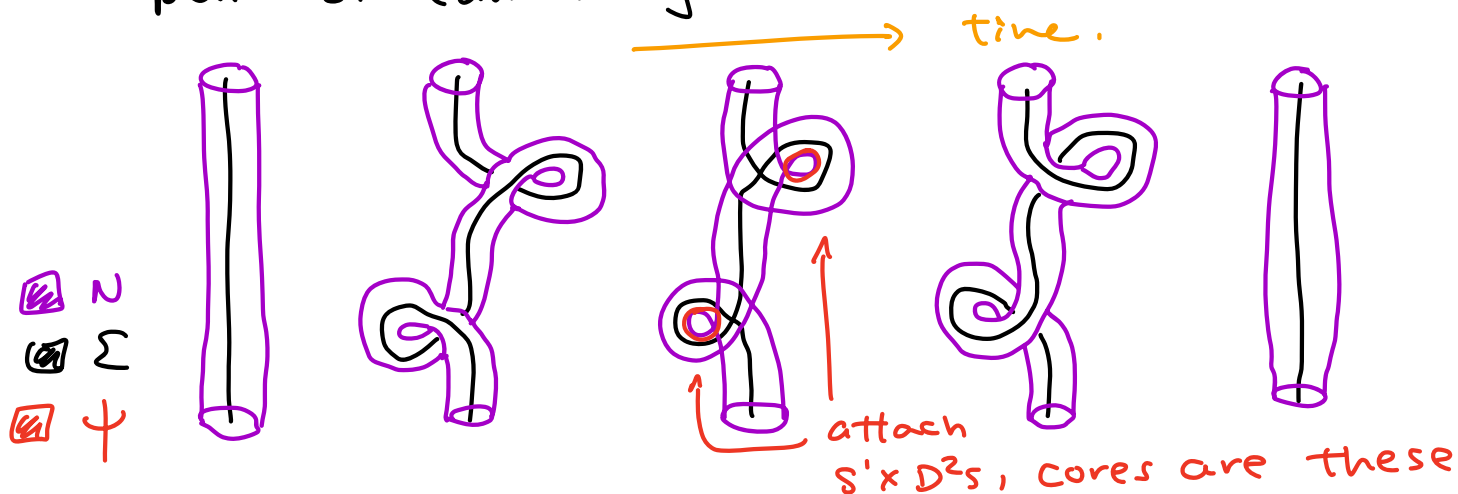
$$2) \quad \begin{array}{ccc} D^2 \times D^2 & \longrightarrow & D^2 \times D^2 \\ x, y & & -y, -x \end{array}$$

Plumb a few times, w/ ^{>0} cancelling signs.
let the result of the self plumbings on

$$(D^2 \times D^2, D^2 \times \{0\}) \text{ be } (N, \Sigma).$$

Set the attaching region ϕ as $S^1 \times D^2 \subset \partial(D^2 \times D^2)$.

For the tips: here's a local model for a pair of cancelling plumbings:



accessory circles shown.

To thicken in solid tori, take 1 direction tangent to ∂N shown, take 1 direction in time.

3) Cap Block

This is basically a disc block, except

- 1) There's no restriction on the signs of self $\cap$ s / self plumbings
- 2) There's no tip region.

A block is a surface / disc / cap block if under some diffeomorphism $\cong$, I can identify it w/ the std surface / disc / cap block

4.2 Towers (still DET 12.1)

Ok, we'll now start building skyscrapers.

Let's start w/ towers. Here are a lot of def's.

Def A stage is a fin union of surface, disc, cap blocks, s.t. the surface, disc blocks are disjoint from all other blocks.

→ cap blocks can $\cap$ each other.

• Given our blocks $N_i = (N_i, \Sigma_i, \phi_i, \psi_i)$, write

stage $Q = (G = \cup N_i, \cup \Sigma_i, \underline{\Phi} = \cup \phi_i, \underline{\Psi})$.

• Here, use a subset of ψ_i in tip $\underline{\Psi}$.
stage tip region
stage attach region

Terminology

- o full: if $\underline{\Psi}$ uses all ψ_i
- o homogeneous: uses ONLY blocks of ONE type
- o symmetric = full + hom.

