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Bordered Heegaard Floer homology

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Introduction

In the last decade Heegaard Floer homology, first introduced by Ozsváth and Szabó in their seminal paper [**OS04c**], has been one of the central topics of study in low dimensional topology. In its simplest version (called the hat version), it assigns to each closed oriented and connected 3-manifold Y a $\mathbb{F}_2$-vector space $\widehat{HF}(Y)$ obtained as the homology of a chain complex constructed by counting holomorphic curves in a certain space associated to Y. This construction and its variants turn out to be very effective when studying 3-manifolds and the geometric structures they carry. For example the Heegaard Floer invariants determine the genus and fiberedness of knots ([**OS04a**],[**Ni07**]), give a Dehn surgery characterization of the unknot ([**KMOS07**]) and provide invariants of contact structures ([**OS05**]). Heegaard Floer invariants also have a very fruitful interaction with the 4-dimensional world, and using them one can for example define powerful invariants of closed 4-manifolds ([**OS06**]) conjecturally equivalent to the Seiberg-Witten invariants and numerical invariants of knots providing lower bounds on their slice genus ([**OS03b**]).

Anyway, like all the Floer homology theories, these Heegaard Floer invariants have the big drawback of being terribly difficult to compute. In fact, also for a simple manifold like the Poincaré homology sphere it is not clear how to compute those invariants in a direct way (i.e. without using surgery exact sequences or other tools), as it involves the count of really complicated holomorphic curves.

A milestone in the study of such invariants was the algorithm designed by Sarkar and Wang ([**SW10**]), providing a combinatorial way to compute the hat version of Heegaard Floer homology. This was a great achievement for the whole theory, making it the first effectively computable Floer homology theory.

An interesting problem that arises naturally from the whole construction is how to extend such invariants of closed 3-manifolds to invariants of 3-manifolds with boundary. A practical motivation for this has a computational nature. In fact, even though it is purely combinatorial, the algorithm by Sarkar and Wang is far from being effective, as also for simple examples as the Poincaré homology sphere one already has to deal with hundreds and hundreds of generators and differentials. Furthermore, it works ad hoc for each example, so at the end of the day it turns out not to be really useful nor for single specific manifolds nor for studying infinite families of them.

In this sense, a natural way to address this problem would be to define easily computable invariants of 3-manifolds with boundary which interact in a nice algebraic way with respect to gluings along boundaries. Having such invariants, in order to compute the one associated to a closed 3-manifold one would cut it along surfaces in simple pieces, and then reconstruct everything on the algebra side.

Another important motivation for the extension of Heegaard Floer homology to manifolds with boundary comes from the world of quantum invariants. In fact the hat version of Heegaard Floer homology associates ([**OS06**]) to every smooth oriented and connected cobordism W between Y_1 and Y_2 a linear map $F_W : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$ which composes in a functorial way, for instance if $\partial W_1 = Y_1 \cup (-Y_2)$ and $\partial W_2 = Y_2 \cup (-Y_3)$ then

$$F_{W_1 \cup_{Y_2} W_2} = F_{W_2} \circ F_{W_1} : \widehat{HF}(Y_1) \to \widehat{HF}(Y_3).$$

These properties make $\widehat{HF}$ a restricted $3+1$ topological quantum field theory (TQFT), where by restricted we mean that it is defined only on connected and non empty manifolds and cobordisms.

Given $3+1$ a (non restricted) TQFT, it is natural to ask in the framework of quantum invariants if this can be categorified to a local $2+1+1$ theory, i.e. if the assignment

$$\{4\text{-manifolds}\} \longrightarrow \{\text{elements of } \mathbb{F}_2\}$$
$$\{3\text{-manifolds}\} \longrightarrow \{\text{vector spaces over } \mathbb{F}_2\}$$

can be extended one dimension below, for example by

$$\{2\text{-manifolds}\} \longrightarrow \{\text{algebras over } \mathbb{F}_2\}$$

in a way such that the composition axioms of a 2-category are satisfied. In particular one would recover invariants for 3-manifolds with boundary considering them as cobordisms between 2-manifolds.

In the present work we describe a construction moving towards this direction due to Lipshitz, Ozsváth and Thurston, called bordered Heegaard Floer homology ([**LOT11b**]). This associates

- to each surface with a suitable parametrization a differential algebra $\mathcal{A}(\mathcal{Z})$;
- to each compact 3-manifold Y with one boundary component parametrized by such a surface two bordereded invariants over such differential algebra, respectively a left differential $\mathcal{A}(-\mathcal{Z})$-module $\widehat{CFD}(Y)$ and a right $\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$ over $\mathcal{A}(\mathcal{Z})$, well defined up to respectively homotopy equivalence and $\mathcal{A}_\infty$ homotopy equivalence.

Furthermore these objects will come with a pairing theorem which will permit to reconstruct the Heegaard Floer homology of a closed manifold from the knowledge of the bordered invariants of the two parts in which it is cut by a separating surface.

The definition of such invariants is a natural prosecution for the work done for Heegaard Floer homology, and involves the count of holomorphic curves in some spaces obtained by cutting the spaces we used to define the closed invariants. There will be some serious analytical and algebraic complications with respect to the closed case, but the spirit of the construction (involving holomorphic curves with assigned asymptotics) is exactly the same. For this reason, many of the proofs and constructions in the bordered case will be (more or less) simple adaptations of the ones in the closed case, and for this reason we will not usually go too deep into the details, which are often pretty long and technical. We prefer to pay more attention to the new geometric phenomena that happen in the bordered case, and see (also with practical examples) how everything fits in the algebraic context.

Organization. This is the plan for the rest of the paper.

In chapter 1 we construct Heegaard Floer homology in the cylindrical reformulation due to Lipshitz [**Lip06**]. This is meant to be an introduction to the whole topic, providing both the basic topological and analytical tools of the subject and the key examples we will always have in mind for the rest of the work. In particular, after having introduced Heegaard diagrams and the topological constructions we are interested in, we will expose the theory of moduli spaces of holomorphic curves we will need, discussing transversality and index theory and gluing and compactness results. After this, we will be able to construct the Floer chain complex and sketch the proof that its homology is indeed an invariant of the 3-manifold we started with.

In chapter 2 we discuss the construction of the bordered invariants $\widehat{CFD}$ and $\widehat{CFA}$. While the topological constructions are a straightforward generalization of those in the closed case, much more complicated algebraic objects and analytical results are involved in the definition of these invariants. In particular, while the type D module will be a classic differential module, the type A invariant will be an $\mathcal{A}_\infty$ module, which is an algebraic structure where (loosely speaking) associativity holds only up to homotopy. Furthermore, the compactifications of the moduli spaces we will construct will not have a nice structure as in the closed case, making proofs much trickier (but the same in the spirit).

In chapter 3 we finally discuss the pairing theorem, which allows to reconstruct the Heegaard Floer homology of the global manifold by the knowledge of the bordered invariants of the two parts in which it is cut by a separating surface. We will discuss two proofs of this key result, the first (more geometric) via time dilation, the second (more combinatorial) via the aforementioned nice Heegaard diagrams.

Finally we briefly discuss some very recent developments and refinements of the theory presented in this work.

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Contents

CHAPTER 1

Heegaard Floer homology

This chapter has the aim to introduce the reader to Heegaard Floer homology, first defined by Ozsváth and Szabó in their seminal paper [**OS04c**]. There are many variants of this construction, and we will focus only on the simplest one (called the *hat* version) which associates to each closed oriented 3-manifold Y a $\mathbb{F}_2$-vector space $\widehat{HF}(Y)$. This is obtained as the homology of a chain complex constructed by counting holomorphic curves in an almost complex manifold associated to the 3-manifold. Such an almost complex manifold will depend on many choices, and so will the chain complex, but its homology will turn out to be independent of all of them (as in the general spirit of counting invariants), returning us an invariant of the original 3-manifold Y.

