Journal of Hyperbolic Differential Equations
Vol. 16, No. 2 (2019) 379–400

DOI: 10.1142/S0219891619500140

The wave equation near flat Friedmann–Lemaître–Robertson–Walker and Kasner Big Bang singularities

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Received 18 June 2018
Accepted 21 March 2019
Published 31 July 2019

Communicated by Philippe G. LeFloch

Abstract. We consider the wave equation, $\Box_g \psi = 0$, in fixed flat Friedmann–Lemaître–Robertson–Walker and Kasner spacetimes with topology $\mathbb{R}_+ \times \mathbb{T}^3$. We obtain generic blow up results for solutions to the wave equation toward the Big Bang singularity in both backgrounds. In particular, we characterize open sets of initial data prescribed at a spacelike hypersurface close to the singularity, which give rise to the solutions that blow up in an open set of the Big Bang hypersurface $\{t = 0\}$. The initial data sets are characterized by the condition that the Neumann data should dominate, in an appropriate L^2-sense, up to two spatial derivatives of the Dirichlet data. For these initial configurations, the $L^2(\mathbb{T}^3)$ norms of the solutions blow up toward the Big Bang hypersurfaces of FLRW and Kasner with inverse polynomial and logarithmic rates, respectively. Our method is based on deriving suitably weighted energy estimates in physical space. No symmetries of solutions are assumed.

Keywords: Wave equation; blow up; big bang singularities.

Mathematics Subject Classification 2010: 35Q75, 35Q76, 83C75, 83F05

1. Introduction and Main Theorems

In this note, we analyze the behavior of solutions to the wave equation on cosmological backgrounds toward the initial singularity. Our spacetimes of interest are the spatially homogenous, isotropic flat Friedmann–Lemaître–Robertson–Walker

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(FLRW) backgrounds and the anisotropic vacuum Kasner spacetimes. The former plays an important role in physics, since observational evidence suggests that at sufficiently large scales the universe seems to be spatially homogeneous and isotropic. The Kasner solutions also play an important role in the theory of general relativity, since they form the past attractor of Bianchi type I spacetimes, the *Kasner circle*, which in turn are the basic building blocks in the "BKL conjecture" [5] concerning spacelike cosmological singularities, see [7] for recent developments on this subject in the setting of spatially homogeneous solutions to the Einstein-vacuum equations.

The spacetimes have the topology $\mathbb{R}_+ \times \mathbb{T}^3$ and are endowed with the metrics:

$$g_{\mathrm{FLRW}} = -dt^2 + t^{\frac{4}{3\gamma}}(dx_1^2 + dx_2^2 + dx_3^2), \quad \frac{2}{3} < \gamma < 2, \tag{1.1}$$

$$g_{\mathrm{Kasner}} = -dt^2 + \sum_{j=1}^{3} t^{2p_j} dx_j^2, \quad \sum_{j=1}^{3} p_j = 1, \quad \sum_{j=1}^{3} p_j^2 = 1, \;\; p_j < 1, \tag{1.2}$$

respectively. Both metrics $g_{\mathrm{FLRW}}, g_{\mathrm{Kasner}}$ have a Big Bang singularity at $t = 0$, where the curvature blows up $|\mathrm{Riem}| \sim t^{-2}$, as $t \to 0$. Metrics of the form (1.1) are solutions of the Einstein–Euler system for ideal fluids with linear equation of state $p = (\gamma - 1)\rho$, where p is the pressure and ρ is the energy density. The case $\gamma = 2$ corresponds to stiff fluids, i.e. $p = \rho$, where incompressibility is expressed by the velocity of sound c_s equating the velocity of light $c = 1$. For the stiff case, the dynamics of the Einstein equations toward the singularity are completely understood by the work of [30–32]. The other endpoint $\gamma = \frac{2}{3}$ corresponds to the coasting universe which does not have a spacelike singularity. On the other hand, the Kasner metric (1.2) is a solution to the Einstein-vacuum equations. When one of the $p_j = 1$ equals one, and the other two vanish (flat Kasner), it corresponds to the Taub form of Minkowski space and there is no singularity as the spacetime is flat.

Our goal is to understand the behavior of smooth solutions to the wave equation toward these singularities from the initial value problem point of view and by deriving appropriate energy estimates in physical space, which may also prove useful for dynamical studies. According to the references in the literature, such as [1, 24, 27, 29], these waves are shown to blow up in certain cases. We wish to characterize open sets of initial data at a given time $t_0 > 0$ for which such blow up behavior occurs at $t = 0$. Denote the constant t hypersurfaces by Σ_t.

First, we give the general asymptotic profile of all solutions.

Theorem 1.1. *Let ψ be a smooth solution to the wave equation, $\Box_g \psi = 0$, for either of the metrics $g_{\mathrm{FLRW}}, g_{\mathrm{Kasner}}$, arising from initial data $(\psi_0, \partial_t \psi_0)$ on Σ_{t_0}. Then, ψ can be written in the following form*:

$$\psi_{\mathrm{FLRW}}(t, x) = A_{\mathrm{FLRW}}(x) t^{1-\frac{2}{\gamma}} + u_{\mathrm{FLRW}}(t, x), \tag{1.3}$$

$$\psi_{\mathrm{Kasner}}(t, x) = A_{\mathrm{Kasner}}(x) \log t + u_{\mathrm{Kasner}}(t, x), \tag{1.4}$$

where $A(x), u(t, x)$ are smooth functions and $u_{\mathrm{FLRW}} t^{\frac{2}{\gamma}-1}, u_{\mathrm{Kasner}}(\log t)^{-1}$ tend to zero, as $t \to 0$.

We prove the preceding theorem by deriving appropriate stability estimates for renormalized variables, which as a corollary imply the continuous dependence of $A(x)$ on initial data. For instance, solutions coming from initial configurations close to those of the homogeneous solutions $t^{1-\frac{2}{\gamma}}, \log t$ in FLRW and Kasner, respectively, will blow up with leading order coefficients $A(x) \sim 1$. Hence, the set of all blowing up solutions to the wave equation is open and dense.[a]

Our next theorem yields a characterization of open sets of initial data for which the corresponding solutions to the wave equation blow up in $L^2(\mathbb{T}^3)$ at the Big Bang hypersurface $t = 0$.

Theorem 1.2. *Let ψ be a smooth solution to the wave equation, $\Box_g \psi = 0$, for either of the metrics $g_{\text{FLRW}}, g_{\text{Kasner}}$, arising from initial data $(\psi_0, \partial_t\psi_0)$ on Σ_{t_0}, $t_0 > 0$. If $\partial_t\psi_0$ is non-zero in $L^2(\mathbb{T}^3)$, t_0 is sufficiently small such that*

$$\frac{2t_0^{2-\frac{4}{3\gamma}}}{1-(\frac{2}{3\gamma})^2}\sum_{i=1}^{3}\|\partial_t\partial_{x_i}\psi_0\|^2_{L^2(\mathbb{T}^3)} < \epsilon\|\partial_t\psi_0\|^2_{L^2(\mathbb{T}^3)}, \qquad \text{(FLRW)}, \tag{1.5}$$

$$\sum_{i=1}^{3}\frac{2t_0^{2-2p_i}}{(1-p_i)^2}\|\partial_t\partial_{x_i}\psi_0\|^2_{L^2(\mathbb{T}^3)} < \epsilon\|\partial_t\psi_0\|^2_{L^2(\mathbb{T}^3)}, \qquad \text{(Kasner)}, \tag{1.6}$$

and $\psi_0, \partial_t\psi_0$ satisfy the open conditions

$$\begin{aligned}(1-\epsilon)\|\partial_t\psi_0\|^2_{L^2(\mathbb{T}^3)} > {} & t_0^{-\frac{4}{3\gamma}}\sum_{i=3}^{3}\|\partial_{x_i}\psi_0\|^2_{L^2(\mathbb{T}^3)} \\ & + \frac{2t_0^{2-\frac{8}{3\gamma}}}{1-(\frac{2}{3\gamma})^2}\sum_{i,j=1}^{3}\|\partial_{x_j}\partial_{x_i}\psi_0\|_{L^2(\mathbb{T}^3)}, \quad \text{(FLRW)}\end{aligned} \tag{1.7}$$

$$\begin{aligned}(1-\epsilon)\|\partial_t\psi_0\|^2_{L^2(\mathbb{T}^3)} > {} & \sum_{i=3}^{3} t_0^{-2p_i}\|\partial_{x_i}\psi_0\|^2_{L^2(\mathbb{T}^3)} \\ & + \sum_{i,j=1}^{3}\frac{2t_0^{2-2p_i-2p_j}}{(1-p_i)^2}\|\partial_{x_j}\partial_{x_i}\psi_0\|_{L^2(\mathbb{T}^3)}, \quad \text{(Kasner)}\end{aligned} \tag{1.8}$$

for some $0 < \epsilon < 1$, then $\|A(x)\|_{L^2(\mathbb{T}^3)} > 0$.

Remark 1.3. Given a blowing up solution to the wave equation in either FLRW or Kasner, having non-vanishing leading order coefficient $A(x)$, it is easy to see, using the expansions (1.3), (1.4), that the solution satisfies the conditions in Theorem 1.2 for $t_0 > 0$ sufficiently small.