Def A gen. tower T of ht h is a cpt 4-manif w/ corners, given by

$$G_0 \cup G_1 \cup \dots \cup G_n$$

attaching region

of G_i , so $= \text{im } \underline{\Phi}_i$, These are stages

inherits framing

Such that:

$$1) G_i \cap G_j = \emptyset \quad |i-j| > 1$$

$$2) \underbrace{\Phi}_{i+1 \text{th attach}}_{i+1} = \underbrace{\Psi}_{i \text{th tip}}_i \quad \text{for } i \geq 1 \text{ (up to permutation of pts)}$$

3) The corners of $\mathcal{T}$ are

$$\underline{\Phi}_1(\sqcup S' \times S') \cup \underline{\Psi}_n(\sqcup S' \times S')$$

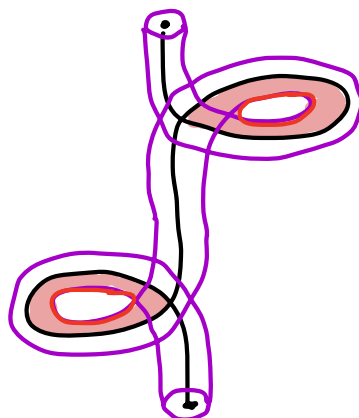
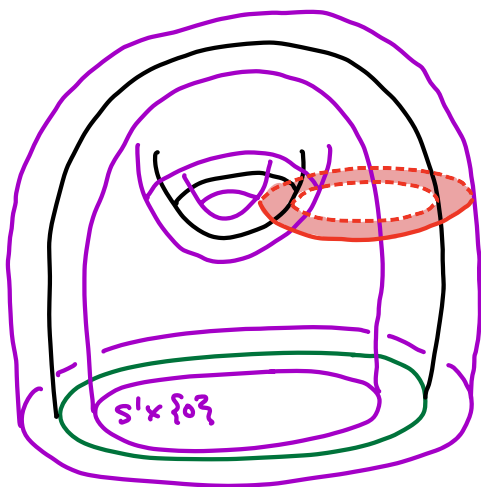
• We'll also set $G_0 = \partial_- \mathcal{T}$.

• We can also call gen' towers hom/full/sym.

What is the spine of $\mathcal{T}$?

→ connect spines Σ_{i+1}, Σ_i via annuli starting at $\partial \Sigma_{i+1}$.

Think abt std models:



Many def'ns:

1) A grope $G(h)$ of ht n is a sym gen' tower, w/ only surface stages. of ht $\hat{n}$

2) A capped grope $G^c(n)$ of ht n is a sym gen' tower of ht $n+1$

$\left\{ \begin{array}{l} G_1, \dots, G_n \text{ surface stages} \\ G_{n+1} \text{ is a cap stage} \end{array} \right. \quad] \text{ body}$

3) A tower T_n w/ n storeys is a sym gen tower

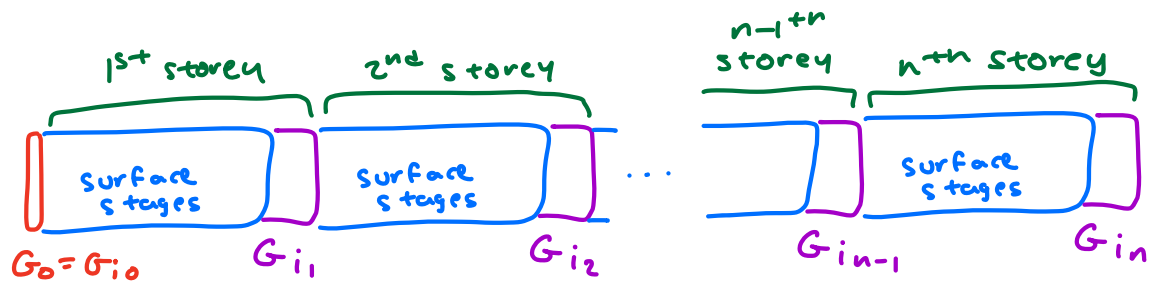
$G_0 \dots G_n$, where each G_i is a surface or disc stage, and there are n disc stages, including G_n . Say the disc stages are $G_{i_1}, \dots, G_{i_n} = G_n$,
† let $i_0 = 0$.

