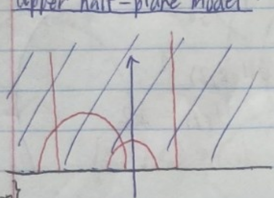
Intro to hyperbolic mfds

upper half-plane model: $\mathbb{H}^2 = \{x+iy \mid y>0\}$.

$$ds^2 = \frac{dx^2+dy^2}{y^2}$$



$\partial\mathbb{H}^2 = \mathbb{R}\cup\{\infty\}$

(geodesics: semicircles and lines perpendicular to $\mathbb{R}$)

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

$A, -A$ act the same way: $PSL(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$

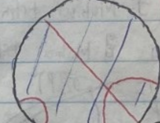
↑ this is the full orientation-preserving isometry group

$\mathrm{Isom}^+(\mathbb{H}^2) \cong PSL(2,\mathbb{R})$

an ordered triple of distinct pts in $\mathbb{R}\cup\{\infty\}$ is positive if they're oriented CCW, like $0, 1, \infty$. $PSL(2,\mathbb{R})$ acts freely and transitively on $\{$pos. triples in $\mathbb{R}\cup\{\infty\}\}$

Poincare disk model: $\Delta^2 = \{x+iy \mid x^2+y^2<1\}$

$$ds^2 = 4\,\frac{dx^2+dy^2}{(1-r^2)^2}$$



(geodesics: circles and lines perpendicular to S^1)

$f: \mathbb{H}^2 \to \Delta^2$ isometry.

$z \mapsto \frac{z-i}{z+i}$

$\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial\mathbb{H}^2$.

say this

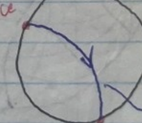
$f: \mathbb{H}^2 \to \mathbb{H}^2$ isometry, $f(z) = \frac{az+b}{cz+d}$. Still makes sense if $z \in \mathbb{R}\cup\{\infty\}$,

extends to $\bar{f}: \overline{\mathbb{H}^2} \to \overline{\mathbb{H}^2}$ homeo

skip

Fixed pts. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$, $A \neq \pm I$.

$$f(z) = \frac{az+b}{cz+d} \quad (=z)$$

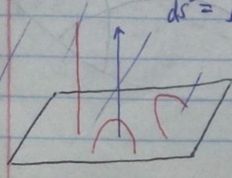


- $(\mathrm{tr}\,A)^2 > 4$: exactly two fixed pts, both on $\partial\mathbb{H}^2$ — hyperbolic/loxodromic.
- $(\mathrm{tr}\,A)^2 = 4$: one, $\partial\mathbb{H}^2$ — parabolic, conjugate to $z \mapsto z+1$ transl.
- $(\mathrm{tr}\,A)^2 < 4$: one, $\mathbb{H}^2$ — elliptic, conjugate to rotation about the fixed pt

dimension 3

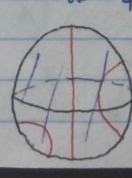
upper half-space $\mathbb{H}^3 = \{(x,y,z) \mid z>0\}$

$$ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$$



disk model $\Delta^3 = \{(x,y,z) \mid x^2+y^2+z^2<1\}$

$$ds^2 = 4\,\frac{dx^2+dy^2+dz^2}{(1-\rho^2)^2}$$



Every isometry $f: \mathbb{H}^3 \to \mathbb{H}^3$ extends uniquely to a homeo $\bar{f}: \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$.

(Brouwer $\Rightarrow$ $\bar{f}$ has a fixed pt. in $\overline{\mathbb{H}^3}$.

<u>elliptic</u>: $\bar{f}$ has at least one fixed pt. in $\mathbb{H}^3$ $\quad$ $\operatorname{tr} A \in (-2,2)$

<u>parabolic</u>: $\bar{f}$ has no fixed pts in $\mathbb{H}^3$, exactly one in $\partial\mathbb{H}^3$ $\quad$ $\operatorname{tr} A = \pm 2$

<u>hyperbolic</u>: two $\quad$ $\operatorname{tr} A \in \mathbb{C} \setminus [-2,2]$)

$$\operatorname{Isom}^+(\mathbb{H}^3) \cong PSL(2,\mathbb{C}).$$

(an isometry is determined by its action on the boundary $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$, $PSL(2,\mathbb{C})$ acts on Riemann sphere $\mathbb{C} \cup \{\infty\}$ by Möbius tsfm's).