[a] In the sense that if a solution blows up in a compact set at Σ_0, i.e. $A(x) \neq 0$ in that compact set, then this property persists under sufficiently small perturbations. On the contrary, if $A(x) = 0$ in an open subset of Σ_0, one can always add a small multiple of $t^{1-\frac{2}{\gamma}}, \log t$ to produce a new solution with $A(x) \neq 0$ in that open set.

We also prove a local version of Theorem 1.2, giving open initial conditions in a neighborhood of Σ_{t_0}, U_{t_0}, whose domain of dependence intersects the singular hypersurface Σ_0 at a neighborhood U_0, where the $L^2(U_0)$ norm of the corresponding solutions blows up.

Theorem 1.4. *Let U_{t_0} be an open neighborhood in Σ_{t_0}, $t_0 > 0$, whose domain of dependence intersects Σ_0 in $U_0 = (0,\delta)^3$, and let ψ be a smooth solution to the wave equation, $\Box_g\psi = 0$, for either of the metrics $g_{\text{FLRW}}, g_{\text{Kasner}}$, arising from initial data $(\psi_0, \partial_t\psi_0)$ on U_{t_0}. If $\partial_t\psi_0$ is non-zero in U_{t_0}, t_0 is sufficiently small such that*

$$\begin{aligned}
&\frac{2t_0^{2-\frac{4}{3\gamma}}}{1-(\frac{2}{3\gamma})^2}\sum_{i=1}^3 \|\partial_t\partial_{x_i}\psi_0\|^2_{L^2(U_{t_0})} \\
&\quad + \frac{4t_0^{1-\frac{2}{3\gamma}}}{1-\frac{2}{3\gamma}}\left[3\|\partial_t\psi_0\|^2_{L^2(U_{t_0})} + \sum_{l=1}^3\|\partial_t\partial_{x_l}\psi_0\|^2_{L^2(U_{t_0})}\right] \qquad (1.9)\\
&\quad + 6\log\left(1+\frac{2}{1-\frac{2}{3\gamma}}\frac{t_0^{1-\frac{2}{3\gamma}}}{\delta}\right)\|\partial_t\psi_0\|^2_{L^2(U_{t_0})} < \epsilon\|\partial_t\psi_0\|_{L^2(U_{t_0})}, \qquad \text{(FLRW)}
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^3\frac{2t_0^{2-2p_i}}{(1-p_i)^2}\|\partial_t\partial_{x_i}\psi_0\|^2_{L^2(U_{t_0})} \\
&\quad + \sum_{l=1}^3\frac{4t_0^{1-p_l}}{1-p_l}[\|\partial_t\psi_0\|^2_{L^2(U_{t_0})} + \|\partial_t\partial_{x_l}\psi_0\|^2_{L^2(U_{t_0})}] \qquad (1.10)\\
&\quad + \sum_{l=1}^3 2\log\left(1+\frac{2}{1-p_l}\frac{t_0^{1-p_l}}{\delta}\right)\|\partial_t\psi_0\|^2_{L^2(U_{t_0})} < \epsilon\|\partial_t\psi_0\|^2_{L^2(U_{t_0})}, \qquad \text{(Kasner)}
\end{aligned}$$

and $(\psi_0, \partial_t\psi_0)$ satisfy the open conditions:

$$\begin{aligned}
&(1-\epsilon)\|\partial_t\psi_0\|^2_{L^2(U_{t_0})} \\
&\quad > t_0^{-\frac{4}{3\gamma}}\sum_{i=1}^3\|\partial_{x_i}\psi_0\|^2_{L^2(U_{t_0})} + \frac{2t_0^{2-\frac{8}{3\gamma}}}{1-(\frac{2}{3\gamma})^2}\sum_{i,j=1}^3\|\partial_{x_j}\partial_{x_i}\psi_0\|^2_{L^2(U_{t_0})} \qquad (1.11)\\
&\quad + \frac{4t_0^{1-\frac{2}{\gamma}}}{1-\frac{2}{3\gamma}}\left[3t_0^{-\frac{4}{3\gamma}}\sum_{i=1}^3\|\partial_{x_i}\psi_0\|^2_{L^2(U_{t_0})} + t_0^{-\frac{4}{3\gamma}}\sum_{i,l=1}^3\|\partial_{x_i}\partial_{x_l}\psi_0\|^2_{L^2(U_{t_0})}\right]\\
&\quad + 6\log\left(1+\frac{2}{1-\frac{2}{3\gamma}}\frac{t_0^{1-\frac{2}{\gamma}}}{\delta}\right)t_0^{-\frac{4}{3\gamma}}\sum_{i=1}^3\|\partial_{x_i}\psi_0\|^2_{L^2(U_{t_0})}, \qquad \text{(FLRW)}
\end{aligned}$$

$$
\begin{aligned}
&(1-\epsilon)\|\partial_t\psi_0\|^2_{L^2(U_{t_0})}\\
&\quad > \sum_{i=1}^3 t_0^{-2p_i}\|\partial_{x_i}\psi_0\|^2_{L^2(U_{t_0})} + \sum_{i,j=1}^3 \frac{2t_0^{2-2p_i-2p_j}}{(1-p_i)^2}\|\partial_{x_j}\partial_{x_i}\psi_0\|^2_{L^2(U_{t_0})}\\
&\qquad + \sum_{i,l=1}^3 \frac{4t_0^{1-p_l}}{1-p_l}[t_0^{-2p_i}\|\partial_{x_i}\psi_0\|^2_{L^2(U_{t_0})} + t_0^{-2p_i}\|\partial_{x_i}\partial_{x_l}\psi_0\|^2_{L^2(U_{t_0})}] \qquad (1.12)\\
&\qquad + \sum_{l=1}^3 2\log\left(1+\frac{2}{1-p_l}\frac{t_0^{1-\frac{2}{\gamma}}}{\delta}\right)\sum_{i=1}^3 t_0^{-2p_i}\|\partial_{x_i}\psi_0\|^2_{L^2(U_{t_0})}, \quad \text{(Kasner)}
\end{aligned}
$$

for some $0 < \epsilon < 1$*, then* $\|A(x)\|_{L^2(U_0)} > 0$.

The blow up behavior of linear waves observed near Big Bang singularities is reminiscent of the behavior of waves in black hole interiors containing spacelike singularities [8, 13]. Examples are the Schwarzschild singularity or black hole singularities occurring in spherically symmetric solutions to the Einstein-scalar field model [10], where a logarithmic blow up behavior has been observed for spatially homogeneous waves. Such logarithmic blow up behavior was recently confirmed [15] for generic linear waves in the Schwarzschild black hole interior. The aforementioned blow up behaviors, however, are in contrast to the behavior of waves observed near null boundaries, where linear and dynamical waves have been shown in general to extend continuously past the relevant null hypersurfaces [11, 16–22], see also [25, 26].

Lastly, we should note that although we only deal with spatially homogeneous spacetimes, our method of proof is applicable to cosmological spacetimes with Big Bang singularities exhibiting asymptotically velocity term-dominated (AVTD) behavior [2–4, 6, 9, 12, 14, 20, 23, 28].

2. Proof of Main Theorems

2.1. *Energy argument and notation*

In order to derive the energy estimates required for the proof of Theorem 1.2, we will apply the vector field method and define certain energy currents constructed from the *stress–energy tensor*

$$T_{ab}[\psi] = \partial_a\psi\partial_b\psi - \frac{1}{2}g_{ab}\partial^c\psi\partial_c\psi, \qquad (2.1)$$

of the scalar field ψ. The divergence of $T_{ab}[\psi]$ reads

$$\nabla^a T_{ab}[\psi] = \partial_b\psi \cdot \Box_g\psi, \qquad (2.2)$$

where ∇ stands for the spacetime covariant connection. Hence, if ψ satisfies the homogeneous wave equation

$$\Box_g\psi = 0, \qquad (2.3)$$

it follows that we have energy–momentum conservation $\nabla^a T_{ab}[\psi] = 0$. Contracting the stress–energy tensor with a vector field multiplier X, then defines the associated current

$$J_a^X[\psi] = X^b T_{ab}[\psi], \tag{2.4}$$

whose divergence, according to (2.2), equals

$$\nabla^a J_a^X[\psi] = (\nabla^a X^b) T_{ab}[\psi] + X\psi \cdot \Box_g \psi. \tag{2.5}$$

Applying the divergence theorem to (2.5), over the spacetime domain $\{U_s\}_{s\in[t,t_0]}$ (Fig. 1), we thus obtain

$$\begin{aligned} &\int_{U_t} J_a^X[\psi] n_{U_t}^a \mathrm{vol}_{U_t} + \sum_{l=1}^{3} \int_{\cup\mathcal{N}_l^\pm} J_a^X[\psi] n_{\mathcal{N}_l^\pm}^a \mathrm{vol}_{\mathcal{N}_l^\pm} \\ &\quad = \int_{U_{t_0}} J_a^X[\psi] n_{U_{t_0}}^a \mathrm{vol}_{U_{t_0}} - \int_t^{t_0} \int_{U_s} \nabla^a J_a^X[\psi] \mathrm{vol}_{U_s} ds, \end{aligned} \tag{2.6}$$

where $n_{U_t} = -\partial_t$, vol_{U_t} is the intrinsic volume form of U_t and

$$\left.\begin{aligned} n_{\mathcal{N}_l^\pm} &= -\partial_t \pm t^{-\frac{2}{3\gamma}} \partial_{x_l}, \\ \mathrm{vol}_{\mathcal{N}_l^\pm} &= t^{\frac{4}{3\gamma}} dt dx_i dx_j, \\ t^{-\frac{2}{3\gamma}} dt &= \pm dx_l, \quad \text{on } \mathcal{N}_l^\pm, \end{aligned}\right\} \quad \text{(FLRW)},$$

$$\left.\begin{aligned} n_{\mathcal{N}_l^\pm} &= -\partial_t \pm t^{-p_l} \partial_{x_l}, \\ \mathrm{vol}_{\mathcal{N}_l^\pm} &= t^{1-p_l} dt dx_i dx_j, \\ t^{-p_l} dt &= \pm dx_l, \quad \text{on } \mathcal{N}_l^\pm, \end{aligned}\right\} \quad \text{(Kasner)}, \tag{2.7}$$

for each $l = 1, 2, 3$; $i < j$; $i, j \neq l$.