• For $j \in [1, n]$,

$$G_{i_{j-1}+1} \cup \dots \cup G_{i_j}$$

is the j^{th} storey of T_n . This storey has $i_j - i_{j-1}$ stages, and forms a capped grope (when ignoring the tip data of G_{i_j}).

• call G_{i_j} the j^{th} storey grope caps.



! skip

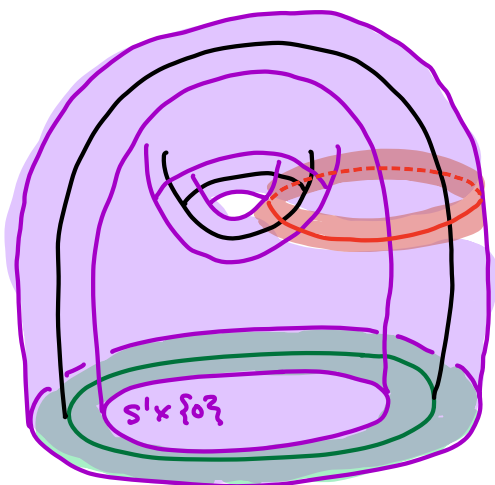
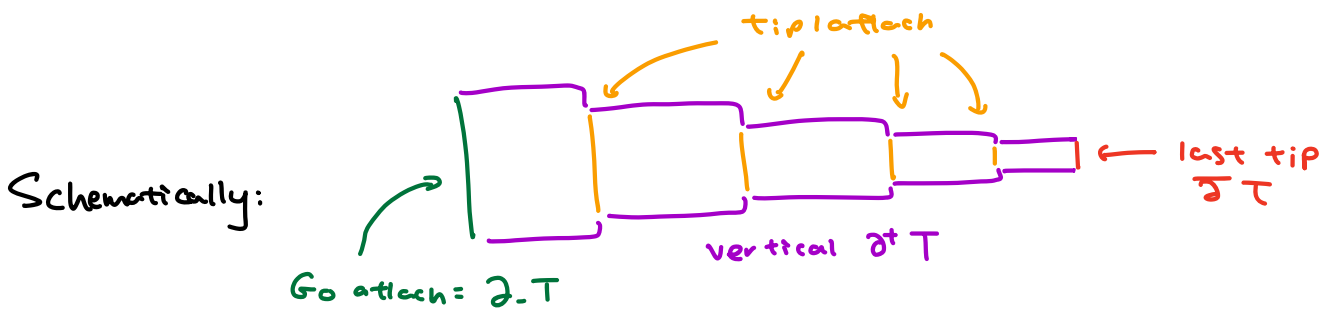
4) A capped tower T_n^c w/ n storeys is a
 sym gen tower $G_0 \cup \dots \cup G_n \cup G_{n+1}$
 (n storey tower) (cap stage)

Also, given a gen' tower T , we'll let $\partial^- T = G_0$.

Denote the tip region of T as $\bar{\partial} T$. Denote
 (also G_n)

the rest of the ∂ as the vertical boundary

$$\partial^+ T := \overline{\partial T} \setminus (\partial^- T \cup \bar{\partial} T)$$



In the case of $T = 1$
 surface block, looks
 like this

- Only one α curve in tip shown
- only outer vertical ∂ shown

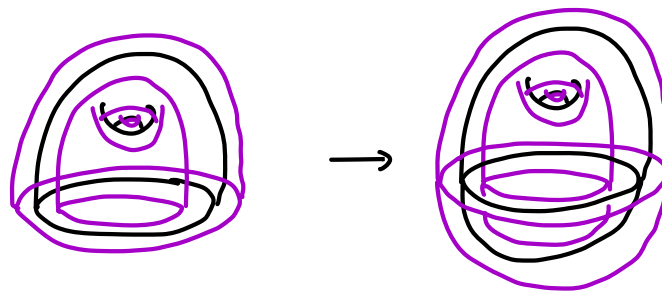
$\partial^- T$ $\bar{\partial} T$ $\partial^+ T$

Extra def's:

- union-of-spheres-like (capped) gropes / towers:
attach $D^2 \times D^2$ s to $S^1 \times D^2$ s in attaching region G_0 ,
so $G_1 \cup G_0$ looks like a bunch of thickened closed surfaces

according to framing

! skip



- In a grope, add $\pm$ data to the pairs of accessory curves in the lowest surface stage. (Let attached blocks in higher stages inherit the sign from their predecessor). This gives a $\pm$ labelling to the entire grope. You can also do this to towers, by doing this to the lowest grope.

