Fun facts about triply-graded homology

Joshua Wang

October 10, 2025

Background on HOMFLYPT

The HOMFLYPT polynomial is a rational function $\overline{P}(L) \in \mathbf{Z}(q, a)$ for each oriented link L satisfying the skein relation

$a\overline{P}\Big($  $\Big) - a^{-1}\overline{P}\Big($  $\Big) = (q - q^{-1})\overline{P}\Big($  $\Big)$

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Torus knot examples:

$$\overline{P}(T(2,3)) = \begin{pmatrix} -a^4(1) \\ +a^2(q^{-2}+q^2) \end{pmatrix} \qquad \overline{P}(T(2,5)) = \begin{pmatrix} -a^6(q^{-2}+q^2) \\ +a^4(q^{-4}+1+q^4) \end{pmatrix}$$

$$\overline{P}(T(3,4)) = \begin{pmatrix} a^{10}\,(1) \\ -a^8\left(q^4+q^2+1+q^{-2}+q^{-4}\right) \\ +a^6\left(q^6+q^2+1+q^{-2}+q^{-6}\right) \end{pmatrix}$$

Background on HOMFLYPT

The triply-graded homology $\overline{\mathrm{H}}(L) = \bigoplus_{i,j,k\in\mathbf{Z}} \overline{\mathrm{H}}^{i,j,k}(L)$ of an oriented link L, defined by Khovanov–Rozansky '05, satisfies

$$\text{the coefficient of } q^i a^j \text{ in } \overline{P}(L) = \sum_{k\in\mathbf{Z}} (-1)^k \operatorname{rk} \overline{\mathrm{H}}^{i,j,k}(L).$$

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$\overline{\mathrm{H}}(T(3,4))$ viewed in “a-layers”:

a^{10}	-6	-4	-2	0	2	4	6
8				$\mathbf{Z}$			
7							
6							
5							
4							
3							
2							
1							
0							

$a^{10}(1)$

a^{8}	-6	-4	-2	0	2	4	6
8							
7					$\mathbf{Z}$	$\mathbf{Z}$	
6							
5			$\mathbf{Z}$	$\mathbf{Z}$			
4							
3		$\mathbf{Z}$					
2							
1							
0							

$-a^8(q^{-4}+q^{-2}+1+q^2+q^4)$

a^{6}	-6	-4	-2	0	2	4	6
8							
7							
6							$\mathbf{Z}$
5							
4				$\mathbf{Z}$	$\mathbf{Z}$		
3							
2			$\mathbf{Z}$				
1							
0	$\mathbf{Z}$						

$+a^6(q^{-6}+q^{-2}+1+q^2+q^6)$

$T(n, n+1)$ and Catalan combinatorics

Today, many algebraic geometers, representation theorists, and algebraic combinatorialists are interested in triply-graded homology.

$T(n, n+1)$ and Catalan combinatorics

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There was a series of conjectures, now mostly proven, relating $\overline{\mathrm{H}}(L)$ to objects outside of topology (q, t-Catalan numbers, Hilbert schemes of plane curve singularities, Cherednik algebras, the Hilbert scheme of points in $\mathbf{C}^2$)

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Gorsky '10 conjectured a relationship between $\overline{\mathrm{H}}(T(n, n+1))$ and Catalan combinatorics, proved by Hogancamp '17.

$T(n, n+1)$ and Catalan combinatorics

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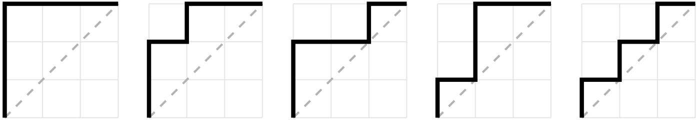
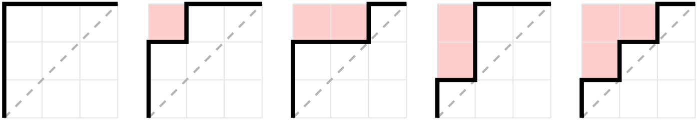
Gorsky '10 conjectured a relationship between $\overline{\mathrm{H}}(T(n, n+1))$ and Catalan combinatorics, proved by Hogancamp '17.

Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count mountain ranges $\mu \subset [0, n] \times [0, n]$.



$T(n, n+1)$ and Catalan combinatorics

For each mountain range $\mu \subset [0, n] \times [0, n]$, let $A(\mu)$ be the area above it:



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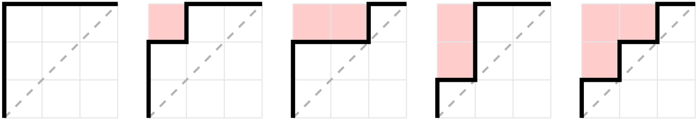
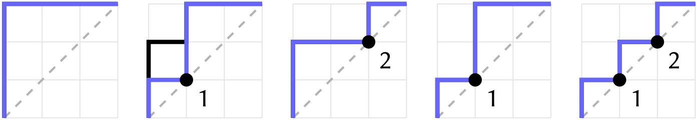


With μ fixed, imagine a blue path starting from the northeast corner, traveling west and south, bouncing between μ and the diagonal.



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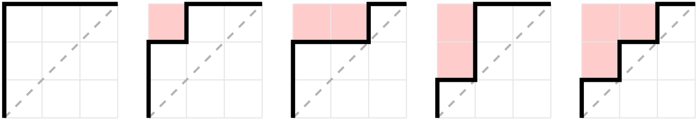

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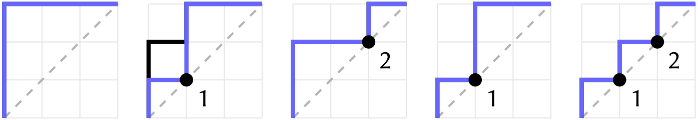

The sum of the locations where it hits the diagonal is the *bounce* $B(\mu)$.

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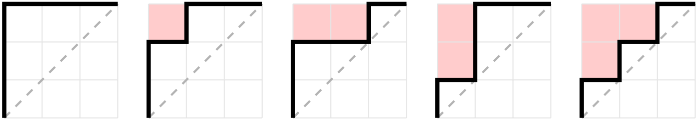

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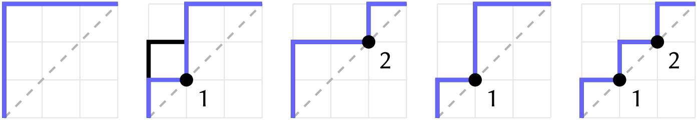

The sum of the locations where it hits the diagonal is the *bounce* $B(\mu)$. The q, t-Catalan number $C_n(q, t)$ is the weighted count

$$C_n(q,t) = \sum_{\mu} q^{\binom{n}{2} - A(\mu)} t^{B(\mu)} \qquad C_3(q,t) = q^3 + q^2 t + q t^2 + qt + t^3$$

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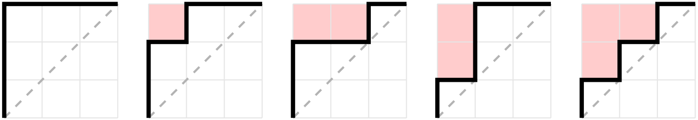

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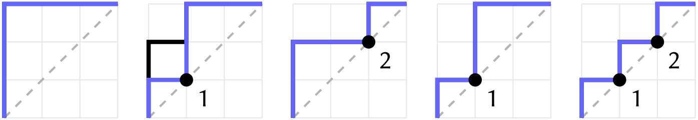

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Originally defined by Garsia–Haiman '96. This version by Haglund '98.

