

D-modules on the Fargues-Fontaine curve



D-modules on The Fargues-Fontaine curve

(joint w/ Anschütz, Le Bras, Rodriguez Camargo)

Ultimate goal Express pro-étale cohomology with $\mathbb{Q}_p$ -coefficients of rigid spaces, over $\mathbb{Q}_p$ or $\mathbb{F}_p$, in terms of de Rham data.

Plan § Background and motivation

§ de Rham-Fargues-Fontaine stacks

§ Applications

§ Background and motivation

$K/\mathbb{Q}_p$ discretely valued with perfect res. field κ , $F := W(\kappa)(\frac{1}{p})$

$\bar{K}$ fixed alg. closure, $C := \hat{\bar{K}}$, $\gamma_K := \text{gal}(\bar{K}/K)$.

Thm (Scholze, Gmez-Niziol, Bhargava-Morrow-Scholze, Česnaičis-Koshikawa)

Let $X/\mathcal{O}_K$ proper p -adic f. scheme with semistable red.

For $i \geq 0$, there is a natural iso. of filtered (φ, N, γ_K) -mod.

$$H_{\text{proet}}^i(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{st}} \cong H_{\text{HK}}^i(X_K) \otimes_F B_{\text{st}}.$$

Here, $H_{\text{HK}}^i(X_K)$ denotes the Hyodo-Kato cohomology
 $(= H_{\text{dR}}^i(X_K/W(K))[\frac{1}{p}])$.

It is endowed with Frobenius φ , and monodromy action N
 (coming from singularities of X_K) satisfying
 $N\varphi = p\varphi N$.

Equivalently, for $i \geq 0$, there is G_K -equiv. pullback square

$$H_{\text{proet}}^i(X_C, \mathbb{Q}_p) \longrightarrow (H_{\text{HK}}^i(X_K) \otimes_F B_{\text{st}})^{N=0, \varphi=1}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \text{Fil}^0(H_{\text{dR}}^i(X_K) \otimes_K B_{\text{dR}}) & \longrightarrow & H_{\text{dR}}^i(X_K) \otimes_K B_{\text{dR}} \end{array}$$

Remark 1) Alternative proof in good reduction case was given by
 Artin - Mather - Morrow - Nikolas via
 "Beilinson fiber square".

2) generalization in good reduction case to crystalline $\mathcal{O}_p$ -ad systems with integral lattice by Grothendieck.

Q What about rigid spaces not having semistable formal model, or not proper?

(local Shimura varieties are interesting examples of rigid spaces which are not proper and in general do not admit a semistable formal model).

Thm (Grothendieck - (Deligne -) Illusie, B.)

Let X/K qcqs smooth rigid space. We have a G_K -equivariant pullback square in $\mathcal{D}(\text{Mod}_{\mathcal{O}_p}^{\text{solid}})$

$$R\Gamma_{\text{proét}}(X_c, \mathcal{O}_p) \longrightarrow (R\Gamma_{\text{HK}}(X_c) \otimes_{F^{\text{ur}}} B_{\text{ST}})^{N=0, q=1}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \text{Fil}^0(R\Gamma_{\text{proét}}(X) \otimes_K^{\mathbb{L}} B_{\text{dR}}) & \longrightarrow & R\Gamma_{\text{proét}}(X) \otimes_K^{\mathbb{L}} B_{\text{dR}} \end{array}$$

Rmk There is a generalization to any rigid space X/K (possibly singular) covered by countable # of affinoids + qcqs.

Here, for Z/C smooth rigid space,

$$R\Gamma_{HK}(Z)$$

denotes the Hyab-Kato ch. def. by Gmiz-Niziot,
(following a construction of Beilinson): it is def. étale
locally, via alterations of Hartl and Temkin, as the
Hyab-Kato ch. of red. mod. p of a semistable t -model.
It is refinement of de Rham ch.: it is endowed
with (φ, N) -mod. structure, and, for Z/C qcqs,

$$R\Gamma_{HK}(Z) \otimes_{F^{nr}}^L C \simeq R\Gamma_{dR}(Z).$$

Using that Hyab-Kato is easier to compute than pro-étale,
one can show for example the following Artin vanishing:

Cor X/C smooth Stein space, $H_{\text{pro-ét}}^i(X, \mathbb{Q}_p) = 0$ for $i > \dim X$.