<u>Ex</u>: $z \mapsto \frac{1}{z}$ in $PSL(2,\mathbb{C})$ $\leftrightsquigarrow$ rotation of Δ^3 by π about x-axis.

<u>action on triples</u>: z_1, z_2, z_3 distinct pts on $\mathbb{C} \cup \{\infty\}$

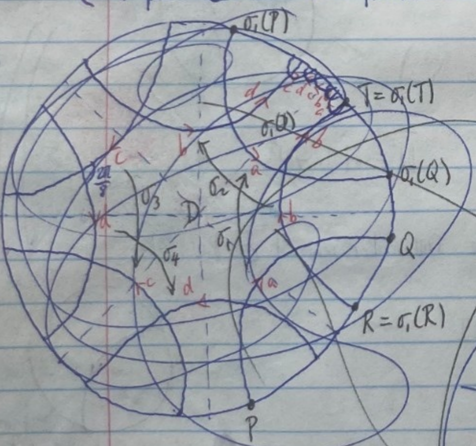
w_1, w_2, w_3 ..

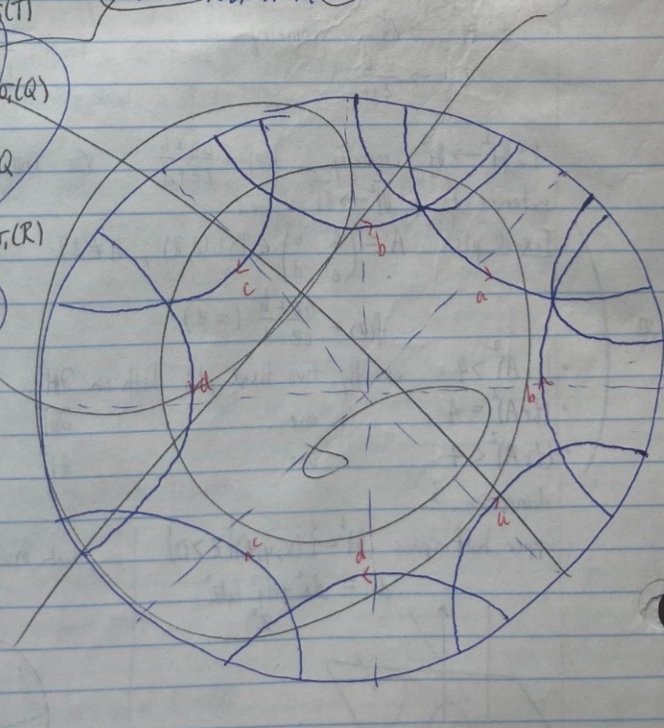
$\exists!$ Mobius tsfm f s.t. $f(z_i) = w_i$.

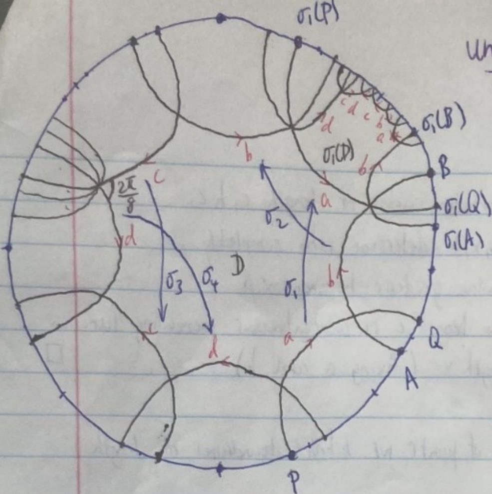
(4 pts: Möbius tsfm preserve cross ratio.)