Our approach will differ from Ozsváth and Szabó's original construction, and will be the so called *cylindrical reformulation* due to Lipshitz [**Lip06**], which has the flavor of symplectic field theory (SFT). Recall that a symplectic manifold (M, ω) ([**MS04**]) is an even dimensional manifold M together with a 2-form ω which is closed and non degenerate. Any symplectic manifold admits a lot of *tame almost complex structures*, i.e. $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{Id}$ and $\omega(JX, X) > 0$ for every $X \in T_pM$, and one of the most powerful techniques to study symplectic manifolds (introduced by Gromov in his celebrated paper [**Gro85**]) consists in the study of J-holomorphic curves inside them, i.e. maps $u : (S, j) \to (M, J)$ from a Riemann surface with J-linear differential, i.e.

$$J \circ du = du \circ j.$$

Symplectic field theory, which analytic foundations were studied in [**BEH$^+$03**], considers J-holomorphic curves in symplectic manifolds with cylindrical ends, which basically means that the ends of the manifold have the form $N \times \mathbb{R}$ and the symplectic form ω is $\mathbb{R}$-invariant on them. Such holomorphic curves may also have boundary on some specified *lagrangian submanifolds*, which are half-dimensional submanifolds $L \subset M$ such that $\omega|_{TL} \equiv 0$.

The two formulations (the original and the cylindrical ones) are completely equivalent and both have their benefits and drawbacks. In the Lipshitz's version one counts holomorphic curves in very simple low dimensional manifolds (namely products of a closed surface and a disk), and so it has a much more visual flavor than the original one (which deals with high dimensional and topologically complicated manifolds). In any case, the main reason we will be using the cylindrical reformulation is that it behaves much better with the theory we are going to develop afterwards, where we will cut 3-manifolds along surfaces.

Our aim is to introduce the reader to the topic giving a geometric intuition of the phenomena involved in the construction, in view of the main body of the present work developed in the subsequent chapters. For this reason, we will usually sketch the interesting proofs without entering the details, which are sometimes quite long and require many analytical tools (we refer the interested reader to the original paper [**Lip06**] for them). In any case, as many of the definitions and proofs in the bordered case are simply slight generalizations of those in the closed case, we will try to make clear the main ideas behind them in this setting, in order to make the latter chapters easier to read.

The plan for the chapter is the following. In section 1 we will introduce Heegaard diagrams, which are the combinatorial way in which we will present our 3-manifold Y. In section 2 we study the topological questions related to Heegaard Floer homology, while section 3 is dedicated to the analytical ones, and it deals in particular with moduli spaces of holomorphic curves. In section 4 we discuss a technical condition on Heegaard diagrams called admissibility. In section 5 we finally define the Floer chain complex, and in section 6 we discuss the proof that its homology is actually an invariant of the 3-manifold Y. In section 7 we discuss some interesting refinements and variants of the construction.

1. Heegaard diagrams

In this section we introduce a classical combinatorial way to present 3-manifolds, see for example [**FM97**]. From now by Y we will always mean a closed, connected and oriented 3-manifold.

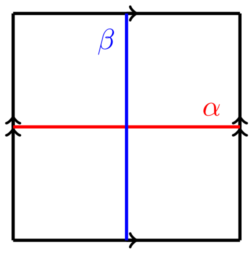
Definition 1.1. A *Heegaard diagram* is a triple $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ where Σ is a closed, connected and oriented surface of genus g, and both $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ are sets of g disjoint homologically independent simple closed curves in Σ.

A Heegaard diagram uniquely specifies a 3-manifold by the following construction.

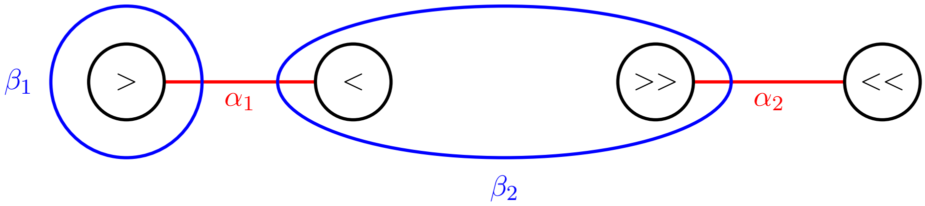
Construction 1.2. Consider the thickened surface $\Sigma \times [0, 1]$. We can do surgery along the g curves $\boldsymbol{\alpha} \times \{0\}$ and the g curves $\boldsymbol{\beta} \times \{1\}$, where doing surgery along a curve in the boundary is the operation of attaching a 2-handle $D^2 \times I$ identifying the $S^1 \times I$ part of its boundary to a regular neighborhood of the curve. By the homological linear independence hypothesis, every surgery does not introduce new boundary components and lowers the genus of the component on which we perform it by 1, so at the end we obtain a 3-manifold with just two boundary components which are both homeomorphic to S^2. Then we can obtain a closed 3-manifold capping them off by attaching two balls D^3.

Let us consider some simple examples of Heegaard diagrams.

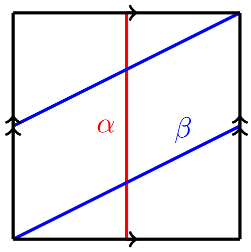
Example 1.3. The following is a genus 1 Heegaard diagram describing the manifold S^3.



The following genus 2 Heegaard diagram also describes S^3 (the picture is drawn on a sphere with 4 holes, and one has to identify the circles as suggested by the figure).

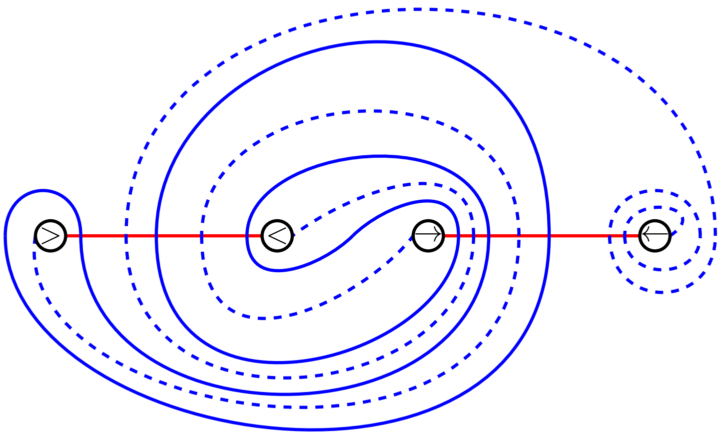


EXAMPLE 1.4. Given coprime integers (p, q), the *lens space* $L(p, q)$ is the 3-manifold obtained by identifying the boundaries of two solid tori $D^2 \times S^1$ by a homeomorphism $\varphi : S^1 \times S^1 \to S^1 \times S^1$ as follows. Consider the longitude $\lambda = \{1\} \times S^1$ and the meridian $\mu = S^1 \times \{1\}$, which homology classes generate $H_1(S^1 \times S^1)$. Then φ is any homeomorphism such that $\varphi_*([\mu]) = p[\lambda]+q[\mu]$, and the resulting manifold does not depend on the actual choice of such a φ. In fact the way in which φ specifies how to attach the meridian disk $D^2 \times \{1\}$ is determined, and after this attaching operation one obtains a manifold with boundary S^2 which can be filled in a unique way in order to get a closed 3-manifold. The following is a genus 1 Heegaard diagram for the lens space $L(2, 1) \cong \mathbb{R}P^3$.



More generally, to get a genus 1 Heegaard diagram for $L(p, q)$ one has just to take as β a curve in the homology class $p[\lambda] + q[\mu]$.

EXAMPLE 1.5. This is a more complicated example.



This Heegaard diagram describes the *Poincaré homology sphere*, which is the 3-manifold

$$Y = \{(z,t,w) \in \mathbb{C}^3 | z^2 + t^3 + w^5 = 0, |z|^2 + |t|^2 + |w|^2 = 1\}.$$

This manifold is really interesting as it is the simplest example of *homology sphere*, i.e. it has the same homology groups as S^3 but non trivial fundamental group.

The following Morse-theoretic approach is really convenient to deal with Heegaard diagrams. We refer the reader to [**Mil65**] for a pleasant introduction to Morse theory.