Hint (analogous with ℓ -adic pro-étale cohomology) this fails w/ $\mathbb{Z}_p$ -coeff.!

For X/K qcqs rigid space, $\ell \neq p$ prime,

$$H_{\text{pro-ét}}^i(X_c, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} K$$

is finite dim. $\mathbb{Q}_\ell$ -v.s.. By ℓ -adic monodromy theorem,

it has (φ, N) -module structure, for any geom. Frob $\varphi \in G_K$.

This ch. has some formal properties as Hyodo-Kato ch. for rigid spaces (e.g. homology invariance w.r.t. open disc) except for the fact that $H_{\text{HK}}(X_c)$ is infinite dim even in the qcqs case: one remedies to this passing to dagger spaces ("rigid spaces with overconvergent structure sheaf")

Remark 1) Colmez-Niziol proof is local to global: it uses log-syntomic ch. in semistable case + alterations.
2) Alternative proof uses a globally def. ch. for rigid spaces Z/C , interpolating between étale and de Rham data: it is (a filtered enhancement of) Fargues-Fontaine ch.

$H_{\text{FF}}(Z)$ (introduced by Le Bras).

Recall for $S = \text{Spa}(R, R^+)$ aff'd perf'd over $\mathbb{F}_p$, the relative FF curve attached to S is def. as: $R \otimes R^+$ present.

$$\text{FF}_S := \mathcal{Y}_{\text{FF}, S} / \varphi^4, \quad \mathcal{Y}_{\text{FF}, S} := \text{Spa}(W(R^+)) \setminus V(p[R^+])$$

Via descent, for Z/C rigid space we def. a diamond $\mathcal{Y}_{\text{FF}, Z^0}$.

For Z/C smooth aff'd $\mathcal{H}_{FF}(Z) \in \mathcal{A}b\mathcal{H}(FF_c^b)$ is the solid quasi-coherent complex on FF_c^b associated to

$$\varphi \leadsto \bigwedge^{\text{top}} R\Gamma(Y_{FF, Z^0}, \mathcal{O}) \in \mathcal{A}b\mathcal{H}(Y_{FF, c^0})$$

décalage operator for $T = \text{"Fontaine } 2\pi i \text{"}$ (Killing jnk T -torsion)

For Z/C smooth qcqs, there is a natural isom.

$$\mathcal{H}_{FF}(Z) \simeq \mathcal{E}(R\Gamma_{\text{HK}}(Z)).$$

quasi-coh. complex
attached to (φ, N) -module

Q 1) Is there a def. of Hyak-Kato ch. avoiding log-geometry / alterations?

Can it be extended to coefficients?

2) Is there a generalization of previous thm. to coeff?

What about a "categorification" of such thm?

($\approx$ Riemann-Hilbert for $\mathcal{O}_p$ -coefficients, cf. Bhatt-Lurie)

Link (Motivic approach)

1) Binda - Galbraith - Vezzani found def. of Hyak-Kato ch. for smooth dagger spaces, avoiding log-geometry.

