SEIBERG-WITTEN THEORY ON FOUR AND THREE-MANIFOLDS

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The aim of this lecture series is to provide an introduction to Seiberg-Witten theory in dimension four and three. We will discuss both the foundational aspects of the subjects and the its topological applications. Here is a tentative synopsis for the lectures.

Lecture 1 - An overview of four-dimensional topology. In the first part of the class, we will introduce of the intersection form of a four-manifold and explain how it plays a central role for topological classification. In the second part, we will discuss several examples of four-manifolds, focusing in particular on elliptic fibrations. We will mention the role of Seiberg-Witten invariants in the classification of such spaces. *References*: Chapters 1 and 3 of [GS99] (see also [Kir89] and [FS09]).

Lecture 2 - Hodge theory and Chern-Weil theory. In this lecture we will start to cover the differential-geometric background needed to write down the Seiberg-Witten equations. We begin by providing a geometric way to describe the invariants of four-manifolds we have encountered so far, via Hodge theory. After this, we quickly review connections of bundles, and then discuss how their curvature can be used to recover characteristic classes. *References*: Chapters 6 of [War83] and Chapters 1 and 2 of [Roe98] (see also [Tau11])

Lecture 3 - Clifford bundles and spinors. Clifford algebras naturally arise when trying to find square roots of Laplacian operators. After discussing the general theory, we focus on the specific (and very explicit) case of spinors on four-manifolds. We'll also talk about the index of elliptic operators, and the Atiyah-Singer index theorem. *References*: Chapter 3 of [Roe98] (see also Chapter 2 and 3 of [Mor96]).

Lecture 4 - The Seiberg-Witten moduli space. Putting pieces together, we'll finally write down the Seiberg-Witten equations - and their group of symmetries. We will discuss how (under suitable circumstances) the set of solutions up to symmetry is a compact smooth manifold, and how in can be used to define smooth invariants of four-manifolds. *References*: Chapter 4 and 6 of [Mor96] (see also [HT])

Lecture 5 - Explicit computations. We will work with examples in which the Seiberg-Witten invariants can be described explicitly using our understanding of the underlying Riemannian geometry: manifolds with positive scalar curvature and Kähler surfaces. We will discuss how the latter example can be suitably generalized to all symplectic manifolds. *References*: Chapter 5 and 7 of [Mor96] (see also [HT])

Lecture 6 - Gluing and Floer homology. Trying to compute the invariants of four-manifolds obtained by cut and paste operation naturally leads to the following question: can one use the Seiberg-Witten equations to define invariants of three-manifolds and four-manifolds bounded by them? We'll see for example that the answer for the three-torus provides the computations to classify elliptic surfaces. We'll study then the Seiberg-Witten equations on a product manifold $\mathbb{R}\times$, and see that they become the gradient flow for the Chern-Simons-Dirac functional $\mathcal{L}$. *References*: Chapter 38 and 5 of [KM07]

Lecture 7 - S^1-equivariant Morse theory. Motivated by the study of the Seiberg-Witten equations on a three-manifold, we'll introduce (finite dimensional) S^1-equivariant homology. We will show how it can be computed (under certain assumptions) using a Morse-theoretic model involving the blow-up of the configuration space. *References*: Chapter 2 of [KM07] (see also [Lin16])

Lecture 8 - Monopole Floer homology and the Frøyshov invariant. We will apply our Morse-theoretic machinery to the case of the Chern-Simons-Dirac functional $\mathcal{L}$ to define Floer homology groups for three-manifolds. We'll discuss the formal properties of these invariants. We then focus on the special case of a homology sphere, and show how they can be used to prove a generalization of Donaldson's theorem to manifolds with boundary. *References*: [Lin16] and Chapters 3 and 39 of [KM07]

Lecture 9 - The Triangulation Conjecture. In special cases, our configuration spaces come with a natural additional symmetry (of quaternionic nature). We'll see how this can be exploited in our setting to give an alternative disproof on the Triangulation conjecture - i.e. that there exists topological manifolds in high dimensions not homeomorphic to a simplicial complex. *References*: [Man16b], [Man16a], [Lin16]

References

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