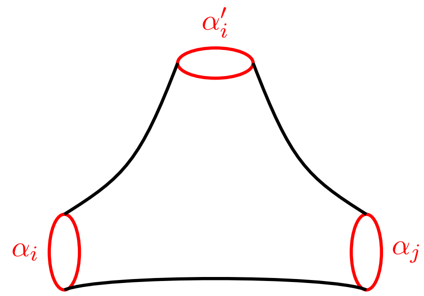
Construction 1.6. Suppose we have a self-indexing Morse function on $f : Y \to [0,3]$, i.e. a Morse function such that for every critical point P we have that $f(P) = \text{ind}(P)$ (the index of the critical point), and suppose also it has only one maximum and one minimum. As $\chi(Y) = 0$, f has the same number g of index 1 and index 2 critical points. Now $f^{-1}([0,3/2])$ is obtained by D^3 by attaching g 1-handles, and so it is a g-handlebody, and the same holds for $f^{-1}([3/2,3])$, so in particular $f^{-1}(3/2)$ is a genus g surface Σ.

After choosing a Riemannian metric on Y, we can obtain a set $\boldsymbol{\alpha}$ of g disjoint homologically independent simple closed curves by intersecting Σ with the ascending manifolds of the index 1 points. Doing the same with the descending manifolds of the index 2 points, we get a collection $\boldsymbol{\beta}$ of g disjoint homologically independent simple closed curves, and so we obtain a Heegaard diagram. It is easy to see that the manifold associated to this diagram following construction 1.2 is exactly Y.

Next we describe some basic operations on Heegaard diagrams. It is simple to see that each of them leaves the associated 3-manifold unchanged.

Definition 1.7. A *Heegaard move* on a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is one of the following operations:

- *isotopy:* moves the families of curves $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ by a one parameter family in a way such the curves of each family remain disjoint at every time;
- *handleslide:* a handleslide of α_i over α_j replaces α_i with a curve α_i' disjoint from $\boldsymbol{\alpha}$ such that α_i, α_j and α_i' bound a pair of pants inside Σ disjoint from the other α-curves.



  Similarly one defines a handleslide of β_i over β_j;
- *stabilization:* chooses a point in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ and makes a connected sum at this point with the genus 1 diagram of S^3 of example 1.3. This operation increases the genus of the surface by 1.

Remark 1.8. As isotopies do not change the associated 3-manifold, we will always suppose without loss of generality that all intersections between our curves are transverse.

The following is the main result regarding Heegaard diagrams.

PROPOSITION 1.9. *Any 3-manifold Y can be represented by a Heegard diagram. Furthermore, any two such Heegaard diagrams can be related by a finite sequence of Heegaard moves.*

The first statement comes from construction 1.6 and the existence of a self-indexing Morse function with only one maximum and one minimum (see [**Mil65**]). As also every Heegaard diagram can be obtained from a self-indexing Morse function by construction 1.6, the second statement follows almost immediately by handle calculus (see [**GS99**]), paying attention to some special issues (we refer the reader to [**OS04c**] for the details).

EXAMPLE 1.10. The second Heegaard diagram of example 1.3 is obtained by the first one by a stabilization and a handleslide.

In the rest of the present work, we will always be interested in *pointed Heegaard diagrams* $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ where z is a point in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. There are clear adaptations of Heegaard moves for pointed Heegaard diagrams. In particular *pointed isotopies* are isotopies which do not cross the basepoint z, and *pointed handleslides* are handleslides such that the pair of pants region involved does not contain z. Proposition 1.9 can be extended to pointed Heegaard diagrams thanks to the following lemma.

LEMMA 1.11. *Given a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and two basepoints $z_1, z_2 \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, the pointed Heegaard diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z_1)$ and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z_2)$ can be connected by a sequence of pointed isotopies and pointed handleslides.*

From now we will always consider only pointed Heegaard diagrams, and drop the 'pointed'.

2. Generators, homology classes and domains

In this section we describe the topological aspects of the construction of Heegaard Floer homology. From now on, given a 3-manifold Y, we will always describe it by means of a Heegaard diagram $\mathcal{H}$. To this, we will associate a chain complex $\widehat{CF}(\mathcal{H}; J)$ constructed by counting special holomorphic curves in $(\Sigma \times [0,1] \times \mathbb{R}, J)$. We now describe the generators of such a complex and the topology of the curves connecting them (in a sense to be defined).

2.1. Generators. Given a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, a *generator* is a g-tuple $\{x_1, \dots, x_g\}$ of points of Σ such that each α-curve and each β-curve contains exactly one of the x_i's. We denote the set of generators of $\mathcal{H}$ by $\mathfrak{S}(\mathcal{H})$.

EXAMPLE 2.1. In example 1.3 both Heegard diagrams for S^3 have only one generator. The diagram for $L(p,q)$ of example 1.4 has p generators, and the diagram for the Poincaré homology sphere of example 1.5 has 19 generators.

2.2. Homology classes of curves. Having fixed two generators $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ we are interested in curves in the product space $\Sigma \times [0,1] \times \mathbb{R}$ with some special boundary and asymptotic conditions, which we make now clear. We will consider complex curves, so the underlying space will have real dimension 2.

Our ambient manifold $\Sigma \times [0,1] \times \mathbb{R}$ comes with the projections $\pi_\Sigma : \Sigma \times [0,1] \times \mathbb{R} \to \Sigma$ and $\pi_{\mathbb{D}} : \Sigma \times [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ (denote by (s,t) the coordinates of the last projection). The α and β-curves determine $2g$ cylinders inside it, $\boldsymbol{\alpha} \times \{0\} \times \mathbb{R}$ and $\boldsymbol{\beta} \times \{1\} \times \mathbb{R}$.

Let $(S, \partial S)$ be a possibly disconnected compact surface with boundary with $2g$ punctures (i.e. points removed) on the boundary $(p_1, \dots, p_g, q_1, \dots, q_g)$. If $\mathbf{x} = (x_1, \dots, x_g)$ e $\mathbf{y} = (y_1, \dots, y_g)$ are two generators, we are interested in proper continuous maps

$$u : (S, \partial S) \to \big((\Sigma \setminus \{z\}) \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{0\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{1\} \times \mathbb{R})\big)$$

such that

- u is asymptotic at the $\{p_i\}$ punctures to the chords $(\mathbf{x} \times [0,1] \times \{-\infty\})$, i.e (up to reordering) $\lim_{w \to p_i} \pi_\Sigma \circ u(w) = x_i$ and $\lim_{w \to p_i} t \circ u(w) = -\infty$;
- u is asymptotic at the $\{q_j\}$ punctures to the chords $(\mathbf{y} \times [0,1] \times \{+\infty\})$, i.e (up to reordering) $\lim_{w \to q_j} \pi_\Sigma \circ u(w) = y_j$ and $\lim_{z \to q_j} t \circ u(w) = +\infty$;
- u extends at the punctures to a well defined map with values in $(\Sigma \setminus \{z\}) \times [0,1] \times \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$ is the compactification of $\mathbb{R}$.

We say that such a curve *connects* $\mathbf{x}$ and $\mathbf{y}$.

Definition 2.2. Denote by $\pi_2(\mathbf{x}, \mathbf{y})$ the set of relative homology classes of maps connecting $\mathbf{x}$ and $\mathbf{y}$, i.e. two such maps are equivalent if they induce the same element of

$$H_2\big(\Sigma \times [0,1] \times \overline{R}, (\boldsymbol{\alpha} \times \{0\} \times \overline{\mathbb{R}}) \cup (\boldsymbol{\beta} \times \{1\} \times \overline{\mathbb{R}}) \cup (\mathbf{x} \times [0,1] \times \{-\infty\}) \cup (\mathbf{y} \times [0,1] \times \{+\infty\})\big)$$

Notation 2.3. The notation $\pi_2(\mathbf{x}, \mathbf{y})$ comes from the original formulation of Heegaard Floer homology, where one considers homotopy classes of disks rather that homology classes of curves.