Below, we will choose the vector field X to be a suitable rescaling n_{U_t}. Note that $n_{U_t}^a J_a^X[\psi] = J_0^X[\psi] = \frac{1}{2}[(\partial_t\psi)^2 + |\overline{\nabla}\psi|^2]$ and $J_a^X[\psi] n_{\mathcal{N}_l^\pm}^a \geq 0$, where $\overline{\nabla}$ is the covariant derivative intrinsic to the level sets of t. Hence, if we can control the second term on the right-hand side of (2.6) (the bulk), in terms of $J_0^X[\psi]$, we obtain an energy estimate for ψ.

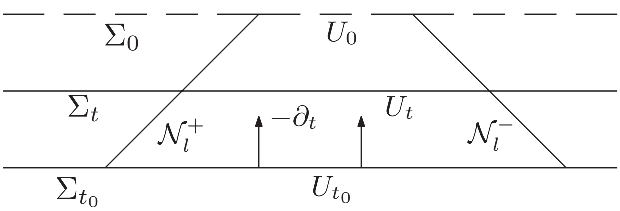

Fig. 1. Domain of dependence of an open neighborhood U_{t_0} of the initial hypersurface Σ_{t_0}.

Note that in the case of the whole torus, $U_{t_0} = \Sigma_{t_0}$, due to the absence of causal boundary terms, (2.6) becomes

$$\int_{\Sigma_t} J_a^X[\psi] n^a_{\Sigma_t} \mathrm{vol}_{\Sigma_t} = \int_{\Sigma_{t_0}} J_a^X[\psi] n^a_{\Sigma_{t_0}} \mathrm{vol}_{\Sigma_{t_0}} - \int_t^{t_0} \int_{\Sigma_s} \nabla^a J_a^X[\psi] \mathrm{vol}_{\Sigma_s} ds. \quad (2.8)$$

In the analysis that follows, we will often require higher-order energy estimates that we can obtain by commuting the wave equation with spatial derivatives and applying the above energy argument. Note that, we are considering homogeneous spacetimes in which the spatial coordinate derivatives $\{\partial_{x_i}\}$ are Killing and hence $[\Box_g, \partial_{x_i}] = 0$, $i = 1,2,3$. This means that the above identities (2.6), (2.8) are also valid for $\partial_x^\alpha \psi$, where we use the standard multi-index notation $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ for an iterated application of spatial derivatives, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. In this notation, the $H^k(\Sigma_t)$ norm of a smooth function $f : (0, +\infty) \times \mathbb{T}^3$ equals

$$\|f\|^2_{H^k(\Sigma_t)} = \sum_{|\alpha| \le k} \int_{\Sigma_t} (\partial_x^\alpha f)^2 \mathrm{vol}_{\mathrm{Euc}}, \quad (2.9)$$

where $\mathrm{vol}_{\mathrm{Euc}} = dx_1 dx_2 dx_3$. We will often omit Σ_t from the norms to ease notation and use H^k, L^2 for the corresponding time-dependent, non-intrinsic norms.

2.2. *Flat FLRW*

Let ψ be a smooth solution to the scalar wave equation

$$\Box_{g_{\mathrm{FLRW}}} \psi = 0. \quad (2.10)$$

Consider the orthonormal frame

$$e_0 = -\partial_t, \quad e_i = t^{-\frac{2}{3\gamma}} \partial_{x_i} \quad (2.11)$$

adapted to the constant t hypersurfaces Σ_t with the past normal vector field e_0 pointing toward the singularity. In this frame, the second fundamental form K_{ij} of Σ_t reads

$$K_{ii} := g(\nabla_{e_i} e_0, e_i) = -\frac{2}{3\gamma} \frac{1}{t}, \quad i = 1,2,3. \quad (2.12)$$

Further, the intrinsic volume form on Σ_t equals

$$\mathrm{vol}_{\Sigma_t} = t^{\frac{2}{\gamma}} \mathrm{vol}_{\mathrm{Euc}}. \quad (2.13)$$

Proposition 2.1. *The following energy inequality holds*:

$$t^{\frac{2}{\gamma}} \int_{\Sigma_t} J_0^{e_0}[\partial_x^\alpha \psi] \mathrm{vol}_{\Sigma_t} \le t_0^{\frac{2}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha \psi] \mathrm{vol}_{\Sigma_{t_0}}, \quad (2.14)$$

for all $t \in (0, t_0]$ and any multi-index α. Moreover, ψ satisfies the pointwise bound

$$|\psi(t,x)| \le C \left(\sum_{|\alpha| \le 2} t_0^{\frac{2}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha \psi] \mathrm{vol}_{\Sigma_{t_0}} \right)^{\frac{1}{2}} \frac{t^{1-\frac{2}{\gamma}} - t_0^{1-\frac{2}{\gamma}}}{\frac{2}{\gamma} - 1} + |\psi(t_0, x)|, \quad (2.15)$$

where $C > 0$ is a constant independent of t_0, γ.

Proof. We compute the divergence of $J_a^{t^{\frac{2}{\gamma}}e_0}[\psi]$:

$$\begin{aligned}\nabla^a J_a^{t^{\frac{2}{\gamma}}e_0}[\psi] &\overset{(2.5)}{=} \nabla^a(t^{\frac{2}{\gamma}}e_0)^b T_{ab}[\psi] = t^{\frac{2}{\gamma}}K^{ab}T_{ab}[\psi] - e_0 t^{\frac{2}{\gamma}}T_{00}[\psi], \qquad (2.16)\\ &= t^{\frac{2}{\gamma}}K_{11}|\overline{\nabla}\psi|^2 - \frac{1}{2}t^{\frac{2}{\gamma}}K^i{}_i|\nabla\psi|^2 + \frac{1}{\gamma}t^{\frac{2}{\gamma}-1}[(e_0\psi)^2 + |\overline{\nabla}\psi|^2],\\ &\overset{(2.12)}{=} \frac{4}{3\gamma}\frac{1}{t}t^{\frac{2}{\gamma}}|\overline{\nabla}\psi|^2.\end{aligned}$$

Hence, utilizing (2.8) for $X = t^{\frac{2}{\gamma}}e_0$ yields

$$\begin{aligned}t^{\frac{2}{\gamma}}\int_{\Sigma_t} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_t} &= t_0^{\frac{2}{\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_{t_0}} - \int_t^{t_0}\int_{\Sigma_s}\frac{4}{3\gamma}s^{\frac{2}{\gamma}-1}|\overline{\nabla}\psi|^2\mathrm{vol}_{\Sigma_s}ds. \qquad (2.17)\\ &\le t_0^{\frac{2}{\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_{t_0}}. \qquad (2.18)\end{aligned}$$

The same identity is valid for $\partial_x^\alpha\psi$, leading to (2.14). In particular, taking into account the volume form (2.13), we have the following bounds for $\partial_t\psi$:

$$t^{\frac{4}{\gamma}}\|\partial_t\partial_x^\alpha\psi\|_{L^2}^2 \le 2t_0^{\frac{2}{\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha\psi]\mathrm{vol}_{\Sigma_{t_0}}, \qquad (2.19)$$

for all $t \in (0, t_0]$ and α. Integrating $\partial_t\psi$ in $[t, t_0]$ and employing the Sobolev embedding $H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$ we derive

$$\begin{aligned}|\psi(t,x)| &= \Big|\int_{t_0}^t \partial_s\psi(s,x)ds + \psi(t_0,x)\Big| \qquad (2.20)\\ &\le C\int_t^{t_0}\|\partial_s\psi\|_{H^2}ds + |\psi(t_0,x)|\\ &\le \frac{C}{\frac{2}{\gamma}-1}\left(t^{1-\frac{2}{\gamma}} - t_0^{1-\frac{2}{\gamma}}\right)\left(\sum_{|\alpha|\le 2} t_0^{\frac{2}{\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha\psi]\mathrm{vol}_{\Sigma_{t_0}}\right)^{\frac{1}{2}} + |\psi(t_0,x)|,\end{aligned}$$

for $\gamma < 2$. □

Remark 2.2. The bounds (2.14), (2.15) are saturated by the homogeneous function $t^{1-\frac{2}{\gamma}}$, which is an exact solution of (2.10).

