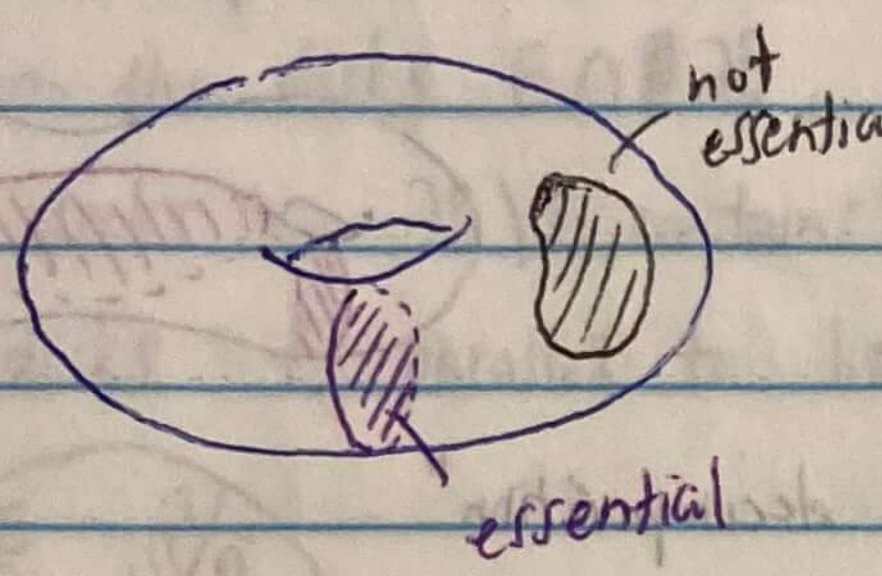
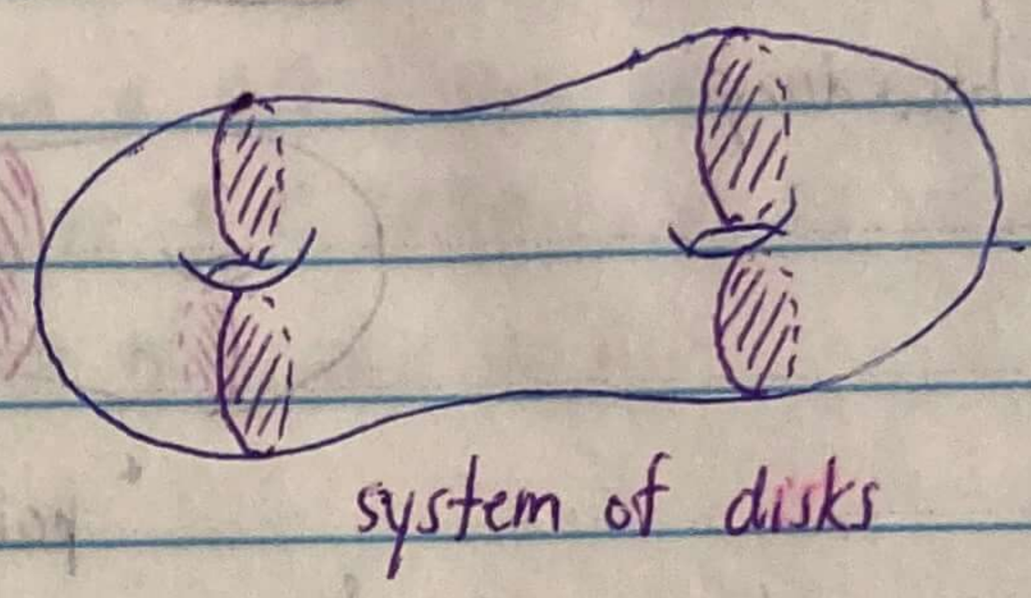
Handlebodies

H: handlebody

Def A properly embedded disk $D \subset H$ is essential if its boundary does not bound a disk in ∂H.

Def A collection $\{D_1, \dots, D_m\}$ of properly embedded, essential disks is called a system of disks for H if the complement of a regular neighborhood of $\cup D_i$ is a collection of balls.

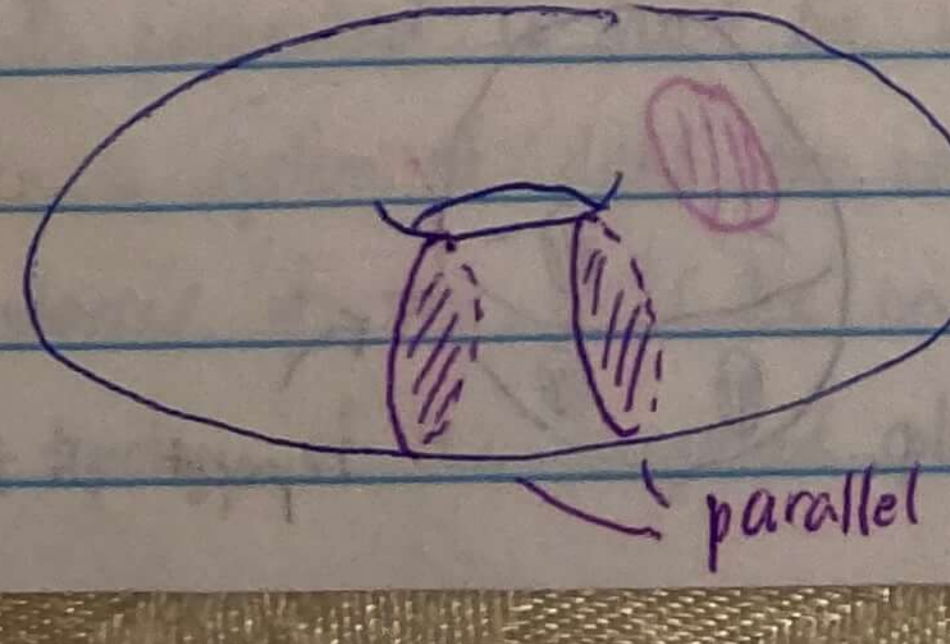




Lemma Given a handlebody H, there is a system of disks for H.

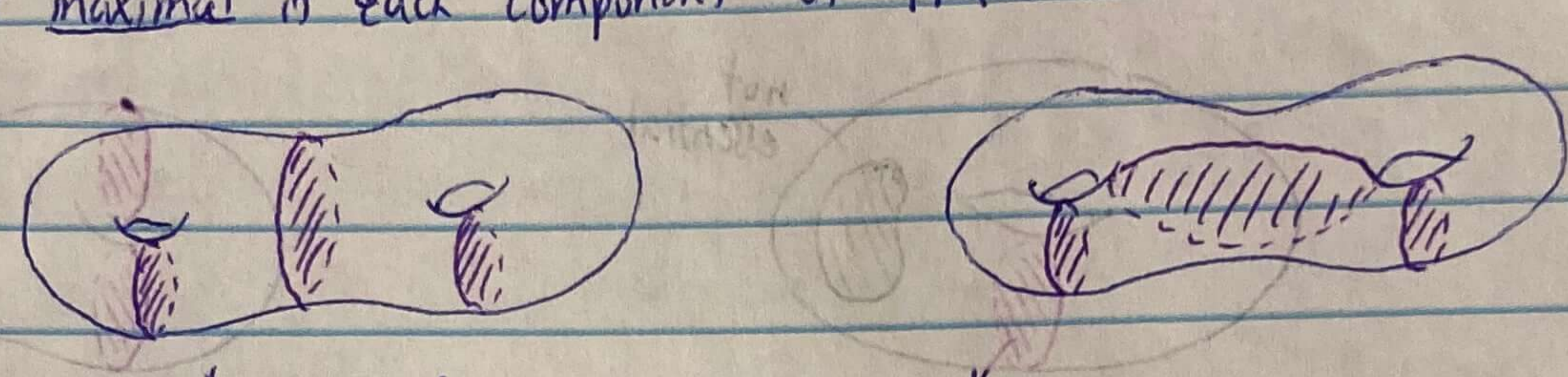
Proof Let $B_1, \dots, B_n$ be a collection of balls, $E_1, \dots, E_m, E_1', \dots, E_m'$ disks in their boundaries such that H is the result of gluing. $\phi: \sqcup B_j \to H$ natural inclusion.
For each pair E_i, E_i', $\phi(E_i) = \phi(E_i') =: D_i$. N: regular nbhd of $\cup D_i$, $\phi^{-1}(N)$ is regular nbhd of $\cup(E_i \cup E_i')$.
Given a ball B and a disk D in its boundary, the complement in B of a regular open nbhd of D is a closed ball. $\Rightarrow$ complement of $\phi^{-1}(N)$ in $\sqcup_j B_j$ is a collection of closed balls. ϕ is one-to-one in the complement of N, so the image of these closed balls is also a collection of closed balls, which is precisely $H \setminus N$.
Remains to show that $D_1, \dots, D_n$ are essential. Let D_i be a disk such that ∂D_i bounds a disk F in ∂H. If F contains the boundary of another disk in the system, replace D_i by this disk. Repeat until F disjoint from all other disks.
F is in the boundary of a ball B_j in the complement of N. Interior of F disjoint from the system of disks $\Rightarrow$ exactly one disk E_i in the boundary of B_j, and this is associated to D_i. B_k: ball containing the second disk E_i' associated to D_i.
Removing D_i from the system has the effect on $B_1, \dots, B_m$ of gluing B_j and B_k together along ~~$B_j B_k$~~ E_i, E_i'. Still get a ball. $\Rightarrow$ Can remove all inessential disks. □