Remark that given generators $\mathbf{x}, \mathbf{y}$ and $\mathbf{w}$ in $\mathfrak{S}(\mathcal{H})$, there is an obvious concatenation map

$$* : \pi_2(\mathbf{x}, \mathbf{y}) \times \pi_2(\mathbf{y}, \mathbf{w}) \to \pi_2(\mathbf{x}, \mathbf{w})$$

obtained by juxtaposing topological curves. Indeed, given a curve connecting $\mathbf{x}$ to $\mathbf{y}$ and a curve connecting $\mathbf{y}$ to $\mathbf{w}$, one can construct a curve connecting $\mathbf{x}$ to $\mathbf{w}$ by gluing the $+\infty$ end of the first with the $-\infty$ end of the second.

2.3. Domains. There is a nice interpretation of homology classes of curves $\pi_2(\mathbf{x}, \mathbf{y})$ in terms of their projection on the surface Σ.

Let $\{D_i\}_{i=1}^n$ be the closures of the regions in which the α and β-curves cut the surface Σ, and choose $z_i \in \mathrm{int}(D_i)$ (choose z in the region containing it). A *domain* is a formal linear combination with $\mathbb{Z}$ coefficients of such regions. Given a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$, for every $i = 1, \dots, n$ there is an intersection number

$$n_{z_i}(B) = \#\big(u^{-1}(\{z_i\} \times [0,1] \times \mathbb{R})\big)$$

where u is any representative of B for which the count makes sense, and it is well-defined because of the boundary conditions. Note that by definition we always have $n_z(B) = 0$. Thus to each homology class we can associate its domain

$$B \mapsto \mathcal{D}(B) = \sum n_{z_i}(B) D_i$$

which is clearly independent of the choices of $z_i \in D_i$. This map obviously respects the concatenation product, i.e $\mathcal{D}(B_1 * B_2) = \mathcal{D}(B_1) + \mathcal{D}(B_2)$.

Given a domain D, we can consider its boundary ∂D as a 1-chain in Σ, and in turn this can be decomposed as a sum over its α and β-components:

$$\partial D = \sum_{i=1}^{g} \partial^{\alpha_i} D + \sum_{i=1}^{g} \partial^{\beta_i} D.$$

Definition 2.4. Given generators $\mathbf{x} = (x_1, \dots, x_g)$ and $\mathbf{y} = (y_1, \dots, y_g)$ be generators one can suppose that after relabeling $x_i, y_i \in \alpha_i$ and $y_i, x_{\sigma(i)} \in \beta_i$ for some permutation $\sigma \in S_g$. The set of *domains connecting* $\mathbf{x}$ *and* $\mathbf{y}$, which we denote $P(\mathbf{x}, \mathbf{y})$, consists of the domains D such that

- $\partial^{\alpha_i} D$ is a 1-chain with boundary $y_i - x_i$.
- $\partial^{\beta_i} D$ is a 1-chain with boundary $x_{\sigma(i)} - y_i$.

In fact, it turns out that this is a good way to describe homology classes.

Lemma 2.5. *If* $\pi_2(\mathbf{x}, \mathbf{y})$ *is non empty, then for* $B \in \pi_2(\mathbf{x}, \mathbf{y})$ *we have that* $\mathcal{D}(B) \in P(\mathbf{x}, \mathbf{y})$, *and* $\mathcal{D} : \pi_2(\mathbf{x}, \mathbf{y}) \to P(\mathbf{x}, \mathbf{y})$ *is a bijection with the subset of domains* $D \in P(\mathbf{x}, \mathbf{y})$ *with* $n_z(D) = 0$.

With this explained, for the rest of the chapter we will always confuse a homology class with its domain and viceversa. We conclude this section with two definitions regarding domains which will turn out to be relevant in what follows.

Definition 2.6. A domain D is *positive* if all its coefficients are non negative. A domain D is *periodic* if $\partial_{\alpha_i} D$ and $\partial_{\beta_i} D$ are all 1-chains with trivial boundary.

Remark 2.7. In particular, if there exists a generator $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, we can identify the set of periodic domains with $P(\mathbf{x}, \mathbf{x})$.

3. Moduli spaces of holomorphic curves

Having studied the topology of curves in $\Sigma \times [0,1] \times \mathbb{R}$, we now turn to the analytical aspects of the construction. We first define the special types of almost complex structures on this ambient manifold we are interested in, and then discuss the properties of holomorphic curves. We then turn our attention to moduli spaces of such curves, addressing transversality issues, index theory and how these can be compactified.

3.1. Almost complex structures. Given a Heegaard diagram $\mathcal{H}$, choose a point z_i in the interior of each component of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ as in the previous section. Let (s,t) be coordinates on $[0,1] \times \mathbb{R}$ and A be a volume form on Σ. Let $\omega = A + ds \wedge dt$ be the split symplectic form on $\Sigma \times [0,1] \times \mathbb{R}$.

Definition 3.1. An almost complex structure J on $\Sigma \times [0,1] \times \mathbb{R}$ is *admissible* if:

(1) J is tame with respect to ω;
(2) in a small cylindrical neighbourhood (such that it does not intesect $(\boldsymbol{\alpha} \cup \boldsymbol{\beta}) \times [0,1] \times \mathbb{R}$) of each $\{z_i\} \times [0,1] \times \mathbb{R}$, J is a split complex structure $j_\Sigma \times j_\mathbb{D}$;
(3) J is translation invariant in the $\mathbb{R}$ direction;

(4) $J(\frac{\partial}{\partial t}) = \frac{\partial}{\partial s}$;
(5) for every $(s,t) \in [0,1] \times \mathbb{R}$, $T(\Sigma \times \{(s,t)\})$ is a J-invariant subspace.

Let us explain the motivation behind such a definition. The first condition assures bounds on the energy of holomorphic curves depending only on their homology classes. Recall that given a holomorphic curve $u : S \to (M, J)$, its *energy* is the integral

$$E(u) = \int_S u^*\omega$$

which can also be interpreted as the area of the curve with respect to the Riemannian metric

$$g_J(X,Y) = \frac{1}{2}[\omega(X,JY) + \omega(Y,JX)].$$

This is a fundamental concept in symplectic geometry because energy bounds permit to construct nice compactifications of moduli spaces via the celebrated *Gromov's compactness theorem* [**Ye94**]. We will return on this later.

The second condition assures that homology classes of holomorphic curves are positive (in the sense of definition 2.6), see lemma 3.3.

The last conditions are required to work in the setting of symplectic field theory. In particular, condition (3) and (4) make $\Sigma \times [0,1] \times \mathbb{R}$ a symplectic manifold with cylindrical ends, while condition (5) is a technical condition needed to apply the results of [**BEH$^+$03**].

Remark 3.2. The cylinders $\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}$ and $\boldsymbol{\beta} \times \{0\} \times \mathbb{R}$ are lagrangian submanifolds.

3.2. Holomorphic curves. We are now ready to define holomorphic curves with respect to an admissible almost complex structure J. Fix two generators $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$.

Let S be a compact possibly disconnected Riemann surface with boundary with punctures on the boundary labeled by $+$ or $-$, and fix an admissible almost complex structure J on $\Sigma \times [0,1] \times \mathbb{R}$. We consider J-holomorphic maps

$$u : (S, \partial S) \to \big((\Sigma \setminus \{z\}) \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})\big)$$

such that:

(1) for every $+$ puncture q, $\lim_{w \to q} t \circ u(w) = +\infty$;
(2) for every $-$ puncture q, $\lim_{w \to q} t \circ u(w) = -\infty$;
(3) for every $t \in \mathbb{R}$ and $i = 1, \dots, g$, $u^{-1}(\alpha_i \times \{1\} \times \{t\})$ and $u^{-1}(\beta_i \times \{0\} \times \{t\})$ consist each of exactly one point;
(4) u has finite *energy* in the symplectic field theory sense;
(5) $\pi_{\mathbb{D}}$ is non constant on every component of S;
(6) u is an embedding.

Let us make some comments on the conditions we have just defined. The first three conditions imply that every such curve defines an element of $\pi_2(\mathbf{x}, \mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$. In particular the third one implies that the limits at $-$ and $+$ infinity are chords of the form $(\mathbf{x} \times [0,1] \times \{-\infty\})$ and $(\mathbf{y} \times [0,1] \times \{+\infty\})$, and in fact is equivalent to it (using the fact that non constant harmonic functions are monotone).