From Proposition 2.1, we understand that $t^{1-\frac{2}{\gamma}}$ is the leading order of ψ at $t = 0$. To prove this rigorously, we derive analogous energy bounds for the renormalized variable $\frac{\psi}{t^{1-\frac{2}{\gamma}}}$ that satisfies the wave equation:

$$\Box\frac{\psi}{t^{1-\frac{2}{\gamma}}} = -\frac{2}{t}\left(1 - \frac{2}{\gamma}\right)e_0\left(\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right). \qquad (2.21)$$

Proposition 2.3. *Let ψ be a smooth solution to the wave equation in FLRW backgrounds with $\frac{2}{3} < \gamma < 2$. Then, the following bounds hold uniformly in $t \in (0, t_0]$:*

$$t^{4-\frac{6}{\gamma}} \int_{\Sigma_t} J_0^{e_0} \left[\partial_x^\alpha \frac{\psi}{t^{1-\frac{2}{\gamma}}} \right] \mathrm{vol}_{\Sigma_t} \leq t_0^{4-\frac{6}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0} \left[\partial_x^\alpha \frac{\psi}{t^{1-\frac{2}{\gamma}}} \right] \mathrm{vol}_{\Sigma_{t_0}}, \quad \frac{4}{3} \leq \gamma < 2, \tag{2.22}$$

$$t^{-\frac{2}{3\gamma}} \int_{\Sigma_t} J_0^{e_0} \left[\partial_x^\alpha \frac{\psi}{t^{1-\frac{2}{\gamma}}} \right] \mathrm{vol}_{\Sigma_t} \leq t_0^{-\frac{2}{3\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0} \left[\partial_x^\alpha \frac{\psi}{t^{1-\frac{2}{\gamma}}} \right] \mathrm{vol}_{\Sigma_{t_0}}, \quad \frac{2}{3} < \gamma \leq \frac{4}{3}, \tag{2.23}$$

for all $t \in (0, t_0]$ and any multi-index α. Moreover, the limit

$$A(x) := \lim_{t \to 0} \frac{\psi}{t^{1-\frac{2}{\gamma}}} \tag{2.24}$$

exists, it is a smooth function and the difference $u(t,x) := \psi - A(x) t^{1-\frac{2}{\gamma}}$ satisfies

$$\lim_{t \to 0} t^{\frac{2}{\gamma}} \int_{\Sigma_t} J_0^{e_0} [\partial_x^\alpha u] \mathrm{vol}_{\Sigma_s} = 0. \tag{2.25}$$

Proof. Let $\eta > 0$. We compute

$$\begin{aligned} \nabla^a \left(J_a^{t^\eta e_0} \left[\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right] \right) &\overset{(2.5)}{=} \nabla^a (t^\eta e_0)^b T_{ab} \left[\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right] + t^\eta e_0 \left(\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right) \cdot \Box_g \frac{\psi}{t^{1-\frac{2}{\gamma}}}, \\ &= t^\eta K^{ab} T_{ab} \left[\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right] - (e_0 t^\eta) T_{00} \left[\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right] - \frac{2}{t} \left(1 - \frac{2}{\gamma} \right) t^\eta \\ &\quad \times \left[e_0 \left(\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right) \right]^2, \\ &\overset{(2.12)}{=} t^{\eta-1} \left[\left(\frac{1}{3\gamma} + \frac{\eta}{2} \right) \left| \overline{\nabla} \frac{\psi}{t^{1-\frac{2}{\gamma}}} \right|^2 + \left(\frac{\eta}{2} + \frac{3}{\gamma} - 2 \right) \right. \\ &\quad \left. \times \left[e_0 \left(\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right) \right]^2 \right]. \end{aligned} \tag{2.26}$$

This leads to different choices of η depending on the value of γ, given by

$$\eta = 4 - \frac{6}{\gamma}, \qquad \text{for} \quad \frac{4}{3} \leq \gamma < 2, \tag{2.27}$$

$$\eta = -\frac{2}{3\gamma}, \qquad \text{for} \quad \frac{2}{3} < \gamma \leq \frac{4}{3}. \tag{2.28}$$

We refer to the former as the stiffest region and the latter as the softest region. The case $\gamma = \frac{4}{3}$ corresponds to radiation, where $\eta = -\frac{1}{2}$. For the two cases, (2.26) reads

$$\nabla^a \left(J_a^{t^{4-\frac{6}{\gamma}} e_0} \left[\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right] \right) = \left(2 - \frac{8}{3\gamma} \right) t^{3-\frac{6}{\gamma}} \left| \overline{\nabla} \frac{\psi}{t^{1-\frac{2}{\gamma}}} \right|^2, \tag{2.29}$$

$$\nabla^a \left(J_a^{t^{-\frac{2}{3\gamma}} e_0} \left[\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right] \right) = \left(\frac{8}{3\gamma} - 2 \right) t^{-1-\frac{2}{3\gamma}} \left[e_0 \left(\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right) \right]^2. \tag{2.30}$$

Then, the energy identity (2.8) for $\frac{\psi}{t^{1-\frac{2}{\gamma}}}$, $X = t^{4-\frac{6}{\gamma}}e_0$, in the stiffest region $\{\frac{4}{3} \leq \gamma < 2\}$, implies that

$$\begin{aligned} t^{4-\frac{6}{\gamma}} \int_{\Sigma_t} J_0^{e_0}\left[\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right] \mathrm{vol}_{\Sigma_t} &\leq t_0^{4-\frac{6}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}\left[\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right] \mathrm{vol}_{\Sigma_{t_0}} \\ &\quad - \left(2 - \frac{8}{3\gamma}\right) \int_t^{t_0} s^{3-\frac{6}{\gamma}} \int_{\Sigma_s} \left|\overline{\nabla} \frac{\psi}{t^{1-\frac{2}{\gamma}}}\right|^2 \mathrm{vol}_{\Sigma_s} ds \\ &\leq t_0^{4-\frac{6}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}\left[\frac{\psi}{t_0^{1-\frac{2}{\gamma}}}\right] \mathrm{vol}_{\Sigma_{t_0}}. \end{aligned} \tag{2.31}$$

Commuting with ∂_x^α yields the bound (2.22). Similarly, we obtain statement (2.23) in the softest region $\{\frac{2}{3} < \gamma \leq \frac{4}{3}\}$. In particular, taking into account the volume form (2.13), we have the bounds:

$$\begin{aligned} \left|\partial_t \frac{\psi}{t^{1-\frac{2}{\gamma}}}\right| &\leq C \left\|\partial_t \frac{\psi}{t^{1-\frac{2}{\gamma}}}\right\|_{H^2} \\ &\leq \frac{C}{t^{2-\frac{2}{\gamma}}} \left(\sum_{|\alpha|\leq 2} t_0^{4-\frac{6}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}\left[\partial_x^\alpha \frac{\psi}{t_0^{1-\frac{2}{\gamma}}}\right] \mathrm{vol}_{\Sigma_{t_0}}\right)^{\frac{1}{2}}, \quad \text{for } \frac{4}{3} \leq \gamma < 2, \end{aligned} \tag{2.32}$$

$$\begin{aligned} \left|\partial_t \frac{\psi}{t^{1-\frac{2}{\gamma}}}\right| &\leq C \left\|\partial_t \frac{\psi}{t^{1-\frac{2}{\gamma}}}\right\|_{H^2} \\ &\leq \frac{C}{t^{\frac{2}{3\gamma}}} \left(\sum_{|\alpha|\leq 2} t_0^{-\frac{2}{3\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}\left[\partial_x^\alpha \frac{\psi}{t^{1-\frac{2}{\gamma}}}\right] \mathrm{vol}_{\Sigma_{t_0}}\right)^{\frac{1}{2}}, \quad \text{for } \frac{2}{3} < \gamma \leq \frac{4}{3}, \end{aligned} \tag{2.33}$$

which imply that $\partial_t\psi(t,x) \in L^1([0,t_0])$, uniformly in x, for all $\frac{2}{3} < \gamma < 2$. Thus, $\frac{\psi}{t^{1-\frac{2}{\gamma}}}$ has a limit function $A(x)$, as $t \to 0$. The smoothness of $A(x)$ follows by repeating the preceding argument for $\partial_x^\alpha \frac{\psi}{t^{1-\frac{2}{\gamma}}}$.

Consider now the energy flux of the difference $\psi - A(x)t^{1-\frac{2}{\gamma}}$,

$$t^{\frac{2}{\gamma}} \int_{\Sigma_t} |e_0(\psi - A(x)t^{1-\frac{2}{\gamma}})|^2 + |\overline{\nabla}(\psi - A(x)t^{1-\frac{2}{\gamma}})|^2 \mathrm{vol}_{\Sigma_t} n \tag{2.34}$$

$$\begin{aligned} &= t^{\frac{2}{\gamma}} \int_{\Sigma_t} \left| t^{1-\frac{2}{\gamma}} e_0 \frac{\psi}{t^{1-\frac{2}{\gamma}}} + \left(1 - \frac{2}{\gamma}\right) t^{-\frac{2}{\gamma}} \left(\frac{\psi}{t^{1-\frac{2}{\gamma}}} - A(x)\right)\right|^2 \\ &\quad + t^{2-\frac{4}{\gamma}} \left|\overline{\nabla}\left(\frac{\psi}{t^{1-\frac{2}{\gamma}}} - A(x)\right)\right|^2 \mathrm{vol}_{\Sigma_t} \end{aligned}$$