Def Two essential, properly embedded disks D, D' are parallel if the complement of a regular neighborhood of $D \cup D'$ in H has a component which is a ball. Equivalently, the complement in ∂H of the boundaries of the disks contains an annulus as a component.



If two disks in a system are parallel, then some ball B_j has exactly two disks of E in its boundary. D' can be isotoped across B_j onto D. Unnecessary to keep both: remove D'. Can always assume we have a system of disks in which no two disks are parallel.

Def A system of disks is *maximal* if each component of $H \setminus \bigcup D_i$ has 3 disks in its boundary.

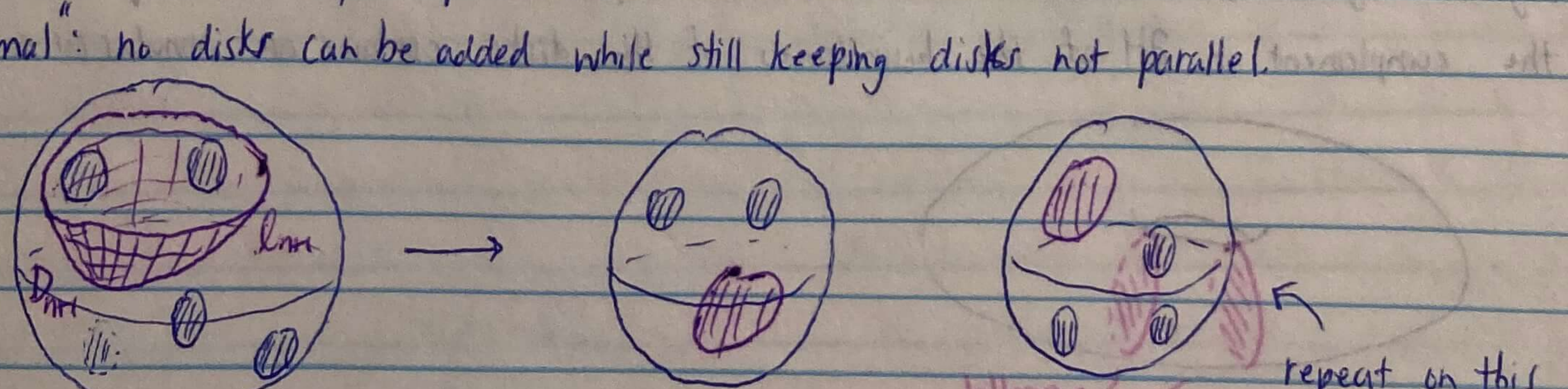


"pair-of-pants decomposition"

Lemma H: handlebody, $\mathbb{D} = \{D_1, \ldots, D_m\}$ system of disks for H. If H has genus ≥ 2 and no two disks in $\mathbb{D}$ are parallel in H, then there is a maximal system of disks for H containing $\mathbb{D}$.

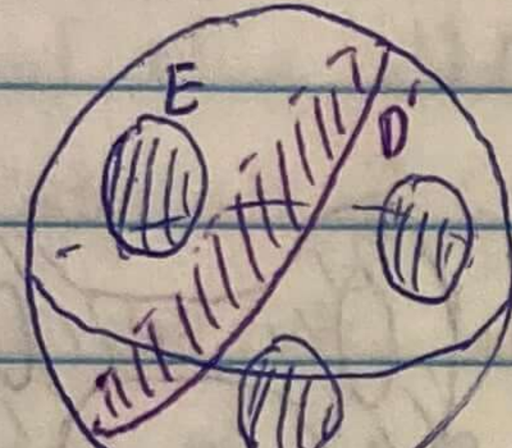
Proof H not a ball: $\mathbb{D}$ nonempty. N: regular open nbhd of $\mathbb{D}$. $H \setminus N$ collection of balls $B_1, \ldots, B_n$. $E = (H \setminus N) \cap \bar{N}$ is a collection of disks in the boundaries of the balls, each D_i corresponds to a pair E_i, E_i' in E. D_i parallel to E_i, E_i'.
For each ball B_j: let $k = \#$ disks of E contained in ∂B_j. If $k=2$, then the complement of E in ∂B_j is an annulus, and E_i, E_h parallel in H. If E_i, E_h come from different disks D_i, D_h in $\mathbb{D}$ then D_i, D_h parallel in H. If E_i, E_h come from the same disk D_i then the complement in ∂H of $\partial E_i \cup \partial E_h \cup \partial D_i$ is a collection of annuli, so H a solid torus. Assumed H has genus ≥ 2 and no two disks of $\mathbb{D}$ parallel, so for each ball B_j, $k \geq 3$.
If $k > 3$, then the complement of E in ∂B_j is a sphere w/ ≥ 4 punctures.
$\ell_{n+1} \subset \partial B_j$: simple closed curve that separates two of the disks from the rest. ℓ_{n+1} not parallel to the boundary of any disk in E since there are at least two disks on either side of ℓ_{n+1} in ∂B_j. Jordan curve thm: ℓ_{n+1} bounds a disk in the sphere. Push the disk into B_j to get D_{n+1} in the handlebody whose boundary is ℓ_{n+1}. Add D_{n+1} to $\mathbb{D}$. This splits B_j into two balls, one of which has exactly 3 disks in its boundary. Repeat for the other. □

"maximal": no disks can be added while still keeping disks not parallel.



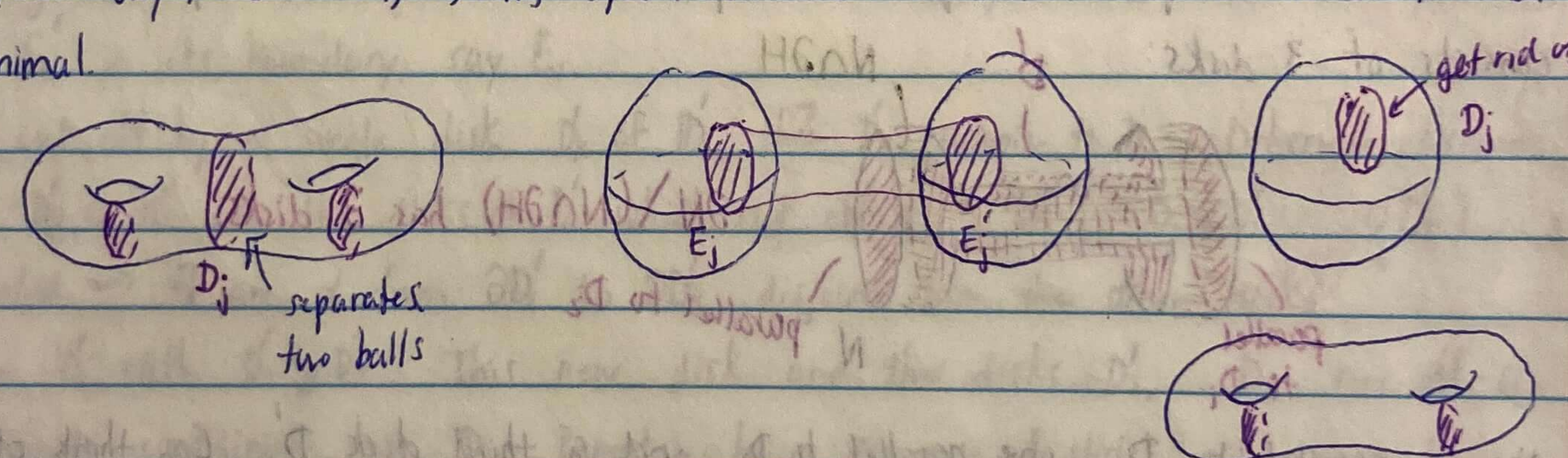
Lemma If $\mathbb{D}$: maximal collection of disks for H, and D' properly embedded disk disjoint from $\mathbb{D}$, then D' parallel to a disk in $\mathbb{D}$.