2) Before that, Le Bras - Vezzoni proposed a def. of
 Forges-Fontaine ch. $A_{FF}(Z)$, for smooth dagger spaces
 Z/C , conceptually as "de Rham ch over FF_C "

$$= \int FF_Z^0 / FF_C^0$$

(Comparison with previous def. of Le Bras still not in the literature)

§ De Rham-Forge-Fontaine stacks

Recall, given a Zariski stack Y cat. of finite type $\mathbb{Q}$ -algebras
 $Y : \text{Ring}_{\mathbb{Q}}^{\text{fin}} \xrightarrow{\quad} \text{Ani} \xleftarrow{\quad} \text{anima} \quad (\text{e.g. } Y/\mathbb{Q}\text{-variety})$

Simpson defined the de Rham prestack as the presheaf

$$Y^{dR} : \text{Ring}_{\mathbb{Q}}^{\text{fin}} \rightarrow \text{Ani}$$

$$R \mapsto Y(R^{\text{red}}), \quad R^{\text{red}} := R/\text{Nil}(R).$$

If $Y/\mathbb{Q}$ is variety, Y^{dR} is a stack for étale top, and
 (in the case Y smooth)

$$(1) \quad \text{Bun}(Y^{dR}) \simeq \{ \text{v.b. with integrable connection } \nabla \text{ on } Y \}$$

$$(\mathcal{E}, \nabla)^{\text{dr}} \leftrightarrow (\mathcal{E}, \nabla)$$

$$(2) \quad R\Gamma(Y^{\text{dr}}, (\mathcal{E}, \nabla)^{\text{dr}}) \simeq R\Gamma_{\text{dr}}(Y, (\mathcal{E}, \nabla)).$$

Recently, Rodriguez Camargo introduced a similar stack in rigid geometry, in the framework of Chacón-Scholze analytic geometry: given a sheaf for analytic top.

any function on such obj. is bdd. $\rightarrow$ $Y: \text{Aff Ring}_{\mathbb{Q}_p}^{\text{bdd}} \rightarrow \text{Ani}$ (e.g. $Y/\mathbb{Q}_p$ analytic adic space)
 category of bounded affinoid rings over $\mathbb{Q}_p$
 (generalization of Tate Huber pairs)

The analytic de Rham presheaf is the presheaf

$$Y^{\text{dr}}: \text{Aff Ring}_{\mathbb{Q}_p}^{\text{bdd}} \rightarrow \text{Ani}$$

$$R \mapsto Y(R^{\text{t-red}}), \quad R^{\text{t-red}} := R/\text{Nil}^+(R)$$

where $\text{Nil}^+(R) = \{f \in R \mid |f| = 0\}$ "spectral norm"

The analytic de Rham stack Y^{dr} is sheafification of this,

w.r.t. to !-descent top def. by Chacón-Scholze. $\leftarrow$ forced by ϕ -units

Def. Let $X/\mathbb{Q}_p$ rigid space, we def. the stack

$$X^{\mathrm{dR}, \mathrm{FF}} := (Y_{\mathrm{FF}, X^0})^{\mathrm{dR}} / \varphi^{\mathbb{Z}}.$$

(Δ) here, we use $\mathrm{Perf}_{\mathbb{F}_p} \ni S \mapsto (Y_{\mathrm{FF}, S})^{\mathrm{dR}}$ has enough descent properties!

Denoting $f: X \rightarrow \mathrm{Spa}(\mathbb{Q}_p)$ structure morph., we def.

$$\mathcal{H}_{\mathrm{dR}, \mathrm{FF}}(X) := Rf_{\star}^{\mathrm{dR}, \mathrm{FF}}(\mathcal{O}) \in \mathrm{Qch}(\mathbb{Q}_p^{\mathrm{dR}, \mathrm{FF}}).$$

Main Thm (in progress, A-B-LB-RC).

$$(1) \mathrm{Perf}(\mathbb{Q}_p^{\mathrm{dR}, \mathrm{FF}}) \simeq \mathcal{D}^b((\varphi, N, G_{\mathbb{Q}_p})\text{-Mod}_{\mathbb{Q}_p}^{\mathrm{fin}})$$

previous def extends to such spaces
↓

(2) Let $X/\mathbb{Q}_p$ (smooth) proper rigid space (or qcqs dagger space).