The fourth condition, which is of purely technical nature, is needed to apply the results of [**BEH$^+$03**] and we will always consider it as implicit. On the other hand, condition (5) implies that $\pi_{\mathbb{D}} : S \to [0,1] \times \mathbb{R}$ is a g-fold branched covering. This follows from the well-known fact in complex analysis that non constant holomorphic maps are open.

The last condition is imposed by technical reasons in order to get manifolds as moduli spaces. We will return on this in the next section.

Given a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$, we denote $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y})$ the space of holomorphic curves connecting $\mathbf{x}$ and $\mathbf{y}$ which induce the class homology B. Here we always consider such maps up to reparameterization of the source, i.e. we consider $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ and $u' : S' \to \Sigma \times [0,1] \times \mathbb{R}$ to be the same if there exists a biholomorphism $\varphi : S \to S'$ such that $u' \circ \varphi = u$. We have the following.

Lemma 3.3. *If $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}) \neq \emptyset$, then B is a positive domain.*

Proof. This comes from the property (2) of the definition of an admissible almost complex structure (definition 3.1). In fact, near all the strips $\{z_i\} \times [0,1] \times \mathbb{R}$ the almost complex structure is integrable, so two curves intersect there positively or are one contained in the other. In particular because of boundary conditions $u(S)$ is not contained in any $\{z_i\} \times [0,1] \times \mathbb{R}$ and viceversa, and so all $n_{z_i}(B)$ are non negative. $\square$

The space of holomorphic curves $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y})$ admits a natural-$\mathbb{R}$ action by translations in the t coordinate (because of property (3) of definition 3.1). So one can define the *moduli space* of holomorphic curves connecting $\mathbf{x}$ and $\mathbf{y}$ in the homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ as the quotient of this action, namely

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}) = \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y})/\mathbb{R}.$$

These are the objects we wish to use to define differentials in $\widehat{CF}(\mathcal{H}; J)$.

3.3. Transversality. First of all we discuss to which extent the spaces $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y})$ can be given a natural smooth manifold structure. The basic idea is easily explained in the finite dimensional case. Given a vector bundle over a manifold $E \to X$, if a section $s : X \to E$ is transverse to the zero section then its zero locus $s^{-1}(0)$ is a smooth manifold (this is basically the implicit function theorem). Note that the transversality condition at a point p can be reformulated as the surjectivity at each point of the linearization $Ds : T_pM \to E_p$.

Our case has the same spirit and in particular one would like to see the space of holomorphic maps $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ from a fixed source as the zero locus of a section of a vector bundle. This is technically much more complicated because one has to work with infinite dimensional spaces (see [**MS04**]). Roughly speaking, one can construct a Banach vector bundle $E \to X$ as a Sobolev completion of the bundle over the space of smooth maps

$$u : (S, \partial S) \to \big((\Sigma \setminus \{z\}) \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})\big)$$

with fiber over u the vector space $\overline{\mathrm{Hom}}^0_{\mathbb{C}}\big(TS, u^*T(\Sigma \times [0,1] \times \mathbb{R})\big)$ of J-antilinear bundle homomorphisms $TS \to u^*T(\Sigma \times [0,1] \times \mathbb{R})$ respecting some conditions at the boundary. The $\overline{\partial}$ operator

$$u \mapsto du + J \circ du \circ j$$

is a section of this Banach bundle, and space of holomorphic maps is exactly the zero set of this section (one has to note that actually by some regularity theory the completion does not enlarge this zero set). Then, in order to apply the implicit function theorem for Banach spaces, one needs to study the linearization $D\overline{\partial} : T_uX \to E_u$. This turns out to be a Fredholm operator (i.e. has finite dimensional kernel and cokernel) with a well defined constant index $\operatorname{ind} D\overline{\partial} = \operatorname{dimKer} D\overline{\partial} - \operatorname{dimCoker} D\overline{\partial}$ depending only on some topological data.

What one expects is that a sufficiently 'generic' choice of the almost complex structure J on $\Sigma \times [0,1] \times \mathbb{R}$ achieves transversality, i.e. $D\overline{\partial}$ is surjective, so that the space of holomorphic maps $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ is a manifold of dimension $\operatorname{ind} D\overline{\partial}$ by the infinite dimensional analogue of the implicit function theorem. For this, there are technical issues that arise for example when dealing with multiply covered components, i.e. curves with components that factor through a branched covering $S \to S'$. In this case one can easily show by index considerations that it is generally impossible to achieve transversality (see [**MS04**]). Furthermore, as we are interested in moduli spaces of such curves, one should put also some attention to the reparameterization group of the source. In any case we have the following positive result.

Proposition 3.4. *For a generic choice of the admissible almost complex structure J, the spaces $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y})$ with $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ and $B \in \pi_2(\mathbf{x}, \mathbf{y})$ are smooth manifolds.*

In this setting a *generic* subset of a complete metric space (and in particular of the space of smooth almost complex structures on $\Sigma \times [0,1] \times \mathbb{R}$, which is a Fréchet manifold) is a subset containing a Baire subset, i.e. a countable intersection of open dense subsets. Baire's lemma tells us that each generic subset is dense, and a countable intersection of generic subsets is still generic.

We now focus on the dimension of these spaces, which as we have said is locally given by the index of the linearization of the $\overline{\partial}$ operator. This has a really nice combinatorial interpretation in terms of domains.

We define the *Euler measure* $e(D)$ of a domain D. Endow Σ with a Riemannian metric such such that α and β-curves are geodesics and always intersect orthogonally. Then for every i we define the Euler measure of a region D_i to be

$$e(D_i) = \frac{1}{2\pi} \int_{D_i} \kappa$$

where κ is the curvature of the metric. It follows from the Gauss-Bonnet formula that if D_i has on the boundary k acute right angles and l obtuse right angles then

$$e(D_i) = \chi(D_i) - k/4 + l/4.$$

We finally extend this notion to all domains by linearity. We have the following result.

Proposition 3.5. *The index of $D\overline{\partial}$ at a (non necessarily embedded) curve $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ in the homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ is given by $g - \chi(S) + 2e(D(B))$.*

When considering embedded curves (which are the curves we are interested in), this index can be expressed purely in terms of the domain B. Given a domain D and an intersection point x between α and β-curves, let $n_x(D)$ be the average of the multiplicities of the four regions (possibly with repetition) which have x as a corner. If $\mathbf{x} = (x_1, \ldots, x_g) \in \mathfrak{S}(\mathcal{H})$, denote $n_{\mathbf{x}}(D) = \sum_i n_{x_i}(D)$.

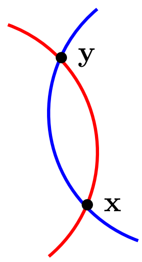
Theorem 3.6. *The index of $D\overline{\partial}$ at an embedded holomorphic curve representing $B \in \pi_2(\mathbf{x}, \mathbf{y})$ is given by*

$$n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + e(B).$$

In particular, $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y})$ has expected dimension $n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + e(B)$, and each curve in it has Euler characteristic $g - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) + e(B)$.

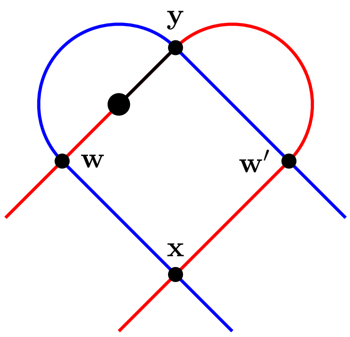
Before studying how sequences of holomorphic curves may degenerate, we provide two simple local examples to get acquainted with holomorphic curves. Here, and in all the rest of the present work, we will always use a product almost complex structure $j_\Sigma \times j_{[0,1]\times\mathbb{R}}$, as when dealing with simple domains (as the ones in our examples) one can prove that transversality can be always achieved by product almost complex structure after a small perturbation of the curves (see [**Lip06**]).

EXAMPLE 3.7. The following index 1 domain determines a unique holomorphic curve in the moduli space connecting $\mathbf{x}$ to $\mathbf{y}$.