$\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \operatorname{Isom}^+(\mathbb{H}^2)$.

$P = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle < \operatorname{Isom}^+(\mathbb{H}^2)$





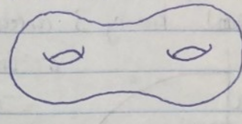


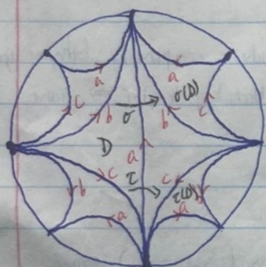
Universal covers of some surfaces

②

$\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \mathrm{Isom}^+(\mathbb{H}^2)$.

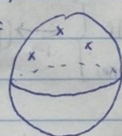
$\Gamma = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle < \mathrm{Isom}^+(\mathbb{H}^2)$

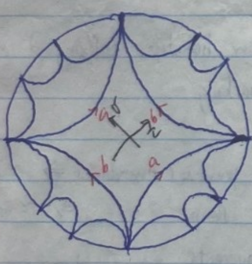
$\mathbb{H}^2/\Gamma =$ 



$\sigma, \tau \in \mathrm{Isom}^+(\mathbb{H}^2)$.

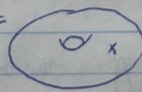
$\Gamma = \langle \sigma, \tau \rangle < \mathrm{Isom}^+(\mathbb{H}^2)$

$\mathbb{H}^2/\Gamma =$  $= \mathbb{C} \setminus \{w_1, w_2\}$ (Little Picard)



$\sigma, \tau \in \mathrm{Isom}^+(\mathbb{H}^2)$

$\Gamma = \langle \sigma, \tau \rangle < \mathrm{Isom}^+(\mathbb{H}^2)$

$\mathbb{H}^2/\Gamma =$ 

A more systematic way of specifying hyperbolic metrics

S : compact oriented connected surface (no punctures - we'll deal w/ them later)

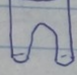
$S_{g,b}$ g - genus, b - boundary

Goal: given $\chi(S) < 0$, construct hyperbolic metrics on S (want boundary to be geodesic)

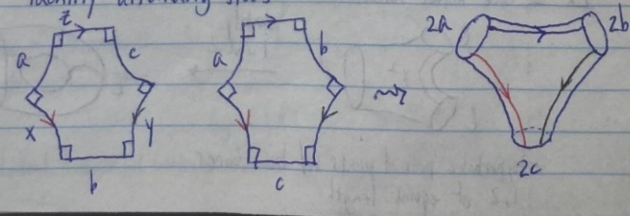
Ex 1 pair of pants P

$S_{0,3}$

$\chi = -1$.



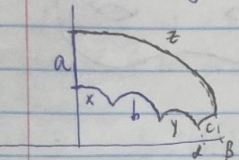
take two identical right-angled hyperbolic hexagons (rahh), identify alternating sides



Fact For any $a,b,c \in \mathbb{R}^+$, $\exists!$ rahh w/ alternating sides of length a, b, c.

<u>Claim 1</u>: knowing 3 consecutive sides a, x, b determine rahh completely.

$\mathbb{H}^2$



c: unique perpendicular geodesic between α, β.

<u>claim 2</u>: The length c is a continuous increasing function of the length x (fixing a and b) $\square$

Corollary For any $a,b,c \in \mathbb{R}^+$, $\exists!$ hyperbolic pair of pants w/ labelled boundaries of length $2a, 2b, 2c$.

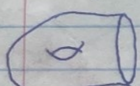
uniqueness:

shortest arcs between the geodesic components (they're geodesics)

Cut. Get two rahh, same alternating lengths $\Rightarrow$ same.

$$\Rightarrow T(P) = \left\{\begin{array}{l}\text{hyperbolic pairs of pants} \\ \text{w/ a labelling of boundaries}\end{array}\right\} \xleftrightarrow[1\text{-}1]{} (\mathbb{R}^+)^3 \cong \mathbb{R}^3.$$

<u>Markings</u>

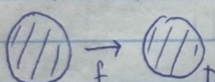


$S_{1,1}$

A <u>marking</u> on a one-holed torus T is an (orient.-pres.) homeo $f: S_{1,1} \to T$.