$$\mathcal{H}_{\mathrm{dR}, \mathrm{FF}}(X) \in \mathrm{Perf}(\mathbb{Q}_p^{\mathrm{dR}, \mathrm{FF}}).$$

Under equivalence in (1), there is a natural isom.

$$\mathcal{H}_{\mathrm{dR}, \mathrm{FF}}(X) \simeq R\Gamma_{\mathrm{HK}}(X_c) \leftarrow \varphi, N, G_{\mathbb{Q}_p}.$$

(3) For $X/\mathbb{Q}_p$ as in (2), pullback of $\mathcal{H}_{\mathrm{dR}, \mathrm{FF}}(X)$ along

(holds in geom.)
(case too)

$$\mathrm{FF}_c^0 \rightarrow Y_{\mathrm{FF}, \mathbb{Q}_p^0} / \varphi^{\mathbb{Z}} \rightarrow \mathbb{Q}_p^{\mathrm{dR}, \mathrm{FF}}$$

identifies with $\mathcal{H}_{\mathrm{dR}}(X_c)$. ($\leftarrow$ previously defined via décalage functor)

Proof sketch of (1) for Bn $\mathcal{O}_p^{qc} := \mathcal{O}_p / \mathcal{O}_p^{\infty}$, $\Gamma := \text{Gal}(\mathcal{O}_p^{qc} / \mathcal{O}_p)$.

(ψ, Γ) -mod. stable by action of $\partial := T^{-1} \text{Lie} \Gamma$.

$$\text{Bn}(\mathcal{O}_p^{\text{dR}, \text{FF}}) \simeq (\psi, \partial, \Gamma) - \text{Bn}^{\text{la}}(\mathcal{Y}_{\text{FF}}, \mathcal{O}_p^{qc, b})$$

loc. an. v.b.,
i.e. loc. free.
 $\mathcal{O}^{\text{la}}$ -modules

Synthesis
(Colmez, Chabrier-Kelly,
etc....) $\simeq (\psi, N, \text{Gal}_p) - \text{Mod}_{\mathcal{O}_p}^{\text{fin}}$

p -adic monodromy thm (Andr , Kedlaye, Mochizuki, Besser)

Remark Have repro of p -adic monodromy thm which shows $\square$
that any v.b. on $\mathcal{O}_p^{\text{dR}, \text{FF}}$ is "quasi-unipotent".

Remark Note that now one can redefine Hyodo-Kato chom.
very similarly to ℓ -adic pro- tble chom., as chom.
with values in (ψ, N) -mod: first one gives a def. purely
in terms of rigid geometry, then one uses the
 p -adic/ ℓ -adic monodromy thm.!

  Applications Recall the following:

Thm (Scholze) Let $X/\mathcal{O}_p$ smooth proper rigid space.
Let $\mathcal{H}$ be a de Rham $\mathcal{O}_p$ -local system on $X_{\text{pro t}}$,

having a $2p$ -loffice. One can functorially attach to $\mathbb{L}$
 a filtered v.b. with integrable connection $\text{Dol}(\mathbb{L})$ on X
 satisfying, for $i \geq 0$, a natural isomorphism

$$H_{\text{proet}}^i(X, \mathbb{L}) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H_{\text{dR}}^i(X, \text{Dol}(\mathbb{L})) \otimes_{\mathbb{Q}_p} B_{\text{dR}}$$

compatible with $G_{\mathbb{Q}_p}$ -action and filtrations.

Prmk $\mathbb{L} \text{ de Rham} \iff \mathbb{L}_{\bar{x}} \text{ de Rham}$ for all classical
 (Liu-Zhu)
 points $x \in X$
 (one point is sufficient if X is connected)

Goal Want to refine above Thm to a pst-comparison.