$$
\begin{aligned}
&\leq 2t^{\frac{2}{\gamma}}\int_{\Sigma_t} t^{2-\frac{4}{\gamma}}\left|e_0\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right|^2+\left(1-\frac{2}{\gamma}\right)^2 t^{-\frac{4}{\gamma}}\left|\frac{\psi}{t^{1-\frac{2}{\gamma}}}-A(x)\right|^2+t^{2-\frac{4}{\gamma}}\left|\overline{\nabla}\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right|^2\\
&\quad+t^{2-\frac{4}{\gamma}}|\overline{\nabla}A(x)|^2\mathrm{vol}_{\Sigma_t}\\
&=4t^2\int_{\Sigma_t}J_0^{e_0}\left[\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right]\mathrm{vol}_{\mathrm{Euc}}+2\left(1-\frac{2}{\gamma}\right)^2\int_{\Sigma_t}\left|\frac{\psi}{t^{1-\frac{2}{\gamma}}}-A(x)\right|^2\mathrm{vol}_{\mathrm{Euc}}\\
&\quad+2t^2\int_{\Sigma_t}|\overline{\nabla}A(x)|^2\mathrm{vol}_{\mathrm{Euc}}\\
&\overset{(2.22),(2.23)}{\leq} o(1)+2\left(1-\frac{2}{\gamma}\right)^2\int_{\Sigma_t}\left|\frac{\psi}{t^{1-\frac{2}{\gamma}}}-A(x)\right|^2\mathrm{vol}_{\mathrm{Euc}}\\
&\quad+2t^{2-\frac{4}{3\gamma}}\sum_{i=1}^{3}\int_{\Sigma_t}|\partial_{x_i}A(x)|^2\mathrm{vol}_{\mathrm{Euc}},
\end{aligned}
$$

for all $\frac{4}{3}\leq\gamma<2$. The third term in the preceding right-hand side clearly tends to zero, as $t\to 0$, and by the definition of $A(x)$, so does the second term. Since the above argument also applies to $\partial_x^\alpha[\psi-A(x)t^{1-\frac{2}{\gamma}}]$, this proves (2.25). □

Remark 2.4. The renormalized estimate (2.22) yields an improved control over the spatial gradient of ψ compared to (2.14). Indeed,

$$
\begin{aligned}
&t^{2-\frac{4}{\gamma}}t^{\frac{2}{\gamma}}\int_{\Sigma_t}|\overline{\nabla}\psi|^2\mathrm{vol}_{\Sigma_t}\leq t_0^{4-\frac{6}{\gamma}}\int_{\Sigma_{t_0}}J_0^{e_0}\left[\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right]\mathrm{vol}_{\Sigma_{t_0}}, \qquad \frac{4}{3}\leq\gamma<2,\\
&t^{\frac{4}{3\gamma}-2}t^{\frac{2}{\gamma}}\int_{\Sigma_t}|\overline{\nabla}\psi|^2\mathrm{vol}_{\Sigma_t}\leq t_0^{-\frac{2}{3\gamma}}\int_{\Sigma_{t_0}}J_0^{e_0}\left[\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right]\mathrm{vol}_{\Sigma_{t_0}}, \qquad \frac{2}{3}<\gamma\leq\frac{4}{3},
\end{aligned}
\tag{2.35}
$$

holds for all $t\in(0,t_0]$, where in the stiffest case $2-\frac{4}{\gamma}<0$, while in the softest $\frac{4}{3\gamma}-2<0$.

Proposition 2.3 validates the asymptotic profile (1.3) of ψ, as stated in Theorem 1.1.

Lemma 2.5. *The following estimate for the L^2 norm of $\partial_{x_i}\psi$ holds*:

$$
\|\partial_{x_i}\psi\|_{L^2(\Sigma_t)}\leq\|\partial_{x_i}\psi\|_{L^2(\Sigma_{t_0})}+\sqrt{2}\frac{(t_0^{1-\frac{2}{\gamma}}-t^{1-\frac{2}{\gamma}})}{1-\frac{2}{\gamma}}t_0^{\frac{2}{\gamma}}\left(\int_{\Sigma_{t_0}}J_0^{e_0}[\partial_{x_i}\psi]\mathrm{vol}_{\mathrm{Euc}}\right)^{\frac{1}{2}},
\tag{2.36}
$$

for all $t\in(0,t_0]$.

Proof. Differentiating in e_0, we have:

$$\frac{1}{2}e_0\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_t)} \leq \|\partial_{x_i}\psi\|_{L^2(\Sigma_t)}\|e_0\psi\|_{L^2(\Sigma_t)} \tag{2.37}$$

$$\leq \frac{1}{t^{\frac{2}{\gamma}}}\left(2t_0^{\frac{2}{\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_{x_i}\psi]\mathrm{vol}_{\Sigma_{t_0}}\right)^{\frac{1}{2}}\|\partial_{x_i}\psi\|_{L^2(\Sigma_t)} \quad \text{(using (2.14))}$$

or

$$e_0\|\partial_{x_i}\psi\|_{L^2(\Sigma_t)} \leq \sqrt{2}\frac{t_0^{\frac{2}{\gamma}}}{t^{\frac{2}{\gamma}}}\left(\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_{x_i}\psi]\mathrm{vol}_{\mathrm{Euc}}\right)^{\frac{1}{2}}. \tag{2.38}$$

Integrating the above on $[t, t_0]$ gives (2.36) for $\gamma < 2$. □

Remark 2.6. The bounds that we have proven so far, stated in Propositions 2.1, 2.3 and Lemma 2.5, are also valid if we replace the integral domains Σ_t, Σ_{t_0} by U_t, U_{t_0}. This can be easily seen from the fact that in the corresponding energy identity (2.6), the null boundary terms have a favorable sign for an upper bound and therefore can be dropped.

Now, we may proceed to derive the blow up criterion given in Theorem 1.2. First, note, that for $\gamma > \frac{2}{3}$, the main contribution of the energy flux generated by $J^{e_0}[\psi]$ comes from the $e_0\psi$ term. Indeed, by (2.35) it follows that

$$t^{\frac{2}{\gamma}}\int_{\Sigma_t} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_t} = \frac{1}{2}\int_{\Sigma_t} t^{\frac{4}{\gamma}}(\partial_t\psi)^2\mathrm{vol}_{\mathrm{Euc}} + O(t^\eta), \tag{2.39}$$

where $\eta = \frac{4}{\gamma} - 2 > 0$, for $\gamma \in [\frac{4}{3}, 2)$ and $\eta = 2 - \frac{4}{3\gamma} > 0$, for $\gamma \in (\frac{2}{3}, \frac{4}{3}]$. Hence, taking the limit $t \to 0$ in the preceding identity and utilizing (2.25) leads to

$$\lim_{t\to 0} t^{\frac{2}{\gamma}}\int_{\Sigma_t} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_t} = \frac{1}{2}\left(1-\frac{2}{\gamma}\right)^2\int_{\Sigma_0} A^2(x)\mathrm{vol}_{\mathrm{Euc}}. \tag{2.40}$$

Combining (2.17), (2.40), we derive

$$\begin{aligned}
&\frac{1}{2}\left(1-\frac{2}{\gamma}\right)^2\int_{\Sigma_0} A^2(x)\mathrm{vol}_{\mathrm{Euc}} \qquad (2.41)\\
&= t_0^{\frac{2}{\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_{t_0}} - \frac{4}{3\gamma}\int_0^{t_0} s^{\frac{2}{\gamma}-1}\int_{\Sigma_s}|\overline{\nabla}\psi|^2\mathrm{vol}_{\Sigma_s}ds\\
&\geq \frac{1}{2}t_0^{\frac{4}{\gamma}}\|\partial_t\psi\|^2_{L^2(\Sigma_{t_0})} + \frac{1}{2}t_0^{\frac{4}{\gamma}-\frac{4}{3\gamma}}\sum_{i=1}^3\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})} \quad \text{(by (2.36))}\\
&\quad - \frac{8}{3\gamma}\int_0^{t_0} s^{\frac{4}{\gamma}-1-\frac{4}{3\gamma}}ds\sum_{i=1}^3\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}\\
&\quad - \frac{16}{3\gamma}\int_0^{t_0} s^{\frac{4}{\gamma}-1-\frac{4}{3\gamma}}\frac{\left(t_0^{1-\frac{2}{\gamma}} - s^{1-\frac{2}{\gamma}}\right)^2}{(1-\frac{2}{\gamma})^2}ds\sum_{i=1}^3 t_0^{\frac{2}{\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_{x_i}\psi]\mathrm{vol}_{\Sigma_{t_0}}
\end{aligned}$$

$$= \frac{1}{2}t_0^{\frac{4}{\gamma}}\|\partial_t\psi\|^2_{L^2(\Sigma_{t_0})} - \frac{1}{2}t_0^{\frac{8}{3\gamma}}\sum_{i=3}^{3}\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}$$

$$- \frac{t_0^{2-\frac{4}{3\gamma}}}{1-(\frac{2}{3\gamma})^2}\sum_{i=1}^{3}\left[t_0^{\frac{4}{\gamma}}\|\partial_t\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})} + t_0^{\frac{8}{3\gamma}}\sum_{j=1}^{3}\|\partial_{x_j}\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}\right].$$

Now is evident now that if the assumptions of Theorem 1.2 for FLRW are satisfied, then $\|A(x)\|_{L^2(\mathbb{T}^3)} > 0$.