Proof $N, B_1, \dots, B_m$ as before. D' disjoint from $\mathbb{D}$: it's properly embedded in a ball B_i in $H \setminus N$. Boundary of B_i contains exactly 3 disks of E.
$\partial D'$ an essential loop ℓ in $\partial B \setminus E$. Jordan curve thm: ℓ separates ∂B into two disks. ℓ disjoint from the disks $E \cap \partial B$, does not bound a disk in their complement $\Rightarrow$ one of the disks in $\partial B \setminus \ell$ contains one of the disks $E \subset \partial B \setminus E$ and the other contains two of the disks. ℓ parallel to boundary of $E \Rightarrow D'$ parallel to E. $\square$



Def A collection of disks is *minimal* if its complement is connected.

Lemma H: handlebody, $\mathbb{D} = \{D_1, \dots, D_m\}$ system of disks. There is a subset of $\mathbb{D}$ that is minimal.



Prop $\mathbb{D} = \{D_i\}$ system of disks for H a handlebody. TFAE:
(1) $\mathbb{D}$ is minimal
(2) No proper subset of $\mathbb{D}$ is a system for H
(3) # disks in $\mathbb{D}$ = genus of H.

Proof $\mathbb{D}$ not minimal $\Rightarrow \exists$ proper subset that is still system. Contrapositive is $(2) \Rightarrow (1)$.
Genus of H is $g = m + 1 - n$, $n = \#$ balls, $m = \#$ pairs of disks.
$\mathbb{D}$ minimal $\Leftrightarrow n = 1 \leftrightarrow g = m$, so $(1) \Leftrightarrow (3)$.
Let $\mathbb{D}$ be minimal system of disks for H, $\#\mathbb{D} = g$. Then H has genus g.
Suppose $\mathbb{D}' \subset \mathbb{D}$ proper subset, $\mathbb{D}'$ system of disks for H. $h := \#\mathbb{D}' < g$. Complement of $\mathbb{D}'$ connected, so must be a single ball. Then genus of H is h. $\downarrow\uparrow$. $(1) \Rightarrow (2)$ ✓ $\square$
Given a system of disks for each handlebody in a Heegaard splitting (Σ, H_1, H_2), the boundaries of the disks lie on Σ. The ways in which these loops intersect determines the topology of the Heegaard splitting and of the ambient manifold. The boundaries of a

system of disks for a handlebody completely determine its topology.

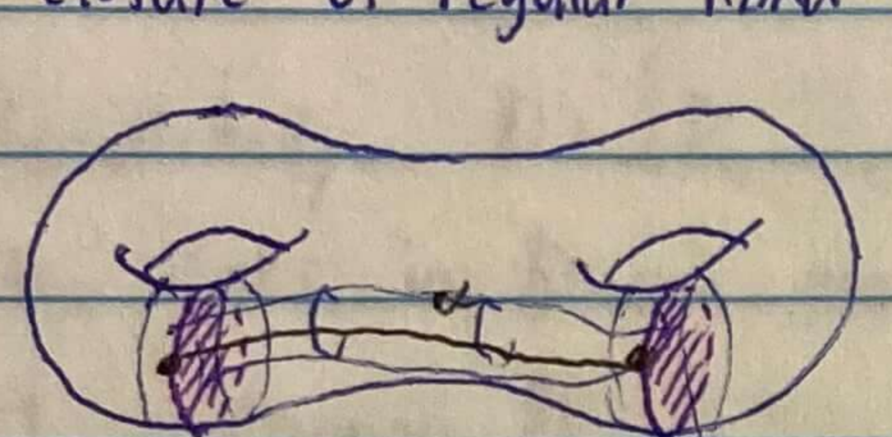
Lemma H, H' handlebodies, $\{D_1, \dots, D_m\}$, $\{D_1', \dots, D_m'\}$ systems of disks for H, H'. Assume there is a homeo $\phi: \partial H \to \partial H'$ such that for each i, $\phi(\partial D_i) = \partial D_i'$. Then ϕ extends to a homeo $\psi: H \to H'$. □

Proof Extend ϕ to interior of disks, then to interior of balls.

Sliding disks

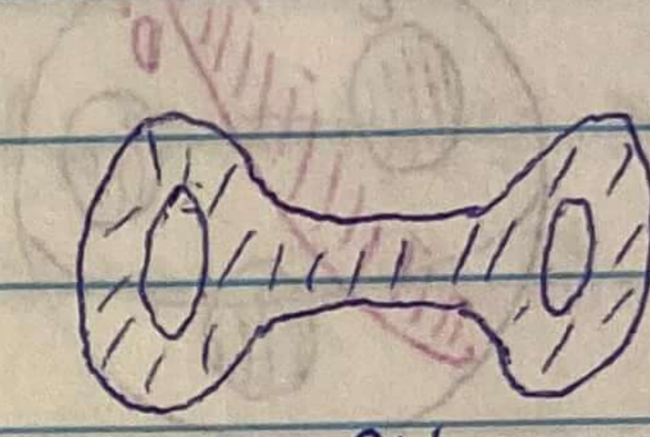
$\mathbb{D} = \{D_1, \dots, D_m\}$ system of disks for H. α: embedded arc in ∂H with one endpoint on ∂D_1 and one endpoint on ∂D_2. Assume interior of α disjoint from all ∂D_i.

N = closure of regular nbhd of $D_1 \cup D_2 \cup \alpha$ in H.



D_1 D_2

N looks like dumbell?



$N \cap \partial H$

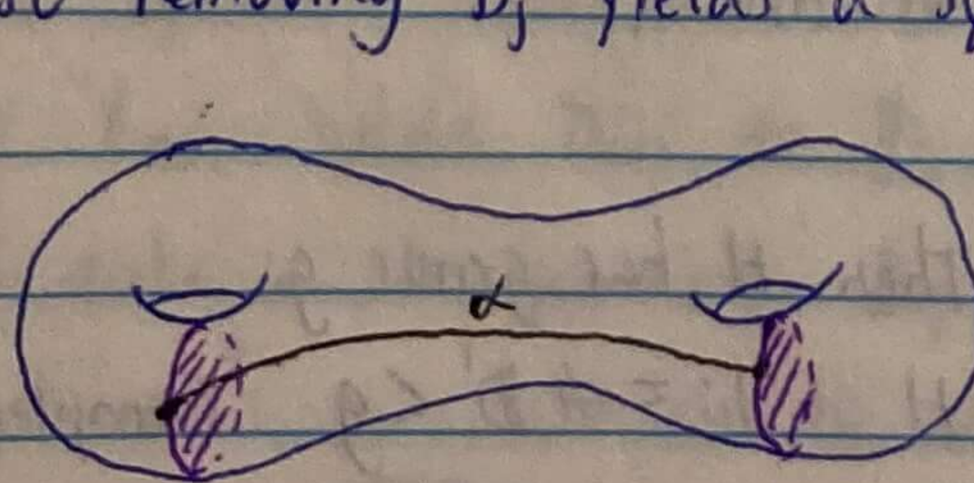
$N \cap \partial H$ is a 3-punctured sphere. Complement in ∂N of this 3-punctured sphere consists of 3 disks:

$\partial N \setminus (N \cap \partial H)$ has 3 disks.