By the Riemann mapping theorem each simply connected proper open set of $U \subset \mathbb{C}$ admits an uniformization from the unit disk $\Delta \subset \mathbb{C} \to U$. Then, to obtain a map $\Delta \to \Sigma \times [0,1] \times \mathbb{R}$ one needs a holomorphic map $\Delta \to [0,1] \times \mathbb{R}$ with the right boundary conditions. i.e. the α-part of $\partial\Delta$ shall to go Re$z = 1$ and the β-part of $\partial\Delta$ to Re$z = 0$. After factoring out the 1-dimensional reparameterization group of the source and of $[0,1] \times \mathbb{R}$, one remains with exactly one holomorphic curve in the moduli space.

EXAMPLE 3.8. The following domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$ has index 2.



There is a 1-parameter family of holomorphic disks connecting $\mathbf{x}$ to $\mathbf{y}$ obtained by uniformizing the domain when it is cut along the black segment (which may also go symmetrically on the β-curve). So in this case $\mathcal{M}^B(\mathbf{x}, \mathbf{y}) \cong (-1, 1)$.

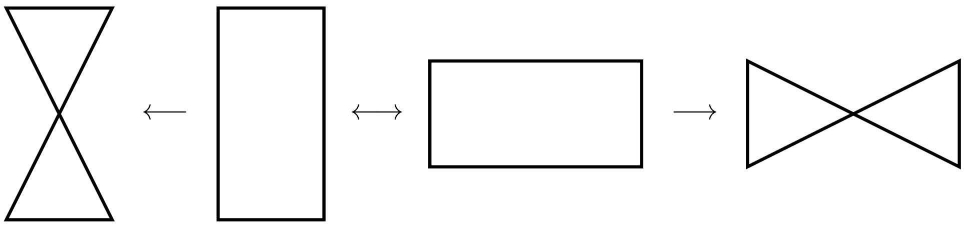
3.4. Compactification and gluing. Finally we discuss how sequences of holomorphic curves may degenerate when approaching the 'ends' of the moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y})$, introducing the compactness and gluing results that we will need.

We first describe some types of degenerate curves. In our setting both the map u and the Riemann surface S can degenerate, giving raise to different phenomena.

A *holomorphic building* is a list of holomorphic curves $v = (u_1, \ldots, u_n)$ (each defined up to $\mathbb{R}$-translation) such that the asymptotics at $+\infty$ of u_i coincide with the asymptotics at $-\infty$ of u_{i+1}, i.e. $u_i \in \mathcal{M}^{B_i}(x_i, x_{i+1})$ for $i = 1, \ldots, n-1$. We call n the number of *stories* of v.

A *Deligne-Mumford degeneration* occurs when the complex structure on the Riemann surface S degenerates along circles and arcs so that they become 'infinitely long necks', giving raise to the so-called *nodal curves.* Those are more precisely collections of Riemann surfaces $(S_1, \ldots, S_k)$ with identifications between some pairs of interior marked points or between some pairs boundary punctures (actually one also requires that the curve is *stable*, i.e. every component has only finitely many automorphisms).

Example 3.9. We study the moduli space of complex rectangles, which are Riemann surfaces which are topologically a disk with four punctures on the boundary. By the Riemann mapping theorem each such surface admits a uniformization as a disk and, actually, as a 'usual' rectangle with edges of length a and b. Two such rectangles are biholomorphic if and only if $\log(a/b) \in \mathbb{R}$ has the same value, so the space of complex rectangles is homeomorphic to $\mathbb{R}$. The degenerations happen when one of the two sides becomes infinitely longer than the other



and in the Deligne-Mumford compactification one obtains a pair of triangles with an identified boundary puncture.

Remark 3.10. Each nodal curve determines a smooth surface by solving all the singularities. For example, for a pair of identified interior marked points this is done by the local operation of solving the nodal singularity

$$\{(z_1, z_2) \in \mathbb{C}^2 | z_1 z_2 = 0\} \qquad \longrightarrow \qquad \{(z_1, z_2) \in \mathbb{C}^2 | z_1 z_2 = \varepsilon\}.$$

A *bubbling* happens when the energy of a holomorphic curve all concentrates is a single point, factoring out a holomorphic sphere in that point. Note that those are generally not Deligne-Mumford degenerations as the source of smooth curve with a bubble is not stable. As we are dealing with curves with boundary, there may also be disks degenerating at the boundary, to which we refer to as *boundary degenerations.*

The notion of convergence for holomorphic curves is somehow complicated, so we will not point it out in detail. Intuitively, it takes account both of the topology on the moduli space of Riemann surfaces and the $\mathcal{C}^\infty$ topology on the set of smooth maps. For example, convergence to a multi-story holomorphic building roughly means that some parts of the curve go to infinity respect to other parts. In any case the phenomena we have just pointed out (maybe more of them at the same time) are all that can happen. This is captured by this version of the celebrated Gromov's compactness theorem.

THEOREM 3.11. *Any sequence of holomorphic curves $u_k : S_k \to \Sigma \times [0,1] \times \mathbb{R}$ inducing the same homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ has a subsequence converging to a holomorphic building of nodal curves with bubblings and boundary degenerations. Furthermore a sequence of holomorphic curves can converge only to one such holomorphic building.*

This exploits in a fundamental way the choice of a tame almost complex structure (see definition 3.1), which assures an energy bound on the curves $\{u_k\}$ depending only on the homology class B. This permits to construct a compactification for each moduli space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y})$, obtained as the closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y})$ in the bigger compact space of possibly degenerate curves. Notice that the last statement of the theorem just states that compactifications of the moduli spaces are compact.

The nice thing of the whole theory is that degenerations in low dimensional moduli spaces are much controlled.

PROPOSITION 3.12. *Fix $B \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\mathrm{ind}(B) \leq 2$, and choose a generic admissible almost complex structure J. If a sequence of holomorphic curves $\{u_k\}$ representing B converges to a nodal holomorphic building $(v_1, \dots, v_l)$, then each curve v_i is a genuine embedded holomorphic curve.*

PROOF. We give a sketch of the proof.

Bubbling and boundary degenerations are impossible because both $\pi_2(\Sigma \times [0,1] \times \mathbb{R})$ and $\pi_2\big(\Sigma \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})\big)$ are trivial, and homotopically trivial curves have zero energy, and hence are constant.

In our case, Deligne-Mumford degenerations along circles do not happen for generic choice of the almost complex structure J. In fact, they have codimension 2 in the moduli space of Riemann surfaces, and so they may appear only in homology classes B with $\mathrm{ind}(B) \geq 3$.

Cusp degenerations (i.e. degenerations along an arc) are ruled out because one would get a component $S' \to \Sigma \times [0,1] \times \mathbb{R}$ with (without loss of generality) all the boundary mapped into $\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}$, which is absurd because of the homological independence of the α_i's.

So the only degeneration that may happen are holomorphic buildings with no nodal curves. We are left to prove that they are actually the kind of genuine holomorphic curves we are interested in. There are quite a few things to verify, but we point out embeddedness which is the most interesting one. First one proves using proposition 3.5 that the index is additive along the levels, i.e. for every k one has $\mathrm{ind}(u_k) = \sum_{j=1}^{l} \mathrm{ind}(v_j)$. Then embeddedness follows from the fact by index formulas analogue to proposition 3.6, near each immersed curve with only k transverse self-intersections there is a $2k$-dimensional family of embedded holomorphic curves, and so the dimension of the moduli space would be too big. □

COROLLARY 3.13. *For a generic choice of J, given a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\mathrm{ind}\, B = 1$ the moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y})$ is a compact 0-manifold, i.e. a finite set of points.*

PROOF. This follow from the previous theorem by the fact that for $\mathrm{ind}(B) \leq 0$ the moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y})$ is empty or is just the trivial curve connecting $\mathbf{x} = (x_1, \dots, x_g)$ to itself (i.e. g copies of the twice punctured disk mapped diffeomorphically to $\{x_1, \dots, x_g\} \times [0,1] \times \mathbb{R}$), and so by the additivity of the index there cannot be degenerations. □

For what concerns 1-dimensional moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y})$, proposition 3.12 lets us construct a nice compactified moduli space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y})$ which is a compact 1-manifold, and because of dimensional count $\partial\overline{\mathcal{M}}^B(\mathbf{x},\mathbf{y})$ consists of special two-story holomorphic buildings. More precisely one has the following result.