$T(n, n+1)$ and Catalan combinatorics

Examples from A. Garsia's "The Macdonald Polynomial Web Page":

$$\boxed{C_3(q,t) \quad \begin{array}{cccc} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{array}} \qquad \text{represents} \qquad \begin{array}{cccc} q^3 & & & \\ & q^2t & & \\ & qt & qt^2 & \\ & & & t^3 \end{array}$$

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$$
C_3(q,t) \quad
\begin{array}{cccc}
1 & & & \\
 & 1 & & \\
 & 1 & 1 & \\
 & & & 1
\end{array}
\qquad \text{represents} \qquad
\begin{array}{cccc}
q^3 & & & \\
 & q^2t & & \\
 & qt & qt^2 & \\
 & & & t^3
\end{array}
$$

$$
C_4(q,t) \quad
\begin{array}{ccccccc}
1 & & & & & & \\
 & 1 & & & & & \\
 & 1 & 1 & & & & \\
 & 1 & 1 & 1 & & & \\
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 & & & 1 & 1 & 1 & \\
 & & & & & & 1
\end{array}
$$

$$
C_5(q,t) \quad
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 & & 1 & 2 & 1 & 1 & & & & & \\
 & & 1 & 2 & 2 & 1 & 1 & & & & \\
 & & & 1 & 2 & 2 & 1 & 1 & & & \\
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Theorem (Haiman '98, Garsia–Haglund '01): The q, t-Catalan number is q, t-symmetric and q, t-unimodal.

$T(n, n+1)$ and Catalan combinatorics

$\overline{\mathrm{H}}(T(3,4))$ viewed in a-layers.

a^{10}	−6	−4	−2	0	2	4	6
8				Z			
7							
6							
5							
4							
3							
2							
1							
0							

a^{8}	−6	−4	−2	0	2	4	6
8							
7					Z	Z	
6							
5			Z	Z			
4							
3		Z					
2							
1							
0							

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6							
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4							
3		$\mathbf{Z}$					
2							
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0							

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8							
7							
6							$\mathbf{Z}$
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4				$\mathbf{Z}$	$\mathbf{Z}$		
3							
2			$\mathbf{Z}$				
1							
0	$\mathbf{Z}$						

There is a nice change of gradings we can apply to triply-graded homology: (also, only record ranks).

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a^{8}	-6	-4	-2	0	2	4	6
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a^{6}	-6	-4	-2	0	2	4	6
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6							$\mathbf{Z}$
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2			$\mathbf{Z}$				
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0	$\mathbf{Z}$						

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1			

1			
1	1		
	1	1	

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	1		
	1	1	
			1

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$\overline{\mathrm{H}}(T(3,4))$ is

$$\begin{array}{|cccc} \hline & & & \\ & & & \\ & & & \\ 1 & & & \end{array} \qquad \begin{array}{|cccc} \hline & & & \\ 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \end{array} \qquad \begin{array}{|cccc} \hline 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{array}$$

$T(n, n+1)$ and Catalan combinatorics

$\overline{\mathrm{H}}(T(3,4))$ is

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$\overline{\mathrm{H}}(T(4,5))$ is

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$\overline{\mathrm{H}}(T(4,5))$ is

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C_3(q,t) & & 1 & 1 & \\
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C_4(q,t) & & 1 & 1 & 1 & & & \\
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\end{array}
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\end{array}
$$

$$
\boxed{C_3(q,t) \quad \begin{array}{cccc}
1 & & & \\
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\qquad
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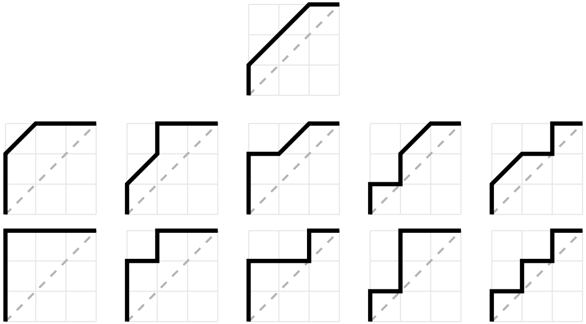
 & & & 1 & 1 & 1 & \\
 & & & & & & 1
\end{array}}
$$

Gorsky conjectured that the bottom a-layer of $\overline{\mathrm{H}}(T(n, n+1))$ is the q, t-Catalan number $C_n(q, t)$!