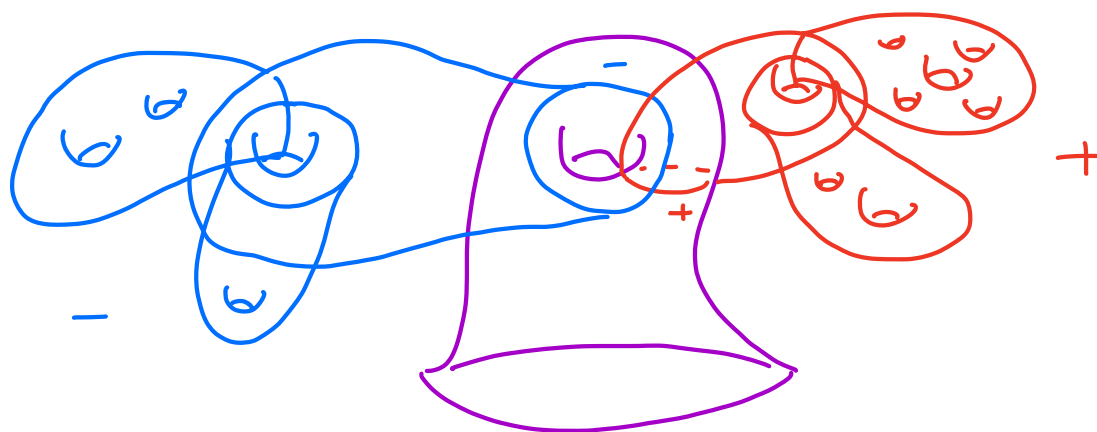
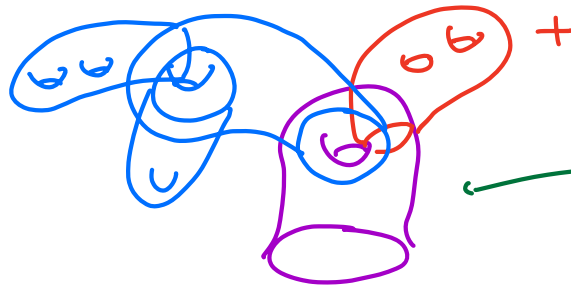


Figure: The spine of one such $\pm$ -labelled grope.

- In this $\pm$ setting, we can make gropes asymmetric (i.e. the $+$, $-$ side have differing heights). $G_0 \cup G_1$ will always be height 1. If the $+$ side has further height a , the $-$ side height b , then say G has height (a, b) .

If $|a-b|=1$, then say G has height $\frac{a+b}{2}$.



note that asym heights are shifted by 1, since they don't count G_1 in the ht.

Ex The spine of a 1.5-ht asym. grope.

Ok, we can now define a good group. The defin kinda sucks.

Def Given an immersion $D^2 \hookrightarrow M^4$, a double pt loop is the image of a path connecting the preimage pts of a double pt.

Def A group Γ is good, if for every

- $\left\{ \begin{array}{l} \bullet \text{ Asym capped grope } G^c \text{ of ht } 1.5 \\ \bullet \text{ gp hom } \phi : \pi_1(G^c) \rightarrow \Gamma \end{array} \right.$

There $\exists$ an immersed disc $D^2 \hookrightarrow G^c$ whose double pt loops def' elts of $\ker \phi \subset \pi_1(G^c)$.

The triv. gp is obv. good. But not rily clear why any other gps are good (lol)

4.3 Skyscrapers

(DET 12.2)

Finally!

Def An ∞ gen tower T_∞ is a cornered ori 4 -manif
$$T_\infty = \bigcup_{i \geq 0} G_i$$

where $G_i = (G_i, \Sigma_i, \underline{\Phi}_i, \underline{\Psi}_i)$ is a stage $\forall i \geq 1$, &
 $G_0 \subset \partial T$ is a $U S^1 \times D^2$. The other axioms are as before.)

Prev. terminology (e.g. hom, full, sym, spine, etc) carry over.