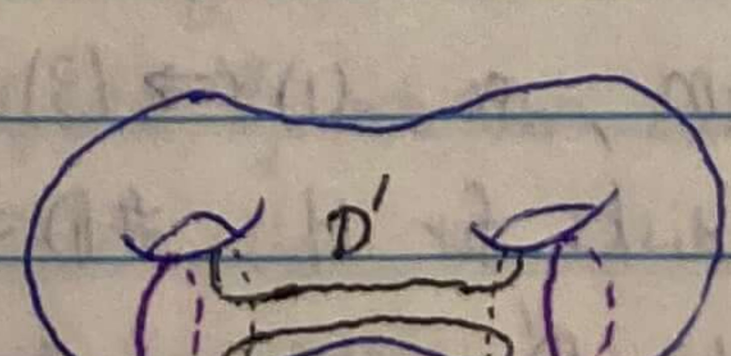
one is parallel to D_1, one parallel to D_2, and a third disk D'. Can think of D' as sliding a piece of D_1 along α, then across D_2. Write $D' = D_1 *_\alpha D_2$

Lemma If $\mathbb{D} = \{D_1, \dots, D_m\}$ a system of disks for H and $D' = D_i *_\alpha D_j$ $(i \neq j)$, then $(\mathbb{D} \cup \{D'\}) \setminus \{D_i\}$ is a system of disks for H.

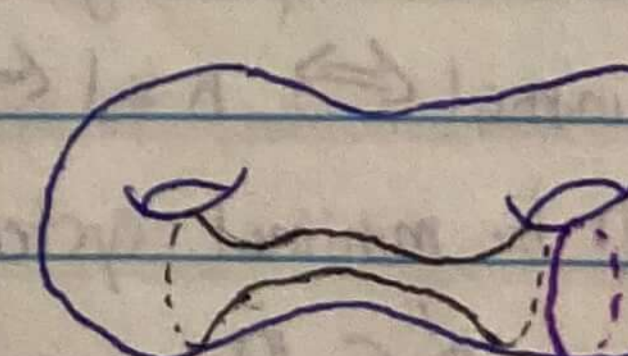
Proof The neighborhood N of $D_i \cup \alpha \cup D_j$ is a ball, so adding D' to the collection yields a system of disks. D_j separates N from some other ball in the complement of $\mathbb{D}$ so removing D_j yields a system of disks for H. □



D_1 D_2



D_1 D_2

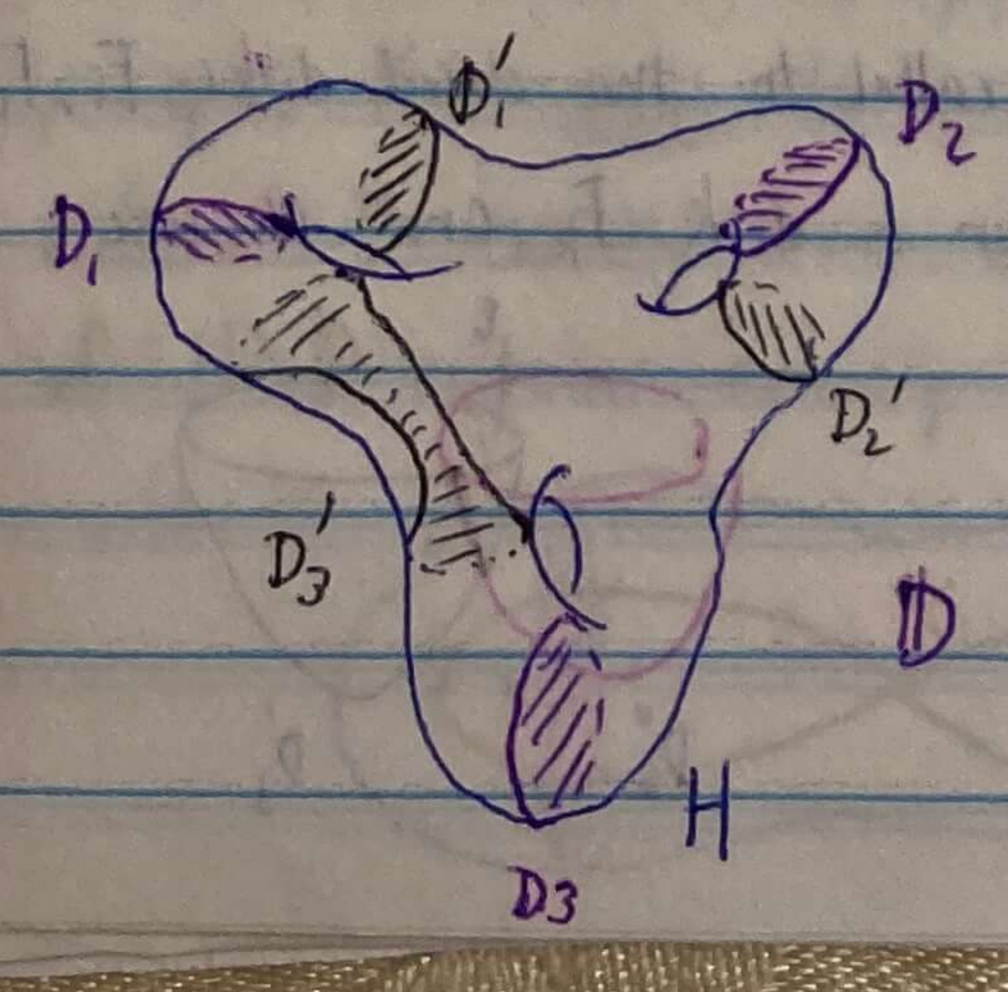


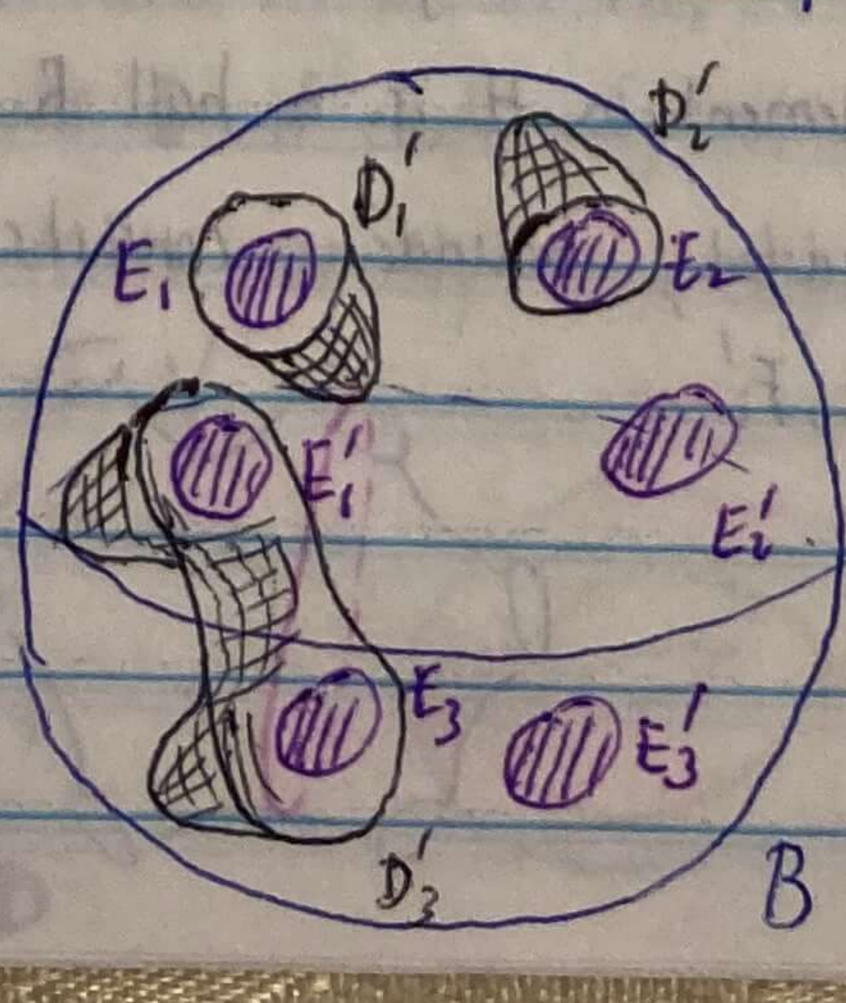
Corollary If $\mathbb{D}$ is a minimal system of disks, then adding D' and removing D_i yields a new minimal system of disks.