To prove Theorem 1.4 for FLRW, we use the local energy identity (2.6) and plug in (2.7), (2.13), (2.16):

$$\begin{aligned} t^{\frac{2}{\gamma}}\int_{U_t} J_0^{e_0}[\psi]\mathrm{vol}_{U_t} &\qquad (2.42)\\ = \frac{1}{2}t_0^{\frac{4}{\gamma}}\|\partial_t\psi\|^2_{L^2(U_{t_0})} + \frac{1}{2}t_0^{\frac{4}{\gamma}-\frac{4}{3\gamma}}\sum_{i=1}^{3}\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})}\\ - \frac{4}{3\gamma}\int_t^{t_0} s^{\frac{4}{\gamma}-1-\frac{4}{3\gamma}}\sum_{i=1}^{3}\|\partial_{x_i}\psi\|^2_{L^2(U_s)}ds\\ - \sum_{l=1}^{3}\int_t^{t_0}\int_{\mathcal{N}_l^\pm\cap\overline{U}_s} s^{\frac{2}{\gamma}+\frac{4}{3\gamma}}\frac{1}{2}\big[|(e_0\pm e_l)\psi|^2\\ + |e_i\psi|^2 + |e_j\psi|^2\big]ds dx_i dx_j \quad (i<j; i,j\neq l). \end{aligned}$$

Since $t^{-\frac{2}{3\gamma}}dt = \pm dx_l$ along $\mathcal{N}_l^\pm$, it follows by integrating that the closure $\overline{U}_t$ of the neighborhood U_t is the cube I^3, where $I = \Big[-\frac{t^{1-\frac{2}{3\gamma}}}{1-\frac{2}{3\gamma}}, \delta + \frac{t^{1-\frac{2}{3\gamma}}}{1-\frac{2}{3\gamma}}\Big]$. We make use of the following one-dimensional Sobolev inequality[b]:

$$f^2(t,x_l) \leq \left(\delta + \frac{2}{1-\frac{2}{3\gamma}}t^{1-\frac{2}{3\gamma}}\right)^{-1}\int_I f^2(t,x_l)dx_l + \|f(t,x_l)\|^2_{H^1(I)}, \qquad (2.43)$$

for $f \in H^1(I), t \in (0,t_0]$.

First, we take the limit $t \to 0$ in (2.42), employing (2.40), and then we apply (2.36) to the integrand in the third line of (2.42) and (2.43) to the integral over $\mathcal{N}_l^\pm \cap \overline{U}_s$ in the last line of (2.42) to deduce the lower bound:

$$\begin{aligned} \frac{1}{2}\left(1-\frac{2}{\gamma}\right)^2\int_{U_0} A^2(x)\mathrm{vol}_{U_0} &\qquad (2.44)\\ \geq \frac{1}{2}t_0^{\frac{4}{\gamma}}\|\partial_t\psi\|^2_{L^2(U_{t_0})} + \frac{1}{2}t_0^{\frac{4}{\gamma}-\frac{4}{3\gamma}}\sum_{i=1}^{3}\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})} \end{aligned}$$

[b]Proof by fundamental theorem of calculus: $f^2(t,x_l) \leq \min_{x_l\in I} f^2(t,x_l) + \int_I 2|f||\partial_{x_l}f|dx_l \leq |I|^{-1}\|f\|^2_{L^2(I)} + \|f\|_{H^1(I)}$.

$$
\begin{aligned}
&-\frac{8}{3\gamma}\int_0^{t_0} s^{\frac{4}{\gamma}-1-\frac{4}{3\gamma}}ds\sum_{i=1}^3\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})}\\
&-\frac{16}{3\gamma}\int_0^{t_0} s^{\frac{4}{\gamma}-1-\frac{4}{3\gamma}}\frac{\left(t_0^{1-\frac{2}{\gamma}}-s^{1-\frac{2}{\gamma}}\right)^2}{(1-\frac{2}{\gamma})^2}ds\sum_{i=1}^3 t_0^{\frac{2}{\gamma}}\int_{U_{t_0}} J_0^{e_0}[\partial_{x_i}\psi]\mathrm{vol}_{U_{t_0}}\\
&-\int_0^{t_0}\left(1+\frac{1}{\delta+\frac{2}{1-\frac{2}{3\gamma}}s^{1-\frac{2}{3\gamma}}}\right)s^{-\frac{2}{3\gamma}}\int_{U_s} s^{\frac{4}{\gamma}}12J_0^{e_0}[\psi]\mathrm{vol}_{\mathrm{Euc}}ds\\
&-\int_0^{t_0} s^{-\frac{2}{3\gamma}}\int_{U_s}4s^{\frac{4}{\gamma}}\sum_{l=1}^3 J_0^{e_0}[\partial_{x_l}\psi]\mathrm{vol}_{\mathrm{Euc}}\\
\geq{}&\frac{1}{2}t_0^{\frac{4}{\gamma}}\|\partial_t\psi\|^2_{L^2(U_{t_0})}-\frac{1}{2}t_0^{\frac{8}{3\gamma}}\sum_{i=1}^3\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})}\\
&-\frac{t_0^{2-\frac{4}{3\gamma}}}{1-(\frac{2}{3\gamma})^2}\sum_{i=1}^3\left[t_0^{\frac{4}{\gamma}}\|\partial_t\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}+t_0^{\frac{8}{3\gamma}}\sum_{j=1}^3\|\partial_{x_j}\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}\right]\\
&-\int_0^{t_0}\left(1+\frac{1}{\delta+\frac{2}{1-\frac{2}{3\gamma}}s^{1-\frac{2}{3\gamma}}}\right)s^{-\frac{2}{3\gamma}}ds\\
&\times\int_{U_{t_0}}12t_0^{\frac{4}{\gamma}}J_0^{e_0}[\psi]\mathrm{vol}_{\mathrm{Euc}}\quad(\text{by (2.14) for }\{U_t\})\\
&-\int_0^{t_0} s^{-\frac{2}{3\gamma}}ds\int_{U_{t_0}}4t_0^{\frac{4}{\gamma}}\sum_{l=1}^3 J_0^{e_0}[\partial_{x_l}\psi]\mathrm{vol}_{Euc}\\
={}&\frac{1}{2}t_0^{\frac{4}{\gamma}}\|\partial_t\psi\|^2_{L^2(U_{t_0})}-\frac{1}{2}t_0^{\frac{8}{3\gamma}}\sum_{i=1}^3\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})}\\
&-\frac{t_0^{2-\frac{4}{3\gamma}}}{1-(\frac{2}{3\gamma})^2}\sum_{i=1}^3\left[t_0^{\frac{4}{\gamma}}\|\partial_t\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}+t_0^{\frac{8}{3\gamma}}\sum_{j=1}^3\|\partial_{x_j}\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}\right]\\
&-\frac{t_0^{1-\frac{2}{3\gamma}}}{1-\frac{2}{3\gamma}}\Bigg[6t_0^{\frac{4}{\gamma}}\|\partial_t\psi\|^2_{L^2(U_{t_0})}+6t_0^{\frac{8}{3\gamma}}\sum_{i=1}^3\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})}\\
&+2t_0^{\frac{4}{\gamma}}\sum_{l=1}^3\|\partial_t\partial_{x_l}\psi\|^2_{L^2(U_{t_0})}+2t_0^{\frac{8}{3\gamma}}\sum_{i,l=1}^3\|\partial_{x_i}\partial_{x_l}\psi\|^2_{L^2(U_{t_0})}\Bigg]\\
&-3\log\left(1+\frac{2}{1-\frac{2}{3\gamma}}\frac{t_0^{1-\frac{2}{3\gamma}}}{\delta}\right)\left[t_0^{\frac{4}{\gamma}}\|\partial_t\psi\|^2_{L^2(U_{t_0})}+t_0^{\frac{8}{3\gamma}}\sum_{i=1}^3\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})}\right].
\end{aligned}
$$

Thus, if the assumptions of Theorem 1.4 for FLRW are satisfied, then the right-hand side of (2.44) gives $\|A(x)\|_{L^2(U_0)} > 0$. This completes the proofs of the main theorems for FLRW.

2.3. *Kasner*

For Kasner the adapted orthonormal frame to the constant t hypersurfaces reads

$$e_0 = -\partial_t, \quad e_i = t^{-p_i}\partial_{x_i}. \tag{2.45}$$

In this frame, the non-zero components of the second fundamental form K of the Σ_t hypersurfaces are

$$K_{ii} := g(\nabla_{e_i} e_0, e_i) = -\frac{p_i}{t}, \quad i = 1, 2, 3. \tag{2.46}$$

Further, the intrinsic volume form vol_{Σ_t} on Σ_t equals

$$\mathrm{vol}_{\Sigma_t} = t\mathrm{vol}_{\mathrm{Euc}}. \tag{2.47}$$

Proposition 2.7 (Upper bound). *Let ψ be a smooth solution to the wave equation, $\Box_g \psi = 0$, in Kasner. Then the following energy inequality holds*:

$$t\int_{\Sigma_t} J_0^{e_0}[\partial_x^\alpha \psi]\mathrm{vol}_{\Sigma_t} \leq t_0 \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha \psi]\mathrm{vol}_{\Sigma_{t_0}} \tag{2.48}$$

for all $t \in (0, t_0]$ and any multi-index α. Moreover, $\partial_x^\beta \psi$ satisfies the pointwise bound

$$|\partial_x^\beta \psi(t,x)| \leq C \sum_{|\alpha|\leq 2} \left(t_0^2 \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha \partial_x^\beta \psi]\mathrm{vol}_{\Sigma_{t_0}} \right)^{\frac{1}{2}} \log\frac{t_0}{t} + |\partial_x^\beta \psi(t_0, x)|, \tag{2.49}$$

for any multi-index β, where C is a constant independent of t_0, p_i.