$T(n, n+1)$ and Catalan combinatorics

Gorsky also conjectured that the higher a-degree q, t-polynomials are weighted counts of Schröder paths (related to q, t-Schröder numbers)



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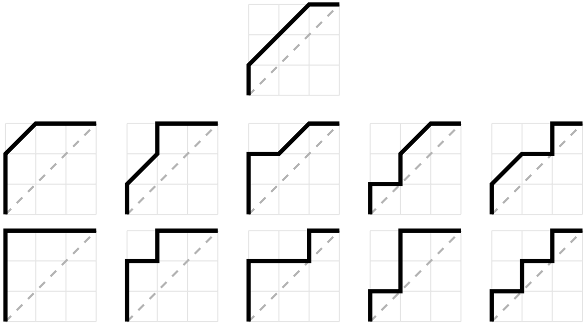

The number in layer $k \in \{0, \ldots, n-1\}$ is equal to the number of k-dim facets of the $(n-1)$-dim associahedron. Total = nth Hipparchus number (150 BCE).

$T(n, n+1)$ and Catalan combinatorics

Here are the ranks of $\overline{\mathrm{H}}$, $\overline{\mathrm{Kh}}$, and $\widehat{\mathrm{HFK}}$ for $T(n, n+1)$.

n	$\operatorname{rk} \overline{\mathrm{H}}$	$\operatorname{rk} \overline{\mathrm{Kh}}$	$\operatorname{rk} \widehat{\mathrm{HFK}}$
1	1	1	1
2	3	3	3
3	11	5	5
4	45	9	7
5	197	17	9
6	903	31	11
7	4 279	57	13
8	20 793	105	15
9	103 049	193	17
10	518 859	355	19
11	2 646 723	653	21
12	13 648 869	1 201	23

The last few computations of $\operatorname{rk} \overline{\mathrm{Kh}}$ are conjectural (Shumakovich–Turner).

Stable triply-graded homology

$\overline{\mathrm{H}}(T(3,4))$ is

1			

1			
1	1		
	1	1	

1			
	1		
	1	1	
			1

Stable triply-graded homology

$\overline{\mathrm{H}}(T(3,4))$ is

1				

1				
1	1			
	1	1		

1				
	1			
	1	1		
			1	

$\overline{\mathrm{H}}(T(3,5))$ is

1				
	1			

1				
1	1			
	2	1		
		1	1	

1				
	1			
	1	1		
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				1

Stable triply-graded homology

$\overline{\mathrm{H}}(T(3,4))$ is

$$\begin{array}{|ccccccc}
\hline
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$\overline{\mathrm{H}}(T(3,5))$ is

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\end{array}
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$\overline{\mathrm{H}}(T(3,7))$ is

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\end{array}$$

Stable triply-graded homology

There is a well-defined stable limit $\overline{\mathrm{H}}(T(n,\infty)) := \lim_{m\to\infty} \overline{\mathrm{H}}(T(n,m))$ with appropriate grading shifts.

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					⋱	⋱	⋱	⋱	⋱

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Hogancamp '15 showed that $\overline{\mathrm{H}}(T(3,\infty)) \cong \mathbf{Z}[x_1, x_2] \otimes \Lambda(\xi_1, \xi_2)$.

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$$
\begin{array}{|ccccccc}
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\begin{array}{|ccccccccc}
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\begin{array}{|cccccccccc}
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\end{array}
$$

Triply-graded to Khovanov

Hogancamp '15 proves that $\overline{\mathrm{H}}_N(T(n,\infty)) \cong \mathbf{Z}[x_1,\ldots,x_{n-1}] \otimes \Lambda(\xi_1,\ldots,\xi_{n-1})$.

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Gorksy–Oblomkov–Rasmussen '12 conjecture that if you equip this graded algebra with the differential

$$d_2(x_k) = 0 \qquad d_2(\xi_k) = \sum_{\substack{i,j\geq 0\\ i+j=k}} x_i x_j$$

its homology is $\overline{\mathrm{Kh}}(T(n,\infty))$.

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If you instead use the differential

$$d(x_k) = 0 \qquad d(\xi_1) = 0 \qquad d(\xi_k) = x_{k-1}$$

the resulting homology should be $\widehat{\mathrm{HFK}}(T(n,\infty))$.