Def An ∞ tower T_∞ is a sym ∞ gen' tower
$$\bigcup_{i \geq 0} G_i$$
, where the $G_i = (G_i, \Sigma_i, \underline{\Phi}_i, \underline{\Psi}_i)$

are again all disc / surface stages. Form storeys
by taking the stages btwn disc stages,
as before.

Def $T_\infty^{\leq k}$ is the fin tower given by the first
 k storeys of T_∞ .

As def'd above, T_∞ is noncpt. We'll compactify it as follows:

Def Consider a loc cpt top space X , & an ascending
chain of cpt sets

$$K_1 \subsetneq K_2 \subsetneq \dots$$

where $K_i \subset \text{int } K_{i+1}$ and $\bigcup K_i = X$.

Let $C_i = X \setminus K_i$, so we have a chain

$$C_1 \supsetneq C_2 \supsetneq \dots$$

Let $E(X)$ be the set of sequences

$$U_1 \supsetneq U_2 \supsetneq \dots$$

where $U_i \subset C_i$ is a conn cpt.

Then:

• call $E(X)$ the "ends" of X

• let the endpt compactification $\hat{X}$ of X

be the topol on $X \cup E(X)$, gen' by

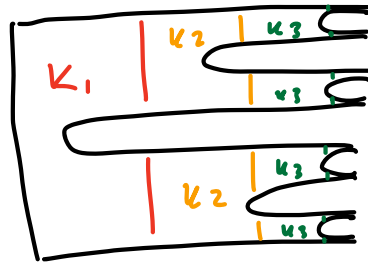
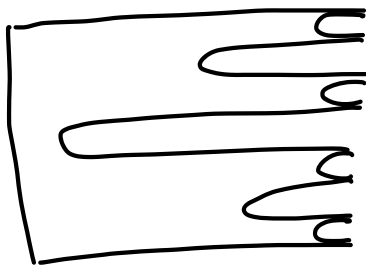
1) the opens of X , and

2) the sets $V \subset \hat{X}$ s.t. $V \cap X = U_j$
for some j , and $V \cap E(X)$ consists
of the seq's $(U_1, U_2, \dots)$ s.t.

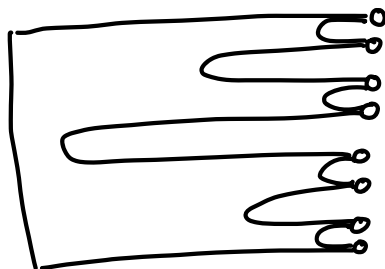
$$V_j, V_{j+1}, \dots \subset U_j.$$

Ex:

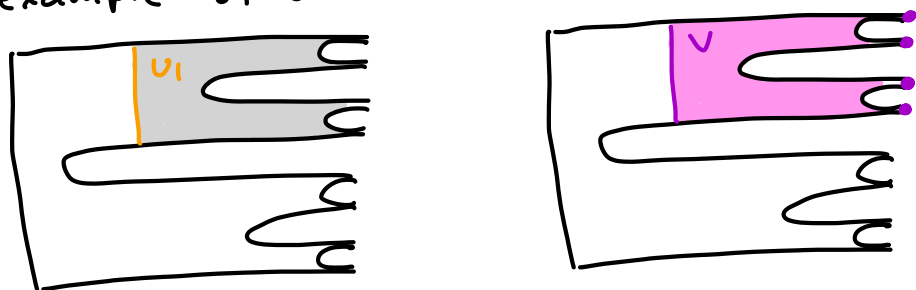
$X =$



$\hat{X} =$



An example of a basis open V :



[Note (Rmk 12.18 DET) The choice of exhausting seq^l of cpt sets doesn't change the homeo type of $\hat{X}$ or the cardinality of $E(X)$.]

Finally, skyscrapers.

Def Consider an ∞ tower S , w/ N_j surface stages in its j^{th} storey. Then S is called an open skyscraper if

- 1) $N_j \geq 4 \quad \forall j$ ("repliable")
- 2) $\sum N_j / 2^j = \infty$ ("a shrinkable")

The endpt compactification of an open skyscraper is a skyscraper, denoted $\hat{S}$.