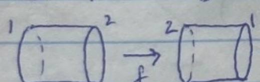
Two markings are equivalent if they are homotopic (fixing ∂ setwise, not nec. pointwise)

$S_{0,1} =$



Alexander trick: unique marking.

$S_{0,2}$:



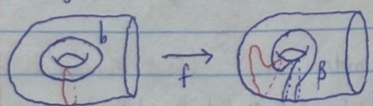
two markings: f preserves boundaries $\Rightarrow \simeq id$

f swaps $\Rightarrow \simeq \pi$ rotation.

$S_{0,3}$:

amounts to labelling of boundary components.

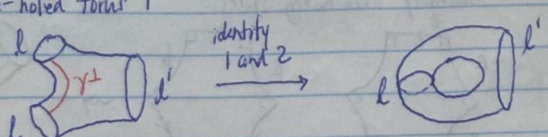
Fact Fix generators a, b of $\pi_1(S_{1,1})$. A marking is determined by the images $\alpha = f(a)$, $\beta = f(b)$.



Pf: cut along $\alpha, \beta \rightsquigarrow$ annulus

f preserves $\partial T \Rightarrow f \simeq id$.

<u>Ex 2</u> one-holed torus T

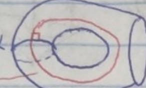


hyperbolic pair of pants w/ boundaries 1, 2 of equal length

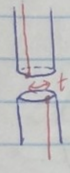
Have some freedom when gluing: can twist.

$\gamma^\perp$: shortest arc between boundaries 1, 2.

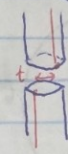
<u>zero twist</u>: endpts of $\gamma^\perp$ line up.



<u>positive twist</u> $t>0$



<u>negative twist</u> $t<0$

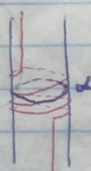


signs: orientation

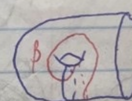
Get a hyperbolic metric on T. Marking: α = identified boundary; $\beta \sim \gamma^\perp \cup$ (arc along α of length $|t|$)

determine a marking   close up $\gamma^\perp$.

$t \gg 0$:

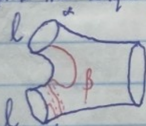


Run in reverse: suppose we're given a marked hyperbolic 1-holed torus.



- cut along α (pick the unique geodesic rep in free htpy class)

$\rightarrow$ pants



$\beta \sim \gamma^\perp \cup$ (arc along α of length t)

twist parameter $= \pm t$.

ℓ, ℓ', t completely specify the marked hyperbolic one-holed torus

$\mathcal{T}(S_{1,1}) = \mathbb{R}^3$.

$\mathcal{T}(S_{g,b}) = \left\{ (X,f) \;\middle|\; \begin{array}{l} X \text{ hyperbolic surface} \\ f: S_{g,b} \to X \text{ homeo} \end{array} \right\} \Big/ \sim$ $(X,f) \sim (Y,h)$:

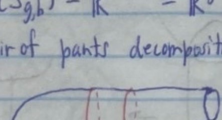
$$\begin{tikzcd} & X \arrow[d, "\exists \text{ isometry } i \text{ st. } i \circ f \simeq h"] \\ S_{g,b} \arrow[ur, "f"] \arrow[r, "h"'] & Y \end{tikzcd}$$

Theorem $\mathcal{T}(S_{g,b}) = \mathbb{R}^{3|\chi(S)|} = \mathbb{R}^{6g-6+3b}$.

count Euler characteristic $\chi = -1$.

pair of pants decomposition: $S_{g,b}$ decomposed into $2g-2+b$ pairs of pants

cut along a maximal collection of disjoint embedded scc, pairwise non-htp'c.