Recall Faltings-Fontaine defined notion of de Rham v.b.
 on $FFC^0/G_{\mathbb{Q}_p} = FF_{\mathbb{Q}_p}^0$:

$B_{\text{dR}}^{\text{dr}}(FF_{\mathbb{Q}_p}^0) \subset B_{\text{dR}}(FF_{\mathbb{Q}_p}^0)$ is the full subcat. spanned
 by v.b. $\mathcal{E}$ st. $W := \mathcal{E} \otimes_{\mathbb{Q}_p} \hat{\mathbb{A}}_{\text{proet}}(G_{\mathbb{Q}_p})$ is

generically flat, i.e.

$$\dim_{\text{Ban}} W(1/\tau) = \dim_{\mathcal{O}_p} W(1/\tau)^{\text{Gap}}.$$

Thm (in progress, A-B-LB-RC, Scholze, Liu-Zhu)

Let $X/\mathcal{O}_p$ smooth rigid space.

There is a natural equivalence of categories

$$\text{RH}: \text{Ban}^{\text{dR}}(\text{FF}_{X^0}) \xrightarrow{\sim} \text{Ban}(X^{\text{dR}, \text{FF}, +}).$$

Here, • $\text{Ban}^{\text{dR}}(\text{FF}_{X^0}) \subset \text{Ban}(\text{FF}_{X^0})$ is the full subcat.
spanned by v.b. which are de Rham at each
geom. point over a classical point of X .

• $X^{\text{dR}, \text{FF}, +}$ is the filtered dR-FF stack defined as the
pushout of analytic stacks

$$\begin{array}{ccc} X^{\text{dR}} & \longrightarrow & X^{\text{dR}, \text{FF}} \\ \downarrow & & \downarrow \\ X^{\text{dR}, +} & \longrightarrow & X^{\text{dR}, \text{FF}, +} \end{array}$$

where $X^{\text{dR},+}$ is the Hodge-filtered analytic de Rham stack;
 $\text{Bun}(X^{\text{dR},+}) \cong \left\{ \begin{array}{l} \text{filtered v.b. } \mathcal{E} \text{ with integrable connection } \nabla \\ \text{on } X, \text{ satisfying Griffiths transversality} \end{array} \right\}$
 (i.e. $\nabla(\text{Fil}^i(\mathcal{E})) \subset \Omega_X^1 \otimes \text{Fil}^{i-1}(\mathcal{E})$)

Remark There is a fully faithful functor

$$\begin{aligned} \text{Loc}_{\mathcal{O}_p}^{\text{dR}}(X_{\text{proét}}) &\hookrightarrow \text{Bun}^{\text{dR}}(\text{FF}_{X^{\circ}}) \\ \mathbb{1} &\mapsto "\mathbb{1} \otimes_{\mathcal{O}_p} \mathcal{O}_{\text{FF}_{X^{\circ}}}" \end{aligned}$$

which identifies the source with the v.b. in the Target which are semistable of slope 0 at each geometric point of X .

Remark (dR = pst) Combining the main thm. of the previous section and the thm above in the case $X = \text{Spa } \mathcal{O}_p$, we obtain

$$\text{Bun}^{\text{dR}}(\text{FF}_{\mathcal{O}_p^{\circ}}) \cong (\varphi, N, G_{\mathcal{O}_p})\text{-Fil Mod}_{\mathcal{O}_p}^{\text{fin}}$$

and then, by the previous remark,

$$\begin{aligned} \mathrm{Rep}^{\mathrm{dr}}_{\mathrm{op}}(G_{\mathrm{op}}) &\cong (\mathcal{U}, N, G_{\mathrm{op}})\text{-Fil Mod}_{\mathrm{op}}^{\mathrm{wo}} \leftarrow \text{weakly admissible} \\ &\cong \mathrm{Rep}^{\mathrm{pst}}_{\mathrm{op}}(G_{\mathrm{op}}) \quad (\text{weakly adm} = \text{adm},) \\ &\quad (\text{Cdmet-Fontaine}) \end{aligned}$$