Def Two systems of disks are *isotopic* if there is an isotopy of H (not fixing the boundary) which takes one system of disks onto the other. A *disk slide* of $\mathbb{D}$ is any system of disks that is isotopic to $(\mathbb{D} \cup \{D'\}) \setminus \{D_i\}$.
Systems $\mathbb{D}$ and $\mathbb{D}'$ are *slide equivalent* if there is a sequence of systems $\mathbb{D}_1, \dots, \mathbb{D}_\ell$ such that $\mathbb{D} = \mathbb{D}_1$, $\mathbb{D}_\ell = \mathbb{D}'$, and for each i, $\mathbb{D}_{i+1}$ is a disk slide of $\mathbb{D}_i$.
This is an equivalence relation.

Lemma If two minimal systems of disks are disjoint, then they are slide equivalent.

Proof $\mathbb{D} = \{D_1, \dots, D_g\}$ $\quad \mathbb{D}' = \{D_1', \dots, D_g'\}$ minimal systems for H.
Complement of a nbhd of $\mathbb{D}$ is a ball B. For each D_i, the closure of the regular nbhd of D_i intersects B in two disks E_i, E_i' in ∂B. Each disk D_i' is properly embedded in B such that its boundary is disjoint from each E_i and E_i'.
The disks of $\mathbb{D}'$ cut B into $g+1$ components. If a component does not contain some E_i or E_i', then two disks in $\mathbb{D}'$ are parallel, impossible. There are $g+1$ components and $2g$ disks in boundary $\Rightarrow$ there is at least one component B' with exactly one E_i or E_i' in its boundary, say E_1.
If B' cut off by a single disk D_k' of $\mathbb{D}'$, then D_k' isotopic to D_1. Otherwise, assume $D_1', \dots, D_k' \subset \partial B'$ are the disks of $\mathbb{D}'$ which cut off B'. $\partial B' \setminus (D_1 \cup D_1' \cup \dots \cup D_k')$ connected $\Rightarrow \exists$ arc α from $\partial D_1'$ to $\partial D_k'$ disjoint from the other disks.
Replace D_1' with $D_1' *_\beta D_k'$. This new disk and the disks $D_2', \dots, D_{k-1}'$ cut off a component containing only D_1. Repeat: by a sequence of slides, can replace D_1' with a disk cutting off a component of B containing only D_1. This new disk is isotopic to D_1, so $\mathbb{D}'$ slide equivalent to a system of disks consisting of D_1 and $D_2', \dots, D_m'$.
The disks $D_2', \dots, D_m'$ cut B into m components, there are $2(m-1)$ disks $E_2, \dots, E_m, E_2', \dots, E_m'$ in the boundary. There is a collection of disks cutting off a single disk, say E_2, in the boundary. This could also contain E_1 or E_1'. But D_1 is a part of the new system of disks, $D_1, D_2', \dots, D_k'$ so can slide any of the disks $D_2', \dots, D_m'$ over E_1 or E_1'.
$\Rightarrow \exists$ sequence of disk slides which replace D_2' with a disk isotopic to D_2. Repeat. $\square$





Theorem (Reidemeister, Singer 1933) Any two ~~[illegible]~~ minimal systems of disks for a handlebody H are slide equivalent.

Proof $\mathbb{D} = \{D_1, \dots, D_m\}$, $\mathbb{D}' = \{D_1', \dots, D_m'\}$ minimal systems of disks. Assume the disks are transverse: $D_i \cap D_j'$ is a (possibly empty) collection of properly embedded arcs and simple closed curves for all i, j.

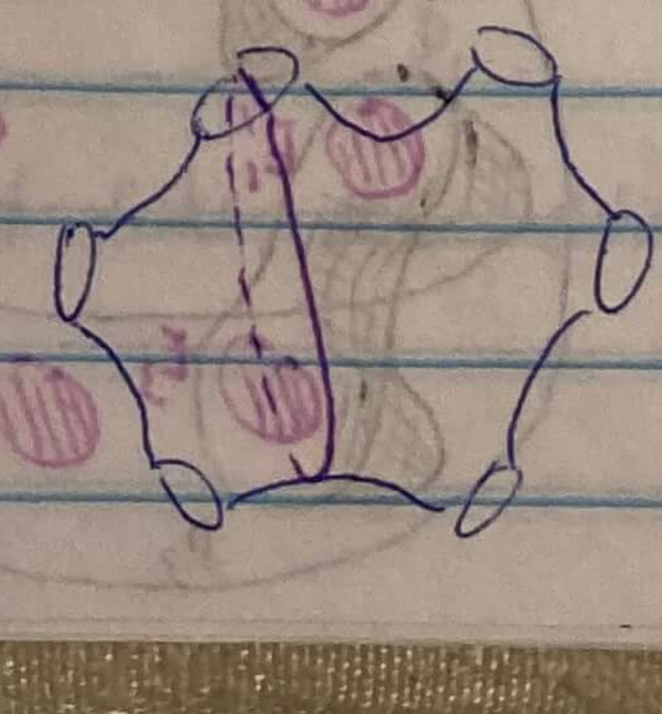
If a component of $D_i \cap D_j'$ is a closed loop, then this bounds a disk in D_j'. An *innermost* loop in D_j' is a loop ℓ in $D_i \cap D_j'$ such that the interior of the disk bounded by ℓ is disjoint from the disks of $\mathbb{D}$. If D_j' intersects a disk of $\mathbb{D}$ in a closed loop, then D_j' contains an innermost loop $\ell \subset D_i \cap D_j'$ for some i. Let ℓ be an innermost loop in D_j', let $E \subset D_j'$ be the disk bounded by ℓ. Complement of $\mathbb{D}$ in H is a ball, $\ell \subset \partial B$ a scc. E properly embedded in B, so $B \setminus E$ is pair of balls. One of these balls has a boundary consisting of E and a disk ~~[illegible]~~ $D_i \in \mathbb{D}$. Isotoping E across this ball into D_i induces an isotopy of D_j' that removes ℓ. Repeat to remove all scc curves of intersection.

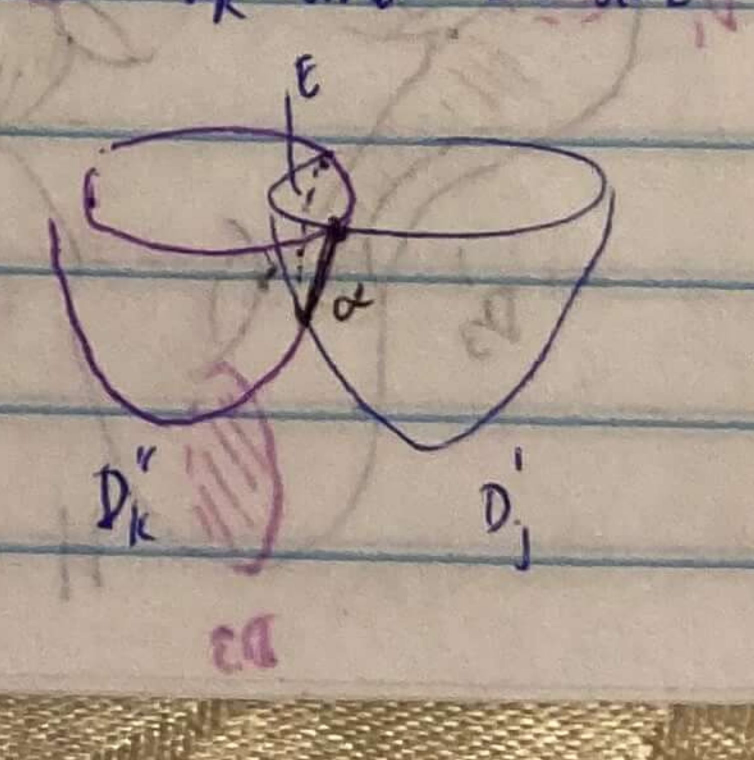
Now $D_i \cap D_j'$ consists of properly embedded arcs for each i, j. Let $I(\mathbb{D}, \mathbb{D}')$ be the number of arcs of intersection over all the disks in $\mathbb{D}$ and $\mathbb{D}'$. By previous lemma, enough to show that $\mathbb{D}$ slide equivalent to a system $\mathbb{D}''$ with $I(\mathbb{D}', \mathbb{D}'') = 0$.