$3g-3+b$ pants curves

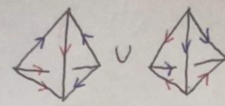
b boundary components

length parameters: $3g-3+b+b = 3g-3+2b$

twist parameters: $3g-3+b$

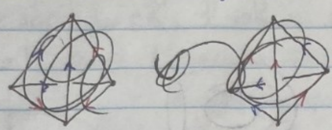
$6g-6+3b$

punctures: send boundary length $\to 0$ (ideal hexagon w/ vertex on $\partial\mathbb{H}^2$) $\mathcal{T}(S_g^n) = \mathbb{R}^{6g-6+2n}$ (n punctures)

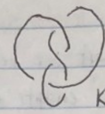


Knot complement of figure 8 knot

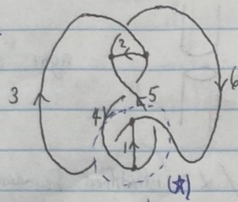
Goal



$= S^3 \setminus$

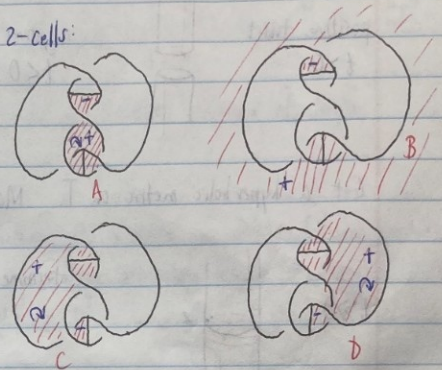


CW complex

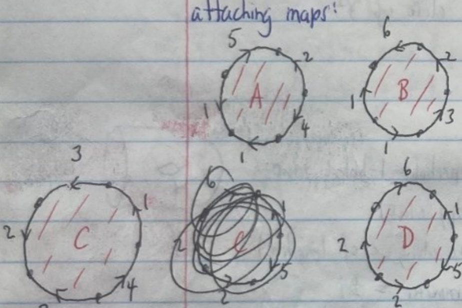


1-skeleton X^1

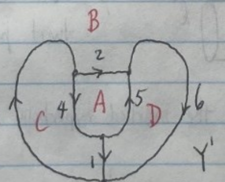
2-cells:



attaching maps:



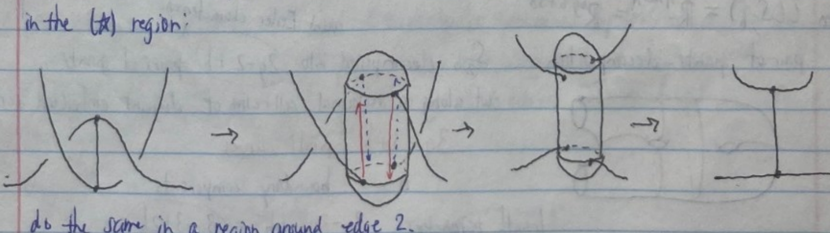
A simpler CW complex



Y^1

describe a homeomorphism $S^3 - X^1 \to S^3 - Y^1$

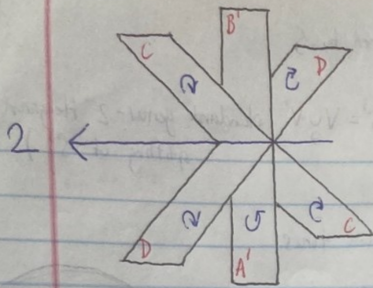
in the (☆) region:



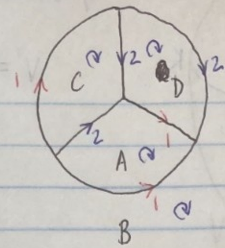
do the same in a region around edge 2.

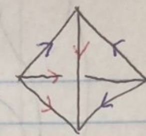
Track the 2-cells A, B, C, D

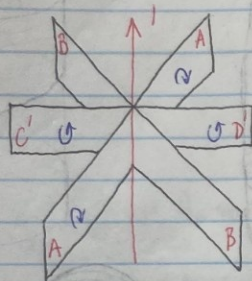
$\Rightarrow S^3 - X^2 \cong B^3 \sqcup B^3$.