Proof sketch of Thm By Beaville-Laszlo, we can

"glue" the following equivalences of categories:

$$1) \mathrm{Bn}(X^{\mathrm{dr}, \mathrm{FF}} \setminus X^{\mathrm{dr}}) \cong \mathrm{Bn}(\mathrm{FF}_X \setminus X)$$

2) (essentially Scholze, Liu-Zhu)

$$\mathrm{Bn}(X^{\mathrm{dr}, \mathrm{FF}}) \cong \left\{ \begin{array}{l} \mathrm{Bn}_{\mathrm{dR}, X}\text{-local systems which are} \\ \text{generically flat at each geom. point} \\ \text{over a classical point } x \in X \end{array} \right\}$$

Finally, we have the following:

Thm (A-B-LB-RC) Let $X/\mathcal{O}_p$ be a smooth proper rigid space. Denote $f: X \rightarrow f_0(\mathcal{O}_p)$ str. morphism.

1) For any $E \in \mathrm{Bn}(X^{\mathrm{dr}, \mathrm{FF}})$, and $i \geq 0$, we have

$$R\Gamma_{\star}^{\text{dR}, \text{FF}}(E) \in \text{Bun}(\mathbb{G}_p^{\text{dR}, \text{FF}}).$$

We denote

$$H_{\text{HK}}^i(X_c, E) := R\Gamma_{\star}^{\text{dR}, \text{FF}}(E) \in (\varphi, N, \mathbb{G}_p)\text{-Mod}_{\mathbb{G}_p}^{\text{fin}}$$

under equivalence of main thm.

2) Let $\mathcal{L}$ be a de Rham $\mathbb{G}_p$ -local system on $X_{\text{proét}}$ having a $\mathbb{Z}_p$ -lattice.

For $i \geq 0$, we have a natural isom. of filtered $(\varphi, N, \mathbb{G}_p)$ -mod.

$$H_{\text{proét}}^i(X_c, \mathcal{L}) \otimes_{\mathbb{Q}_p} B_{\text{st}} \cong H_{\text{HK}}^i(X_c, \text{RH}(\mathcal{L})) \otimes_{\mathbb{Q}_p} B_{\text{st}}.$$

Equivalently, for $i \geq 0$, we have a $\mathbb{G}_p$ -equivariant pullback square

$$H_{\text{proét}}^i(X_c, \mathcal{L}) \longrightarrow (H_{\text{HK}}^i(X_c, \text{RH}(\mathcal{L})) \otimes_{\mathbb{Q}_p} B_{\text{st}})^{N=0, \varphi=1}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \text{Fil}^0(H_{\text{dR}}^i(X, \text{RH}(\mathcal{L})) \otimes_{\mathbb{Q}_p} B_{\text{dR}}) & \longrightarrow & H_{\text{dR}}^i(X, \text{RH}(\mathcal{L})) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \end{array} \quad (*)$$

where, for $\mathcal{E} \in \text{Bun}(X^{\text{dR}, \text{FF}, +})$,

filtered finite $\mathbb{Q}_p$ -v.s.

$$H^i_{\text{dR}}(X, \mathcal{E}) := R^i \Gamma_*^{\text{dR}, +}(\mathcal{E}|_{X^{\text{dR}, +}}) \in \text{Bun}(\mathbb{Q}_p^{\text{dR}, +}).$$

Link As in the trivial coeff. case, there should be a derived variant of $(*)$, for any $X/\mathbb{Q}_p$ qcqs (smooth) dagger space (without even assuming the existence of a $2p$ -lattice for Π).

In this case, to define HK-chandag, one would need to show that for any $\mathcal{E} \in \text{Bun}(X^{\text{dR}, \text{FF}})$, we have

$$R \Gamma_*^{\text{dR}, \text{FF}}(\mathcal{E}) \in \text{Bif}(\mathbb{Q}_p^{\text{dR}, \text{FF}})$$

(in the case the dagger space $X/\mathbb{Q}_p$ has a smooth weak formal model, this finiteness is related to finiteness of rigid cohomology with coeff. proved by Kedlaya).