COROLLARY 3.14. *For a generic choice of J, given a homology class $B \in \pi_2(\mathbf{x},\mathbf{y})$ with* $\operatorname{ind} B = 2$ *the moduli space $\mathcal{M}^B(\mathbf{x},\mathbf{y})$ is an open 1-manifold, and its ends correspond bijectively to*

$$\coprod \mathcal{M}^{B_1}(\mathbf{x},\mathbf{w}) \times \mathcal{M}^{B_2}(\mathbf{w},\mathbf{y})$$

where $\mathbf{w} \in \mathfrak{S}(\mathcal{H})$, and $B_1 \in \pi_2(\mathbf{x},\mathbf{w})$ and $B_2 \in \pi_2(\mathbf{w},\mathbf{y})$ are index one homology classes.

In light of proposition 3.12 the proof of this key result is clear, once one has the right gluing results for holomorphic curves. In the Heegaard Floer context, those can be stated as follows.

PROPOSITION 3.15. *Given a two-story holomorphic building $(u_1,u_2) \in \mathcal{M}^{B_1}(\mathbf{x},\mathbf{w}) \times \mathcal{M}^{B_2}(\mathbf{w},\mathbf{y})$ and sufficiently small neighborhoods U_1 of u_1 and U_2 of u_2 inside these moduli spaces, there is an open neighborhood of (u_1,u_2) in $\overline{\mathcal{M}}^{B_1 * B_2}(\mathbf{x},\mathbf{y})$ homeomorphic to $U_1 \times U_2 \times [0,1)$.*

In particular, this tells us that each pair of curves of corollary 3.14 actually comes from a degeneration of holomorphic curves.

EXAMPLE 3.16. In example 3.8 the points of $\partial\overline{\mathcal{M}}^B(\mathbf{x},\mathbf{y})$ are reached when the cut goes further until $\mathbf{w}$ or $\mathbf{w}'$, and are exactly two two-story holomorphic buildings, one connecting $\mathbf{x}$ to $\mathbf{w}$ and $\mathbf{w}$ to $\mathbf{y}$, and one connecting $\mathbf{x}$ to $\mathbf{w}'$ and $\mathbf{w}'$ to $\mathbf{y}$.

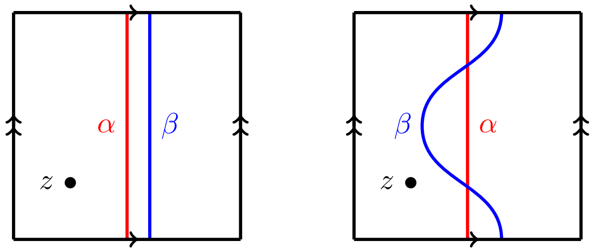
4. Admissibility

This section describes a technical condition that our Heegaard diagrams shall satisfy in order for the whole theory to make sense. In fact, as our invariants will count holomorphic curves connecting generators $\mathbf{x}$ and $\mathbf{y}$, we have to be sure that this count will be well defined. By corollary 3.13 we know that if $\operatorname{ind}(B) = 1$ then $\mathcal{M}^B(\mathbf{x},\mathbf{y})$ consists of a finite set of points, but *a priori* one has to deal with infinitely many homology classes.

DEFINITION 4.1. A pointed Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z)$ is *admissible* if every non trivial periodic domain D (see definition 2.6) with $n_z(D) = 0$ has both positive and negative coefficients.

REMARK 4.2. This condition is usually called *weak admissibility* in literature. Strong admissibility is required to work with other more complicated versions of Heegaard Floer homology (see subsection 7.3), but as we will be dealing only with the hat version we will always drop the ‘weak’.

EXAMPLE 4.3. The obvious Heegaard diagram for $S^2 \times S^1$ on the left is not admissible, while the one on the right is.



Note that the second one is obtained by the first one by isotopy of the β-curve. In fact one can prove that every Heegaard diagram can be turned into an admissible one by performing some special isotopies called *windings* (see [**OS04c**]).

PROPOSITION 4.4. *Any Heegaard diagram is isotopic to an admissible one. Furthermore, any two admissible Heegaard diagrams can be connected by Heegaard moves which preserve admissibility.*

REMARK 4.5. By homological reasons, any Heegaard diagram of a rational homology sphere is admissible. In fact, it is not hard to prove that in such a diagram a periodic domain D with $n_z(D) = 0$ is necessarily trivial.

We now explain how this condition assures that our sums will be finite. The following is a simple lemma in linear algebra.

LEMMA 4.6. *A Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is admissible if and only if there exists an area form on Σ such that every periodic domain D with $n_z(D) = 0$ has total area 0.*

PROPOSITION 4.7. *Fix an admissible Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, and fix generators $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$. There there exist only finitely many homology classes $B \in \pi_2(\mathbf{x}, \mathbf{y})$ with and $\mathcal{M}^B(\mathbf{x}, \mathbf{y}) \neq \emptyset$.*

PROOF. Suppose we have two homology classes $B_1, B_2 \in \pi_2(\mathbf{x}, \mathbf{y})$ supporting holomorphic curves. Now $B_1 * B_2^{-1} \in \pi_2(\mathbf{x}, \mathbf{x})$ so $\mathcal{D}(B_1 * B_2^{-1}) = \mathcal{D}(B_1) - \mathcal{D}(B_2)$ is periodic. Choosing an area form as in the previous lemma, we have then that $\mathcal{D}(B_1)$ has the same area as $\mathcal{D}(B_2)$, so all the domains supporting holomorphic curves have the same area.

As by lemma 3.3 all such domains have non negative coefficients, so there can be only finitely many distinct ones. □

5. The chain complex

We are finally ready to define Heegaard Floer homology. Fix an admissible pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for the 3-manifold Y (which exists by proposition 4.4), and fix a generic (in the sense of proposition 3.4) admissible almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$. Consider the vector space $\widehat{CF}(\mathcal{H}; J)$ generated over $\mathbb{F}_2$ by $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, endowed with the boundary map

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mathrm{ind}(B) = 1}} \#\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y})\big) \cdot \mathbf{y}.$$

Note that the sum is well defined because of corollary 3.13 and proposition 4.7.

THEOREM 5.1. *For a generic almost complex structure J, the map ∂ is a differential, i.e. $\partial^2 = 0$.*

PROOF. This is the archetype of all proofs in Heegaard Floer homology. The coefficient of $\mathbf{y}$ in $\partial^2\mathbf{x}$ is

$$\sum_{\mathbf{w}} \sum_{\substack{B_1\in\pi_2(\mathbf{x},\mathbf{w})\\ \mathrm{ind}(B_1)=1}} \sum_{\substack{B_2\in\pi_2(\mathbf{w},\mathbf{y})\\ \mathrm{ind}(B_2)=1}} \#(\mathcal{M}^{B_1}(\mathbf{x},\mathbf{w}))\cdot\#(\mathcal{M}^{B_2}(\mathbf{w},\mathbf{y})). \tag{1.1}$$

To prove that this is equal to 0, one studies the ends of 1-dimensional moduli spaces of curves connecting $\mathbf{x}$ and $\mathbf{y}$. More precisely, having fixed a homology class $B \in \pi_2(\mathbf{x},\mathbf{y})$ with $\mathrm{ind}(B) = 2$, by corollary 3.14 the ends of $\mathcal{M}^B(\mathbf{x},\mathbf{y})$ correspond bijectively to

$$\coprod \mathcal{M}^{B_1}(\mathbf{x},\mathbf{w}) \times \mathcal{M}^{B_2}(\mathbf{w},\mathbf{y})$$

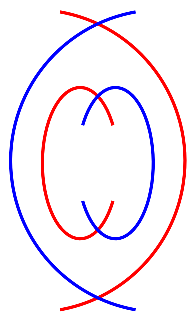
where $\mathbf{w} \in \mathfrak{S}(\mathcal{H})$, $B_1 \in \pi_2(\mathbf{x},\mathbf{w})$ and $B_2 \in \pi_2(\mathbf{w},\mathbf{y})$ are index 1 homology classes with $B_1 * B_2 = B$. As a 1-manifold has an even number of ends, the total number of such pairs of curves is 0. Then, summing over all possible homology classes B, one gets precisely that the sum (1.1) is 0. □

We call the homology of the chain complex $(\widehat{CF}(\mathcal{H};J);\partial)$ the *Heegaard Floer homology* of the pair $(\mathcal{H};J)$. We will prove in the next section that this actually depends only on the 3-manifold Y. We end this section providing some simple examples of computations.