Let $\mathbb{D}'' = \{D_1'', \dots, D_m''\}$ be a system that is slide equivalent to $\mathbb{D}$, such that $I(\mathbb{D}', \mathbb{D}'')$ is minimal.

If $I(\mathbb{D}', \mathbb{D}'') \neq 0$, there is a D_j' such that $D_j' \cap \bigcup D_i''$ nonempty. Each arc of $D_j' \cap D_i''$ (for some i) cuts D_j' into two disks. An arc α is *outermost* if the interior of one of these disks is disjoint from $\mathbb{D}''$. If $D_j' \cap \bigcup D_i''$ contains an arc, it has an outermost one α. Let $E \subset D_j'$ be the subdisk disjoint from $\mathbb{D}''$, and D_k'' the disk such that $\alpha \subset D_j' \cap D_k''$.

$\mathbb{D}''$ minimal $\Rightarrow$ complement in H is a ball B. Each D_i'' parallel to two closed disks $F_i, F_i' \subset \partial B$. $E \cap B$ is properly embedded, boundary consists of an arc in a disk F_k and ~~an~~ arc β disjoint from all F_i, F_i'.





N: regular nbhd in B of $E \cup F_k$. $N \setminus \partial B$ consists of two disks, E_1 and E_2.
Every arc of intersection in $E_i \cap \mathbb{D}'$ will be an arc parallel to $D_k'' \cap \mathbb{D}'$. There is no arc of intersection parallel to α ⇒ # arcs in E_1 < # arcs in D_k'', same for E_2.
$B \setminus (E_1 \cup E_2)$ consists of 3 balls. One contains F_k, one contains F_k'. $B'' \subset B$ is complement of this second ball. WLOG, assume $E_1 \subset \partial B''$. $\partial B''$ contains E_1, F_k and a collection of disks which are parallel to $D_1'', \dots, D_m''$. $\partial B'' \setminus (E_1 \cup D \cup F_1 \cup \dots \cup F_k)$ connected ⇒ $\exists$ an arc β_1 from ∂D to disk F_i in $\partial B''$. Put $G_1 = D *_{\beta_1} F_i$.
∂G_1 separates $\partial B''$ into a component containing F_k and F_i, and a component containing the rest. β_2: arc in $\partial B''$ from ∂G_1 to a second disk F_j in B'', $G_2 = G_1 *_{\beta_2} F_j$.
G_2 separates D, F_i, F_j from rest. Repeat: get disk G_k, separates E_1 from rest.
⇒ G_k isotopic to E_1.
F_k parallel to D_k'' in H. ⇒ $\exists$ sequence of edge slides which replaces D_k'' w/ E_1.
E_1 intersects $\mathbb{D}'$ in fewer arcs ⇒ $I(\mathbb{D}', \mathbb{D}'')$ not minimal. □

Spines

Def A *spine* of a handlebody H is a (PL) graph K embedded in H so that $H \setminus K$ is homeomorphic to $\partial H \times (0,1]$.
If K is a graph embedded in a manifold and H is the closure of a regular neighborhood of K then K is a spine of H.

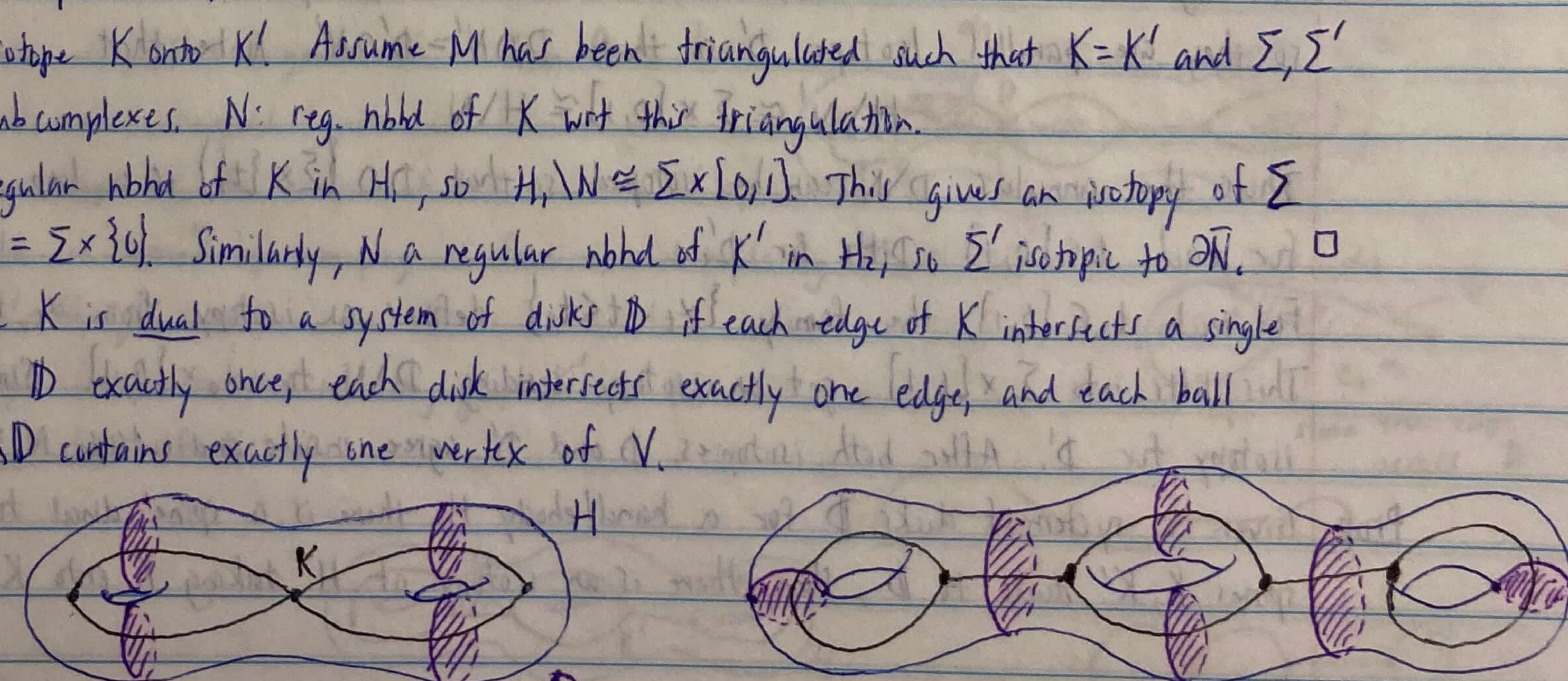
Lemma Every handlebody has a spine.

(✱) **Lemma** Let (Σ, H_1, H_2), (Σ', H_1', H_2') be Heegaard splittings of M. Let K and K' be spines of H_1 and H_1'. If K is isotopic to K' then Σ and Σ' are isotopic.

Lemma If K is a graph in a handlebody H and N is a regular nbhd of K, then $H \setminus N \cong \partial H \times [0,1]$ ⇔ K a spine of H.

Proof of (✱) Isotope K onto K'. Assume M has been triangulated such that $K = K'$ and Σ, Σ' are subcomplexes. N: reg. nbhd of K wrt this triangulation.
N a regular nbhd of K in H_1, so $H_1 \setminus N \cong \Sigma \times [0,1]$. This gives an isotopy of Σ onto $\partial N = \Sigma \times \{0\}$. Similarly, N a regular nbhd of K' in H_1', so Σ' isotopic to ∂N. □

Def A spine K is *dual* to a system of disks $\mathbb{D}$ if each edge of K intersects a single disk of $\mathbb{D}$ exactly once, each disk intersects exactly one edge, and each ball of $H \setminus \mathbb{D}$ contains exactly one vertex of V.