Proof. We compute the divergence of the current $J_a^{e_0}[\psi]$:

$$\begin{aligned}
\nabla^a J_a^{te_0}[\psi] &\overset{(2.5)}{=} \nabla^a (te_0)^b T_{ab}[\psi] = tK^{ab}T_{ab}[\psi] - (e_0 t)T_{00}[\psi] \\
&= \sum_{i=1}^3 K_{ii} t (e_i\psi)^2 - \frac{1}{2}K^i{}_i t|\nabla\psi|^2 + \frac{1}{2}\left[(e_0\psi)^2 + |\overline{\nabla}\psi|^2\right], \\
&\overset{(2.46)}{=} \sum_{i=1}^3 (1 - p_i)(e_i\psi)^2.
\end{aligned} \tag{2.50}$$

Hence, by (2.8), for $X = te_0$, we have

$$\begin{aligned}
t\int_{\Sigma_t} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_t} &= t_0 \int_{\Sigma_{t_0}} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_{t_0}} + \int_t^{t_0}\int_{\Sigma_s} \sum_{i=1}^3 (p_i - 1)(e_i\psi)^2)\mathrm{vol}_{\Sigma_s} ds \\
&\leq t_0 \int_{\Sigma_{t_0}} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_{t_0}}. \quad (p_i \leq 1)
\end{aligned} \tag{2.51}$$

Note that the same inequality holds for $\partial_x^\beta \psi$ by commuting the wave equation with ∂_x^β. In particular, taking into account the volume form, we control

$$\begin{aligned} t^2 \int_{\Sigma_t} (\partial_t \psi)^2 \mathrm{vol}_{\mathrm{Euc}} &\le 2t_0 \int_{\Sigma_{t_0}} J_0^{e_0}[\psi] \mathrm{vol}_{\Sigma_{t_0}}, t^2 \|\partial_t \partial_x^\beta \psi\|_{H^2}^2 \\ &\le \sum_{|\alpha|\le 2} 2t_0 \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha \partial_x^\beta \psi] \mathrm{vol}_{\Sigma_{t_0}}, \end{aligned} \tag{2.52}$$

for all $t \in (0, t_0]$.

Using the fundamental theorem of calculus along e_0 and Sobolev embedding $H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$, we then derive

$$\begin{aligned} |\partial_x^\beta \psi(t,x)| &= |\int_{t_0}^t \partial_s \partial_x^\beta \psi ds + \psi(t_0, x)| \\ &\le \int_t^{t_0} C \|\partial_s \partial_x^\beta \psi\|_{H^2(\mathrm{vol}_{\mathrm{Euc}})} ds + |\partial_x^\beta \psi(t_0, x)| \\ &\le C \sum_{|\alpha|\le 2} \left(t_0^2 \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha \partial_x^\beta \psi] \mathrm{vol}_{\mathrm{Euc}} \right)^{\frac{1}{2}} \log \frac{t_0}{t} \\ &\quad + |\partial_x^\beta \psi(t_0, x)| \quad \text{(by (2.52))}, \end{aligned} \tag{2.53}$$

for all $t \in (0, t_0]$. ☐

Instead of deriving renormalized energy estimates, as in Sec. 2.2 for FLRW (see Proposition 2.3), we prove the validity of the expansion (1.4) by using (2.49) to view the wave equation as an inhomogeneous ODE in t. This procedure is more wasteful in the number of derivatives of ψ that we need to bound from initial data, but it is slightly simpler.

Proof of Theorem 1.1 for Kasner. We express the wave equation for ψ in terms of the (t, x_1, x_2, x_3) coordinate system and treat the spatial derivatives of ψ as error terms:

$$-\partial_t^2 \psi - \frac{1}{t}\partial_t \psi + \sum_{i=1}^3 t^{-2p_i} \partial_{x_i}^2 \psi = 0 \;\Rightarrow\; \partial_t(t\partial_t \psi) = \sum_{i=1}^3 t^{1-2p_i} \partial_{x_i}^2 \psi. \tag{2.54}$$

Integrating in $[t, t_0]$ we obtain the formula

$$t\partial_t \psi = t_0 \partial_t \psi_0 - \int_t^{t_0} \sum_{i=1}^3 s^{1-2p_i} \partial_{x_i}^2 \psi ds$$

$$\psi(t,x) = \psi(t_0, x) + t_0 \partial_t \psi_0 \log \frac{t}{t_0} + \int_t^{t_0} \frac{1}{s} \int_s^{t_0} \sum_{i=1}^3 \bar{s}^{1-2p_i} \partial_{x_i}^2 \psi d\bar{s} ds, \tag{2.55}$$

$$= \psi(t_0,x) + \left(t_0\partial_t\psi_0 + \int_0^{t_0}\sum_{i=1}^3 s^{1-2p_i}\partial^2_{x_i}\psi ds\right)\log\frac{t}{t_0},$$

$$+ \int_t^{t_0}\frac{1}{s}\int_0^s\sum_{i=1}^3 \overline{s}^{1-2p_i}\partial^2_{x_i}\psi d\overline{s}ds$$

$$= A(x)\log t + u(t,x),$$

where

$$A(x) = t_0\partial_t\psi_0 + \int_0^{t_0}\sum_{i=1}^3 s^{1-2p_i}\partial^2_{x_i}\psi ds, \tag{2.56}$$

$$u(t,x) = \psi(t_0,x) - \left(t_0\partial_t\psi_0 + \int_0^{t_0}\sum_{i=1}^3 s^{1-2p_i}\partial^2_{x_i}\psi ds\right)\log t_0 \tag{2.57}$$

$$+ \int_t^{t_0}\frac{1}{s}\int_0^s\sum_{i=1}^3 \overline{s}^{1-2p_i}\partial^2_{x_i}\psi d\overline{s}ds.$$

Note that since by the assumption $1-2p_i > -1$ and by (2.49) $\|\partial^\beta_x\psi\|_{L^\infty} \leq C|\log t|$, $t\in(0,t_0]$, the functions $s^{1-2p_i}\partial^2_{x_i}\psi$, $\frac{1}{s}\int_0^s\sum_{i=1}^3\overline{s}^{1-2p_i}\partial^2_{x_i}\psi d\overline{s}$ are integrable[c] in $[0,t_0]$ and hence the above formulas make sense. Moreover, it is implied by (2.56), (2.57) that $A(x), u(t,x)$ are smooth functions and $u = u_{\text{Kasner}}$ and its spatial derivatives are in fact uniformly bounded up to $t=0$:

$$\|\partial^\alpha_x u\|_{L^\infty(\Sigma_t)} \leq C, \tag{2.58}$$

for all $t\in(0,t_0]$, any multi-index α, where $C>0$ is a constant depending on initial data. □

Next, we prove the blow up result stated in Theorem 1.2 for Kasner. Notice that since $p_i < 1$, the main contribution of the energy flux generated by the current $J^{e_0}[\psi]$, as $t\to 0$, comes from the $\partial_t\psi$ term:

$$t\int_{\Sigma_t} J^{e_0}[\psi]\mathrm{vol}_{\Sigma_s} = \frac{1}{2}\int_{\Sigma_t} t^2(\partial_t\psi)^2 + \sum_{i=1}^3 t^{2-2p_i}(\partial_{x_i}\psi)^2\mathrm{vol}_{\text{Euc}}, \tag{2.59}$$

$$= \frac{1}{2}\int_{\Sigma_t} t^2(\partial_t\psi)^2\mathrm{vol}_{\text{Euc}} + \sum_{i=1}^3 t^{2-2p_i}O(|\log t|^2).$$

Utilizing (1.4), (2.58) it follows that

$$\lim_{t\to 0} t\int_{\Sigma_t} J^{e_0}[\psi]\mathrm{vol}_{\Sigma_s} = \frac{1}{2}\lim_{t\to 0}\int_{\Sigma_t} t^2(\partial_t\psi)^2\mathrm{vol}_{\text{Euc}} = \frac{1}{2}\int_{\Sigma_0} A^2(x)\mathrm{vol}_{\text{Euc}}. \tag{2.60}$$

[c]By Proposition 2.7, their $L^1([0,t_0])$ norm is bounded by initial data.

Thus, returning to (2.51) and taking the limit $t \to 0$ we obtain the identity:

$$\begin{aligned}\int_{\Sigma_0} A^2(x)\mathrm{vol}_{\mathrm{Euc}} &= t_0 \int_{\Sigma_{t_0}} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_{t_0}} + \int_0^{t_0}\int_{\Sigma_s}\sum_{i=1}^3 (p_i-1)(e_i\psi)^2\mathrm{vol}_{\Sigma_s}ds,\\ &= \frac{1}{2}t_0^2\|\partial_t\psi\|^2_{L^2(\Sigma_{t_0})} + \frac{1}{2}\sum_{i=1}^3 t_0^{2-2p_i}\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}\\ &\quad + \int_0^{t_0}\sum_{i=1}^3 (p_i-1)s^{1-2p_i}\int_{\Sigma_s}(\partial_{x_i}\psi)^2\mathrm{vol}_{\mathrm{Euc}}ds. \end{aligned} \tag{2.61}$$

We bound the L^2 norm of $\partial_{x_i}\psi$ as follows.