A big goal of ours will be to show

$$(\hat{S}, \partial \hat{S}) \stackrel{\text{homeo}}{\simeq} (D^2 \times D^2, S^1 \times D^2)$$

yet we don't even know that $\hat{S}$ is a manif!

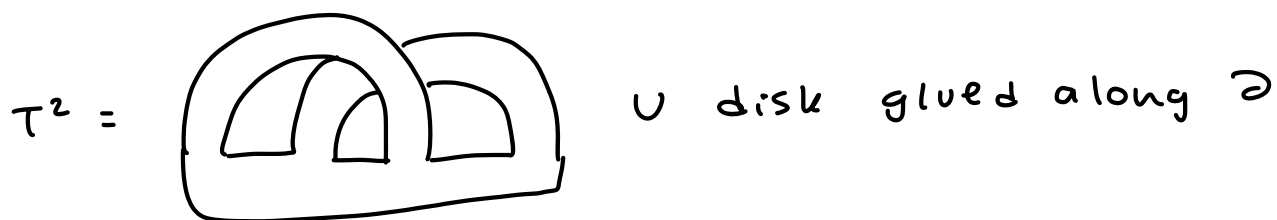
The 2nd cond implies by Bing-Whitehead decomp stuff

that $\partial \hat{S}$ is 1) a manif, & 2) $\simeq S^3$. So that's

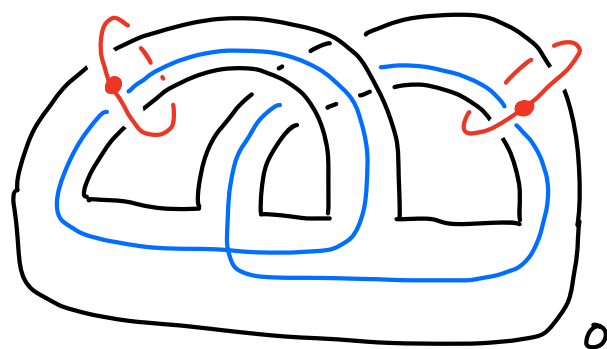
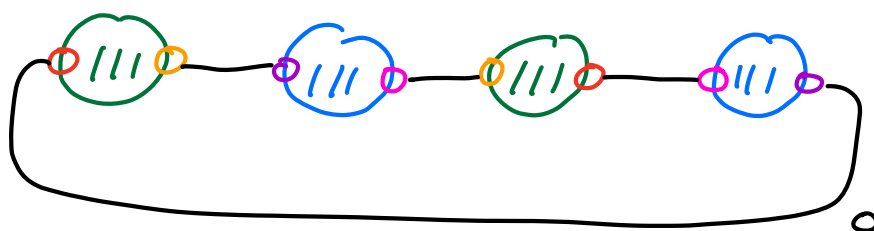
a good start $\ddot{\smile}$.

§5 Kirby diagrams of Towers

Surface blocks



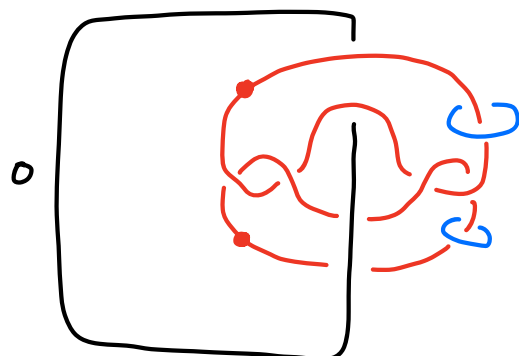
So in handle notation, when you thicken by D^2 , get



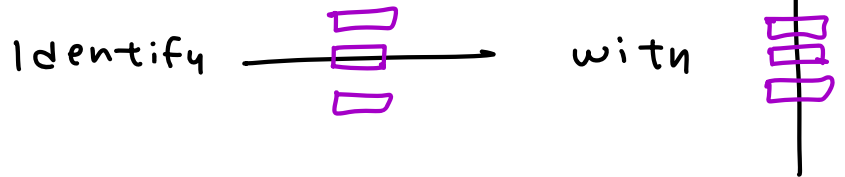
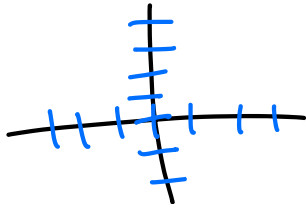
The blue circles form a H_1 basis of the spine.
(we don't interpret them as handles rn).