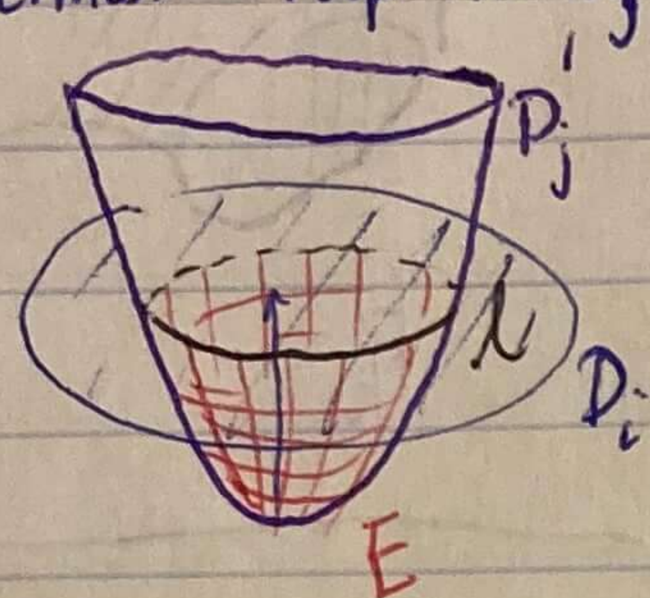
$\mathbb{D} = \{D_1, \dots, D_m\}$ $\quad \mathbb{D}' = \{D_1', \dots, D_m'\}$

$D_i \cap D_j'$

Sps D_j' intersects some disk in $\mathbb{D}$ in a closed loop.

D_j' contains an innermost loop $l \subset D_i \cap D_j'$ for some i.

l: innermost loop in D_j', $E \subset D_j'$ disk b'd by l.



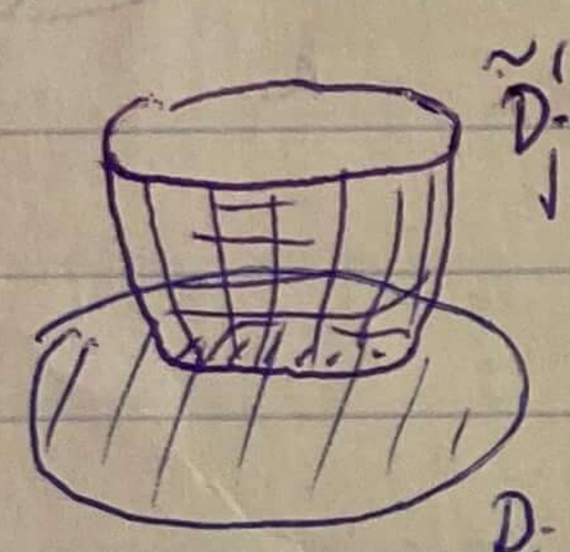
B = complement of $\mathbb{D}$ in H, a ball.

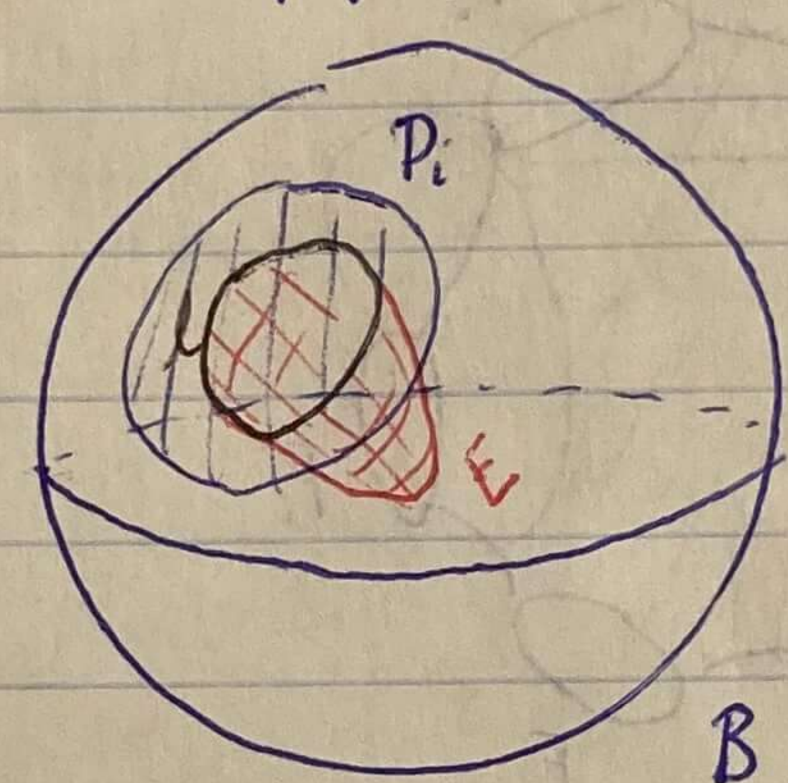
$l \subset \partial B$ scc.

$E \subset B$ properly embedded.

$B \setminus E$ is two balls, one has b'dry E and part of D_i.

isotope E across this ball into D_i.





now: $D_i \cap D_j'$ is properly embedded arcs, $\forall i,j$.

$I(\mathbb{D}, \mathbb{D}')$: # of arcs in $\mathbb{D} \cap \mathbb{D}'$.

$\mathbb{D}$ slide-equiv. to a system $\mathbb{D}''$ w/ $I(\mathbb{D}', \mathbb{D}'') = 0$.

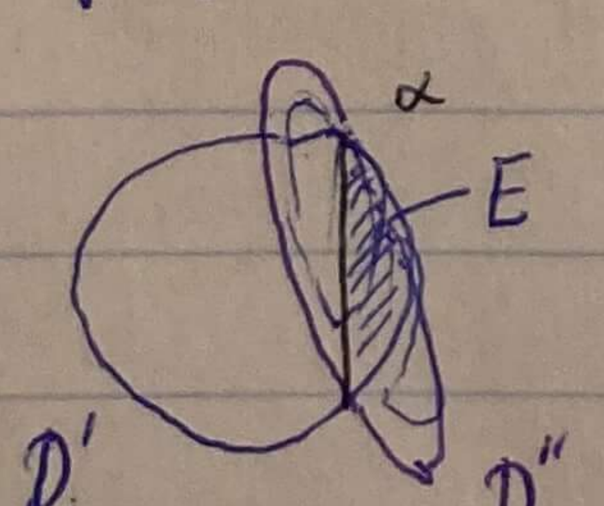
$\mathbb{D}'' = \{D_1'', \dots, D_m''\}$ slide equiv. to $\mathbb{D}$, minimizes $I(\mathbb{D}', \mathbb{D}'')$.

sps $I(\mathbb{D}', \mathbb{D}'') \neq 0$.

$\exists D_j'$ s.t. $D_j' \cap \bigcup D_i''$ nonempty. each arc of $D_j' \cap D_i''$ cuts D_j' into two disks.

α: outermost. $E \subset D_j'$ subdisk disjoint from $\mathbb{D}''$.