Lemma 2.8. *The following estimate for the L^2 norm of $\partial_{x_i}\psi$ holds*:

$$\|\partial_{x_i}\psi\|_{L^2(\Sigma_t)} \le \|\partial_{x_i}\psi\|_{L^2(\Sigma_{t_0})} + \left(2t_0\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_{x_i}\psi]\mathrm{vol}_{\Sigma_{t_0}}\right)^{\frac{1}{2}}\log\frac{t_0}{t}, \tag{2.62}$$

for all $t \in (0, t_0]$.

Proof. We have

$$\begin{aligned} -\frac{1}{2}\partial_t\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_t)} &\overset{C-S}{\le} \|\partial_{x_i}\psi\|_{L^2(\Sigma_t)}\|\partial_t\partial_{x_i}\psi\|_{L^2(\Sigma_t)}\\ -\partial_t\|\partial_{x_i}\psi\|_{L^2(\Sigma_t)} &\le \|\partial_t\partial_{x_i}\psi\|_{L^2(\Sigma_t)}\\ \|\partial_{x_i}\psi\|_{L^2(\Sigma_t)} &\le \|\partial_{x_i}\psi\|_{L^2(\Sigma_{t_0})} + \int_t^{t_0}\|\partial_s\partial_{x_i}\psi\|_{L^2(\Sigma_s)}ds\\ \|\partial_{x_i}\psi\|_{L^2(\Sigma_t)} &\overset{(2.52)}{\le} \|\partial_{x_i}\psi\|_{L^2(\Sigma_{t_0})} + \left(2t_0\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_{x_i}\psi]\mathrm{vol}_{\Sigma_{t_0}}\right)^{\frac{1}{2}}\log\frac{t_0}{t}, \end{aligned} \tag{2.63}$$

where in the last inequality we made use of (2.52). □

Applying (2.63) to (2.61), we derive

$$\begin{aligned}&\int_{\Sigma_0} A^2(x)\mathrm{vol}_{\mathrm{Euc}} \\ &\quad\ge \frac{1}{2}t_0^2\|\partial_t\psi\|^2_{L^2(\Sigma_{t_0})} + \frac{1}{2}\sum_{i=1}^3 t_0^{2-2p_i}\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}\\ &\qquad + \sum_{i=1}^3\int_0^{t_0}(p_i-1)s^{1-2p_i}ds(2\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}) \end{aligned} \tag{2.64}$$

$$+\sum_{i=1}^{3}\int_0^{t_0}(p_i-1)s^{1-2p_i}\left|\log\frac{s}{t_0}\right|^2 ds\left(2t_0^2\|\partial_t\partial_{x_i}\psi\|_{L^2(\Sigma_{t_0})}\right.$$
$$\left.+2\sum_{j=1}^{3}t_0^{2-2p_i}\|\partial_{x_j}\partial_{x_i}\psi\|_{L^2(\Sigma_{t_0})}\right)$$
$$\geq\frac{1}{2}t_0^2\|\partial_t\psi\|^2_{L^2(\Sigma_{t_0})}-\frac{1}{2}\sum_{i=1}^{3}t_0^{2-2p_i}\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}$$
$$-\sum_{i=1}^{3}\frac{t_0^{2-2p_i}}{(1-p_i)^2}\left[t_0^2\|\partial_t\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}+\sum_{j=1}^{3}t_0^{2-2p_i}\|\partial_{x_j}\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}\right].$$

If the assumptions of Theorem 1.2 for Kasner are satisfied, then it is clear from the preceding lower bound that $\|A(x)\|_{L^2(\mathbb{T}^3)}>0$.

To prove the local version of the blow up criterion in Theorem 1.4, we argue similarly, but also take into account the contribution of the null flux terms in (2.8). Note that the upper bounds (2.48), (2.49), (2.63) are also valid for the integral domains U_t, U_0 in place of Σ_t, Σ_{t_0}, since the null flux terms in (2.8) have a favorable sign for an upper bound. Hence, taking the limit $t\to 0$ in (2.6) and employing (2.7), (2.50), (1.4), (2.58) we obtain

$$\int_{U_0}A^2(x)\mathrm{vol}_{\mathrm{Euc}} \tag{2.65}$$
$$\geq\frac{1}{2}t_0^2\|\partial_t\psi\|^2_{L^2(U_{t_0})}+\frac{1}{2}\sum_{i=1}^{3}t_0^{2-2p_i}\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})}$$
$$+\int_0^{t_0}\sum_{i=1}^{3}(p_i-1)s^{1-2p_i}\int_{U_s}(\partial_{x_i}\psi)^2\mathrm{vol}_{\mathrm{Euc}}ds$$
$$-\sum_{l=1}^{3}\int_t^{t_0}\int_{\mathcal{N}_l^{\pm}\cap\overline{U}_s}s^{2-p_l}\frac{1}{2}[|(e_0\pm e_l)\psi|^2+|e_i\psi|^2$$
$$+|e_j\psi|^2]dsdx_idx_j\quad(i<j;i,j\neq l).$$

Since $t^{-p_l}dt=\pm dx_l$ along $\mathcal{N}_l^{\pm}$, it follows by integrating that the closure $\overline{U}_t$ of the neighborhood of U_t is the product $I_1\times I_2\times I_3$, where $I_i=[-\frac{t^{1-p_i}}{1-p_i},\delta+\frac{t^{1-p_i}}{1-p_i}]$. The analogous inequality to (2.43) then reads

$$f^2(t,x_l)\leq\left(\delta+\frac{2}{1-p_l}t^{1-p_l}\right)^{-1}\int_{I_l}f^2(t,x_l)dx_l+\|f(t,x_l)\|^2_{H^1(I_l)}, \tag{2.66}$$
$$\text{with } f\in H^1(I_l),\ \ t\in(0,t_0].$$

Applying the latter bound to the integral over $\mathcal{N}_l^\pm \cap \overline{U}_s$ on the right-hand side of (2.65), along with (2.63), we derive

$$\begin{aligned}
&\int_{U_0} A^2(x)\mathrm{vol}_{\mathrm{Euc}} \\
&\quad \geq \frac{1}{2}t_0^2\|\partial_t\psi\|^2_{L^2(U_{t_0})} + \frac{1}{2}\sum_{i=1}^3 t_0^{2-2p_i}\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})} \\
&\qquad + \sum_{i=1}^3\int_0^{t_0}(p_i-1)s^{1-2p_i}ds(2\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})}) \\
&\qquad + \sum_{i=1}^3\int_0^{t_0}(p_i-1)s^{1-2p_i}\left|\log\frac{s}{t_0}\right|^2 ds\Bigg(2t_0^2\|\partial_t\partial_{x_i}\psi\|_{L^2(U_{t_0})} \\
&\qquad + 2\sum_{j=1}^3 t_0^{2-2p_i}\|\partial_{x_j}\partial_{x_i}\psi\|_{L^2(U_{t_0})}\Bigg) \\
&\qquad - \sum_{l=1}^3\Bigg[\int_t^{t_0}\left(1+\frac{1}{\delta+\frac{2}{1-p_l}s^{1-p_l}}\right)s^{-p_l}\int_{U_s}4sJ_0^{e_0}[\psi]\mathrm{vol}_{U_s}ds \\
&\qquad + \int_t^{t_0}s^{-p_l}\int_{U_s}4sJ_0^{e_0}[\partial_{x_l}\psi]\mathrm{vol}_{U_s}ds\Bigg] \\
&\quad \geq \frac{1}{2}t_0^2\|\partial_t\psi\|^2_{L^2(U_{t_0})} - \frac{1}{2}\sum_{i=1}^3 t_0^{2-2p_i}\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})} \\
&\qquad - \sum_{i=1}^3\frac{t_0^{2-2p_i}}{(1-p_i)^2}\left[t_0^2\|\partial_t\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})} + \sum_{j=1}^3 t_0^{2-2p_i}\|\partial_j\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}\right] \\
&\qquad - \sum_{l=1}^3\frac{2t_0^{1-p_l}}{1-p_l}\Bigg[t_0^2\|\partial_t\psi\|^2_{L^2(U_{t_0})} + \sum_{i=1}^3 t_0^{2-2p_i}\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})} \\
&\qquad + t_0^2\sum_{l=1}^3\|\partial_t\partial_{x_l}\psi\|^2_{L^2(U_{t_0})} + \sum_{i=1}^3 t_0^{2-2p_i}\|\partial_{x_i}\partial_{x_l}\psi\|^2_{L^2(U_{t_0})}\Bigg] \\
&\qquad - \sum_{l=1}^3\log\left(1+\frac{2}{1-p_l}\frac{t_0^{1-p_l}}{\delta}\right) \\
&\qquad \times\left[t_0^2\|\partial_t\psi\|^2_{L^2(U_{t_0})} + t_0^{2-2p_l}\sum_{i=1}^3\|\partial_{x_i}\psi\|^2_{L^2(U_{t_0})}\right].
\end{aligned}$$

Thus, given the assumptions in Theorem 1.4 for Kasner, it follows that $\|A(x)\|_{L^2(U_0)} > 0$, as required.

Acknowledgments

The authors would like to thank Mihalis Dafermos for useful interactions. Further, A. A. and A. F. benefited from discussions with José Natário. This work was partially supported by FCT/Portugal through UID/MAT/04459/2013, grant (GPSEinstein) PTDC/MAT-ANA/1275/2014 and through the FCT fellowships SFRH/BPD/115959/2016 (A. F.) and SFRH/BPD/85194/2012 (A. A.). G. F. was supported by the EPSRC Grant EP/K00865X/1 on "Singularities of Geometric Partial Differential Equations".

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