Ok, well the black knot is an unknot.

Unknotting it gives:



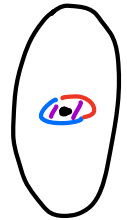
plumbings



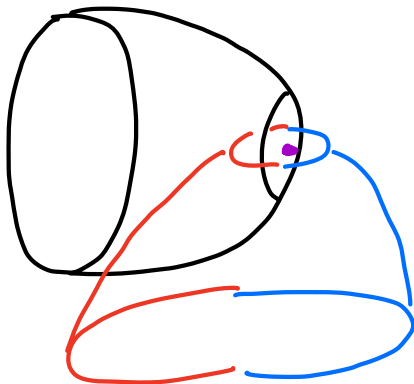
1 dim up: Identify



w/

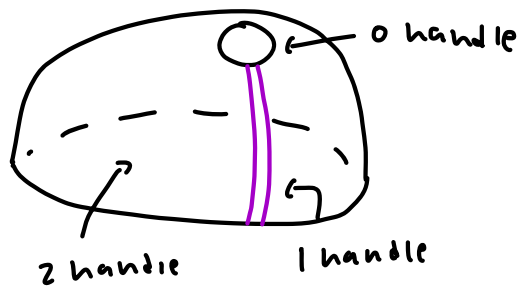


Say we want to plumb two 2-handles

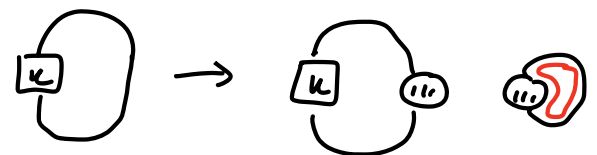


want to cut out a small disk in the center of one 2 handle which forms a meridian of the other 2 handle.

To cut out a disk in the center of the first h^2 , view b^2 as follows:



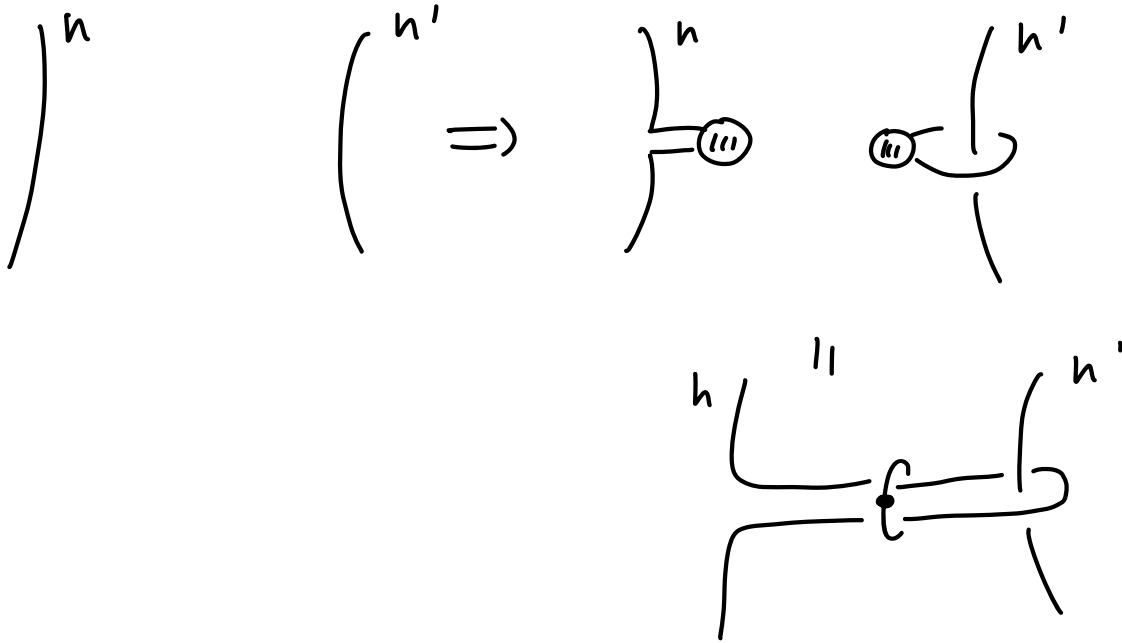
So I can replace



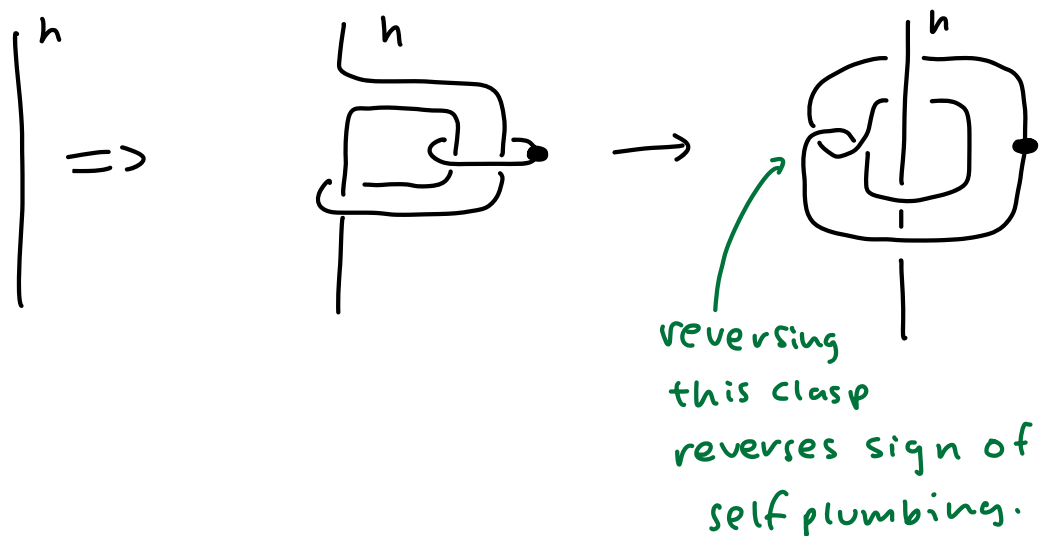
and this deletes a disk, which would have been a 2 handle attached along *here*



So loop this red loop around the other knot:

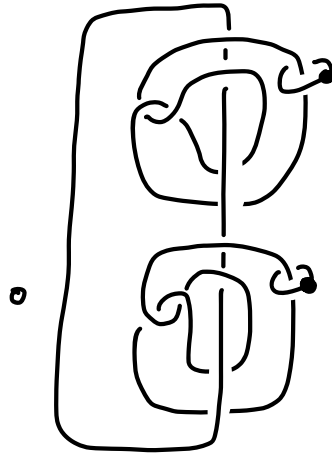


Oh, I won't justify these, but self plumbings look like this:



Disc Blocks

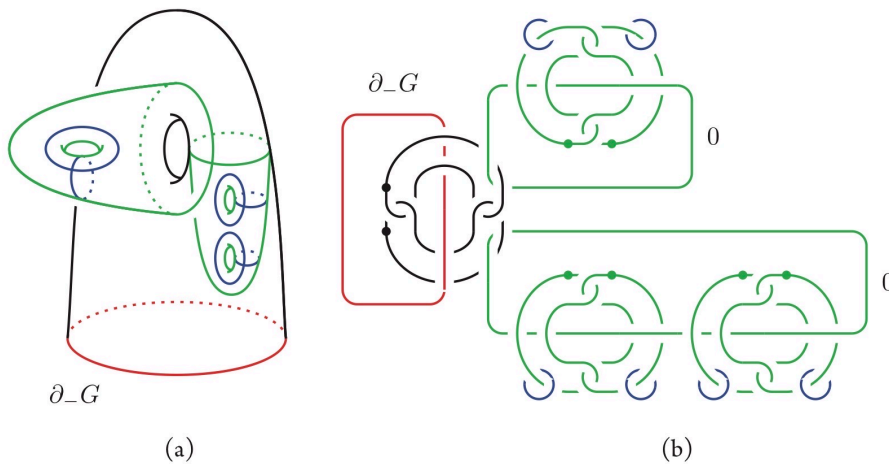
As a result, disc blocks look like this:



Gropes

When you stack blocks on each other, replace tip region w/ new attaching region.

Here's a ht 2 gen tower w/ only surface blocks:



This is DET Fig 13.14.