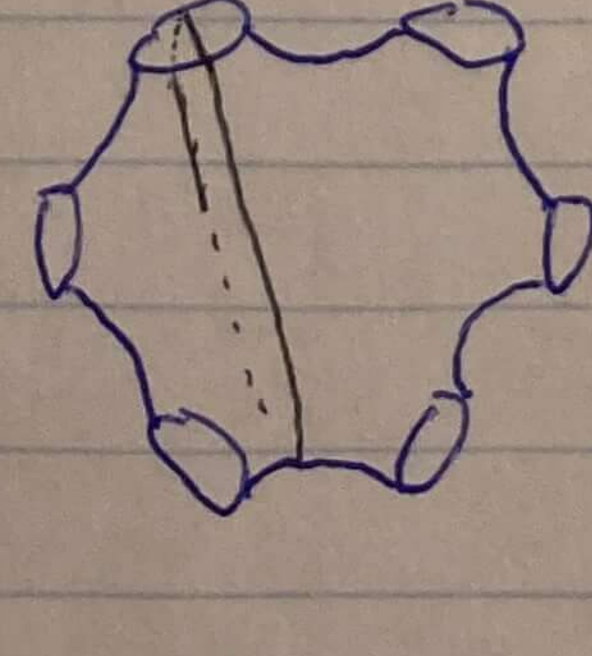
$\alpha \subset D_j' \cap D_k''$



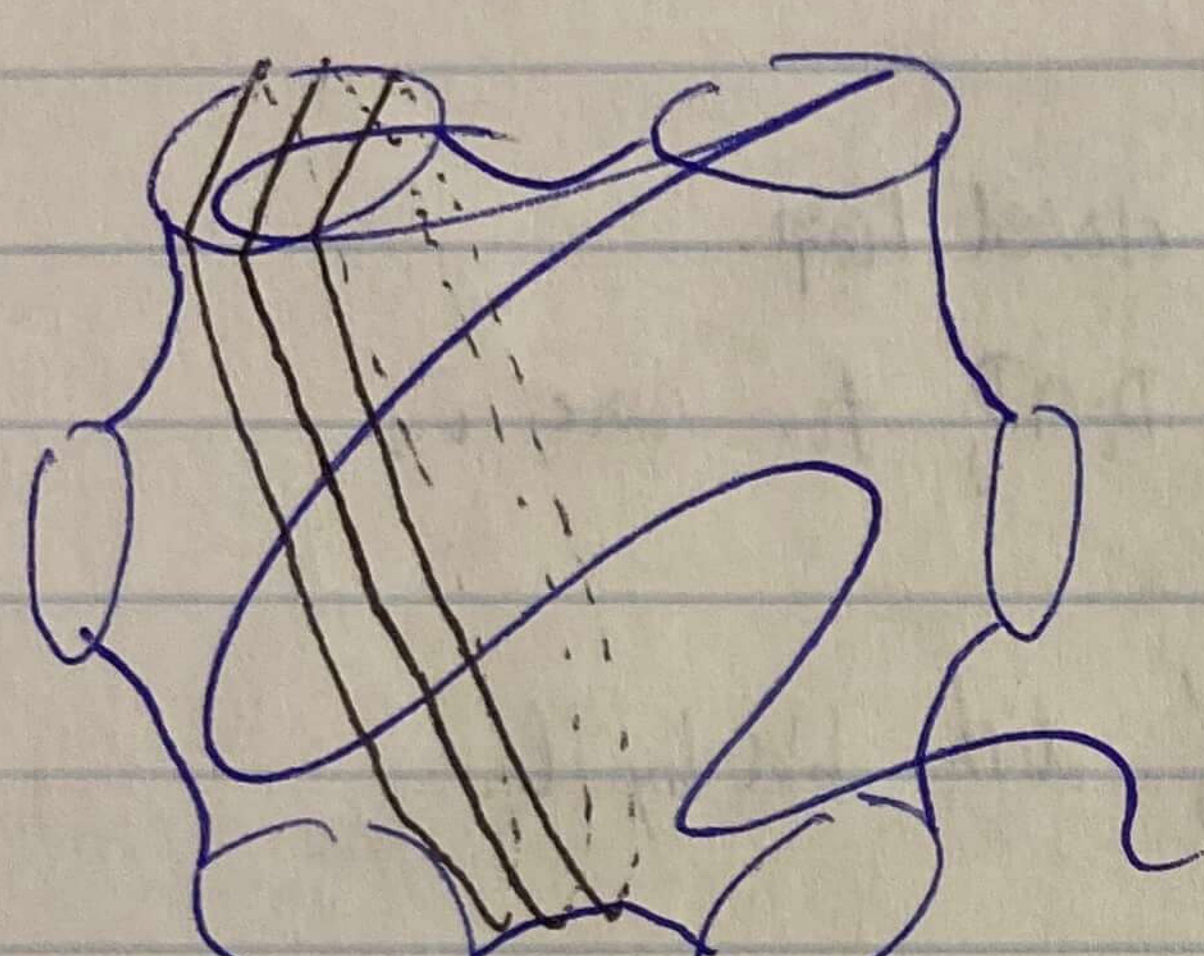
$B = H \setminus \mathbb{D}''$ ball

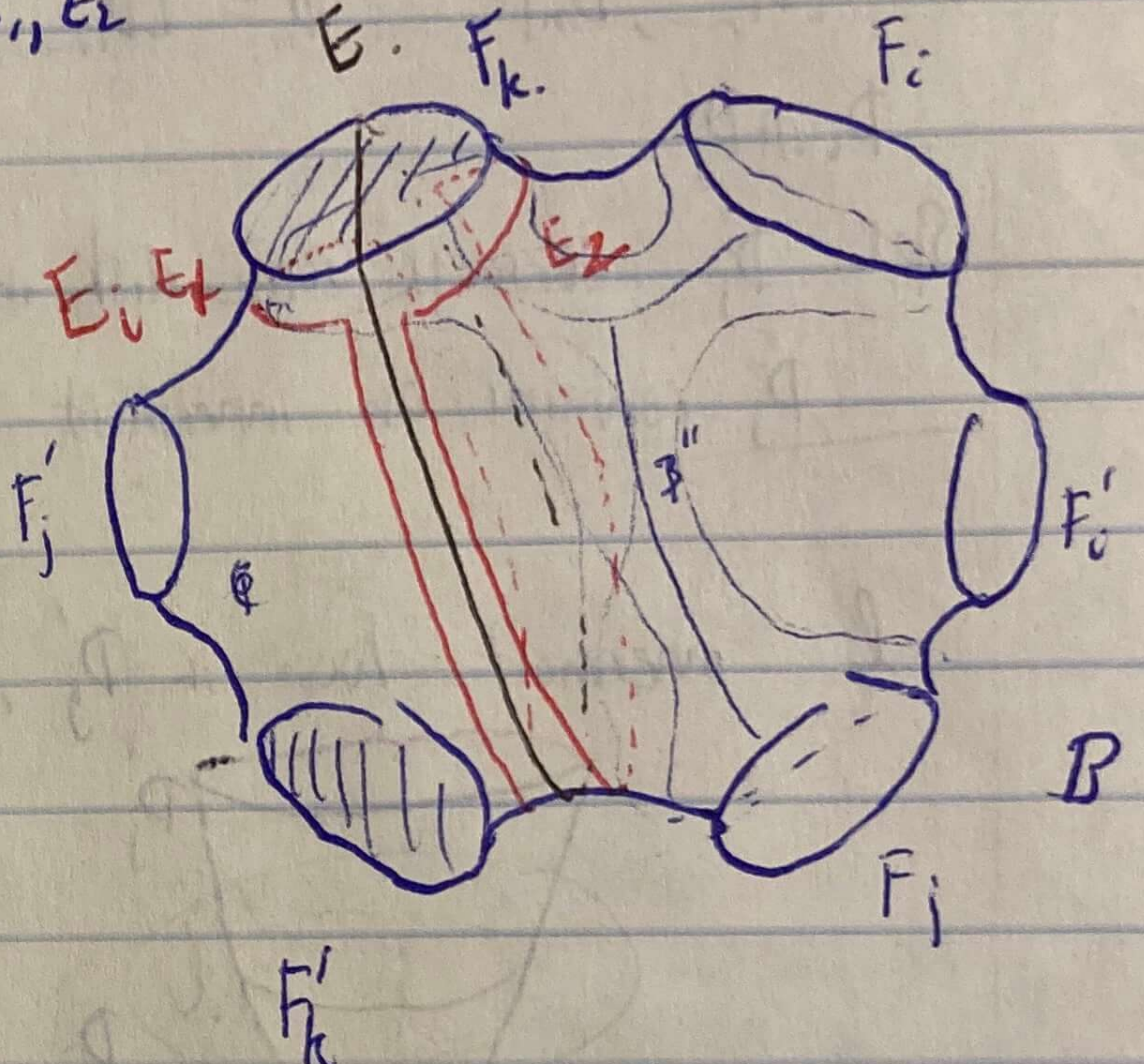
each D_i'' parallel to two closed disks $F_i, F_i' \subset \partial B$.

$E \subset B$ prop. emb., ∂E = ~~[illegible]~~ arc in F_k and arc disjoint from F_i, F_i'.



N: reg. nbhd in B of $E \cup F_k$. $N \setminus \partial B$ is E_1, E_2 disks.





$B \setminus (E_1 \cup E_2)$ is three balls.