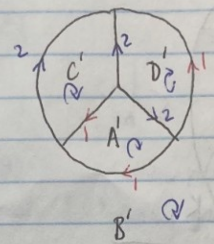
front

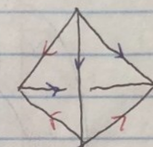






back





Recall: for $S_{0,3}$ thrice-punctured sphere, glued two ideal hyperbolic triangles.
To get hyperbolic structure on S^3-K, glue two ideal hyperbolic tetrahedra.

Historical remark:

First known to Riley [A quadratic parabolic group, 1973].

Recall: $\mathrm{Isom}^+(\mathbb{H}^3) = PSL(2,\mathbb{C})$. Riley constructed a discrete faithful representation

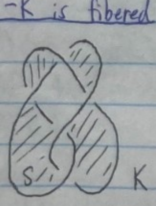
$$\pi_1(S^3-K) \xrightarrow[\rho]{} PSL(2,\mathbb{Z}[\omega]) \subset PSL(2,\mathbb{C})$$

$$\omega = e^{2\pi i/3}$$

Wirtinger presentation: $\pi_1(S^3-K) = \langle a,b \mid a^{-1}b^{-1}abab^{-1}a^{-1}bab=1\rangle$

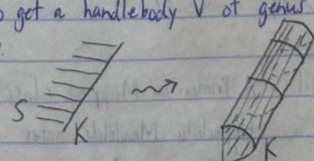
$$\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \rho(b) = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}.$$

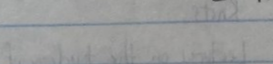
S^3-K is fibered

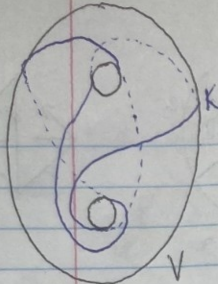


thicken up S to get a handlebody V of genus 2

local picture:



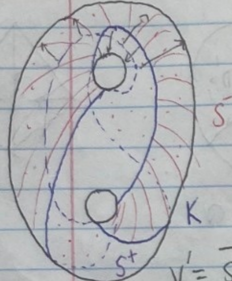




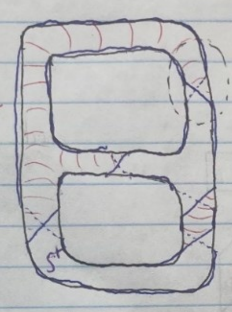
by construction, $V-K$ is fibered by S

$V' = \overline{S^3 - V}$ (so $S^3 = V \cup_{\partial} V'$ standard genus-2 Heegaard splitting of S^3)

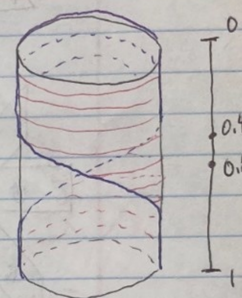
$S^- \cong S^+ \cong$ one-holed torus

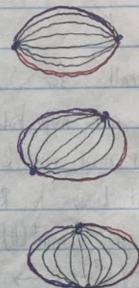


$V' = \overline{S^3 - V}$



V'





these fiberings glue up to fiber $S^3 - K$.

$F = S^+$

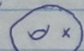
$lk(a, a^+) = -1$ $\quad lk(a, b^+) = 0$

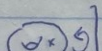
$lk(b, b^+) = 1$ $\quad lk(b, a^+) = 1$

Seifert matrix $S = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$

M: matrix of monodromy $h_*: H_1(F) \to H_1(F)$

$M = (S^T)^{-1} S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ famous Arnold cat map

Mod() $= SL(2, \mathbb{Z})$

$[\varphi:$  $] \leftrightarrow \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

<u>References</u>

Farb-Margalit — A Primer on Mapping Class Groups
Lackenby — Hyperbolic Manifolds notes
Francis — A Topological Picturebook
Riley — A quadratic parabolic group
Burde-Zieschang — Knots
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