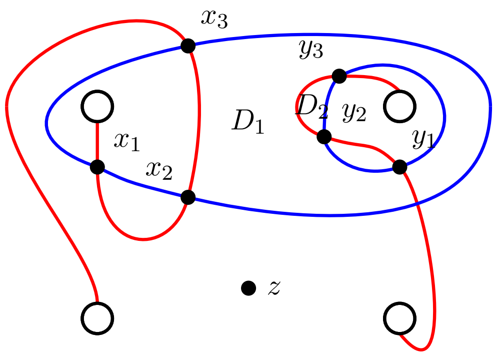
EXAMPLE 5.2. Consider the Heegaard diagram $\mathcal{H}$ for the lens space $L(p,q)$ provided in example 1.4, which is admissible. Then $\widehat{CF}(\mathcal{H};J)$ has exactly p generators, and there are no differentials. In fact it is easy to see that different generators do not have any topological disks connecting them, while a non trivial domain connecting a generator to itself is periodic, and so cannot support holomorphic curves. So $\widehat{HF}(\mathcal{H};J) \cong \mathbb{F}_2^p$.

EXAMPLE 5.3. Consider the admissible Heegaard diagram $\mathcal{H}$ for $S^2 \times S^1$ in example 4.3. There are two generators $\mathbf{x},\mathbf{y}$ (namely the upper and the lower intersection points), and there is exactly one holomorphic disk connecting $\mathbf{x}$ to $\mathbf{y}$ (see example 3.7) in each one of the disk shaped domain, so $\partial\mathbf{x} = 0$, $\partial\mathbf{y} = 0$, and $\widehat{HF}(\mathcal{H};J) \cong \mathbb{F}_2^2$.

EXAMPLE 5.4. As far we have only see examples where the connecting disks are essentialy unique by the Riemann mapping theorem, but this is no longer true when dealing with more complicated domains, where very complicated questions in conformal geometry may show up. Consider for example the following index 1 region with the shape of an annulus.



This region admits a uniformization map from a standard annulus $A = \{r < |z| < R\} \subset \mathbb{C}\}$, where the parameter R/r is determined by the conformal structure on the surface Σ. A holomorphic curve $A \to \Sigma \times [0,1] \times \mathbb{R}$ is then determined by a double branched covering $A \to [0,1] \times \mathbb{R}$ where the α-part of the boundary maps to $\mathrm{Re} z = 1$ and the β-part maps to $\mathrm{Re} z = 0$. As the automorphism group of A is generated by rotations $z \mapsto e^{i\theta} z$ and the inversion $z \mapsto \frac{r}{Rz}$, such a branched covering exists (and is unique up to translations) if and only if the two α-parts of the boundary determine the same angle. This observation can be used to create chain complexes depending on the complex structure. For example the following is a Heegaard diagram for S^3 isotopic to the second one of example 1.3.



Here the annulus D_1 is a domain connecting $\{x_3, y_2\}$ to $\{x_2, y_3\}$. Denote by θ and ϕ respectively the angles in the uniformization determined by the α-arcs connecting x_3 to x_2 and y_3 to y_2. Then there is a holomorphic representative of D_1 if and only if $\theta = \phi$, so for generic choice of the almost complex structure $\#\big(\mathcal{M}^{D_1}(\{x_3, y_3\}, \{x_2, y_2\})\big) = 0$.

On the other hand the domain $D_1 + D_2$ determines a 1-parameter family of annuli connecting $\{x_3, y_3\}$ to $\{x_2, y_3\}$ (obtained by cutting along the α-arc from y_3 to y_2). By analyzing the conformal angles one obtains that

$$\#\big(\mathcal{M}^{D_1+D_2}(\{x_3, y_3\}, \{x_2, y_3\})\big) = \begin{cases} 1 & \text{if } \theta < \phi, \\ 0 & \text{if } \theta > \phi \end{cases}$$

so the differential of $\widehat{CF}(\mathcal{H}; J)$ really depends on J. Anyway, one can check that in both cases $\widehat{HF}(\mathcal{H}; J) \cong \mathbb{F}_2$.

Example 5.5. As we have seen in the previous example, the differentials in the Floer chain complex are not determined in a combinatorial way. This makes Heegaard Floer homology really difficult to compute in general (this is in fact a common aspects of all Floer theories), and for example it is not known how to compute the Floer complex for the diagram of the Poincaré homology sphere of example 1.5. Anyway the homology of such a complex is computable using other techniques (for example the *surgery exact sequence*, see [**OS04b**], or *nice Heegaard diagrams*, see [**SW10**]), and we will return on these later in chapter 3.

6. Invariance

We now turn to the invariance of Heegaard Floer homology. This is a quite mysterious and surprising result, as in the construction of the chain complex $\widehat{CF}(\mathcal{H}; J)$ we have made many choices.

THEOREM 6.1. *The homology of the Floer chain complex $\widehat{HF}(\mathcal{H}; J)$ does not depend on the choice of the admissible Heegaard diagram $\mathcal{H}$ and the generic admissible almost complex structure J, and so defines an invariant of the 3-manifold Y.*

We denote this invariant by $\widehat{HF}(Y)$, the *Heegaard Floer homology* of Y.

We now discuss the main ideas behind the proof of the invariance of Heegaard Floer homology. The details of the proof are quite long and technical, and we refer the reader to [**Lip06**] for a complete treatment.

In light of proposition 4.4, in order to prove invariance one has just to prove that $\widehat{HF}(\mathcal{H}; J)$ is invariant under the following operations (when they preserve admissibility):

- change of the generic admissible almost complex structure J;
- isotopy of the curves;
- handleslides;
- stabilization of the Heegaard diagram.

We address each one of the operations separately in the rest of the section.

6.1. Change of almost complex structure. Suppose we are given two admissible almost complex structures J_1 and J_2 on $\Sigma \times [0,1] \times \mathbb{R}$. In order to prove that $\widehat{HF}(\mathcal{H}; J_1) \cong \widehat{HF}(\mathcal{H}; J_2)$ we construct a chain homotopy $\Phi : (\widehat{CF}(\mathcal{H}; J_1), \partial_{J_1}) \to (\widehat{CF}(\mathcal{H}; J_2), \partial_{J_2})$ by counting some special holomorphic curves.

For a fixed $T > 0$, one can choose an almost complex structure J *connecting* J_1 and J_2 on $\Sigma \times [0,1] \times \mathbb{R}$, i.e. an almost complex structure such that:

- J agrees with J_1 on $\Sigma \times [0,1] \times (-\infty, -T]$ and with J_2 on $\Sigma \times [0,1] \times [T, +\infty)$;
- J satisfies the properties $(1), (2)$ and (4) of definition 3.1;
- J achieves transversality.

Note that in this case J is not $\mathbb{R}$-invariant, so there is not the usual translation action on the moduli spaces. This implies that also index 0 homology classes contain non trivial holomorphic curves, and in fact we use them to define the map

$$\Phi : \widehat{CF}(\mathcal{H}; J_1) \to \widehat{CF}(\mathcal{H}; J_2) \qquad \mathbf{x} \mapsto \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mathrm{ind}(B) = 0}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; J)) \cdot \mathbf{y}.$$

LEMMA 6.2. *Φ is a well defined chain map.*

PROOF. The sum is finite by the usual admissibility argument.

To show that this is a chain map one has to consider the ends of 1-dimensional moduli spaces, i.e. in this case maps representing homology classes $B \in \pi_2(\mathbf{x}, \mathbf{y})$ with index 1.

Again, the only degeneration that may happen is the creation of a two story holomorphic building, and by dimensional considerations and gluing results there is a bijection between $\partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; J)$ and the following two types of singular curves:

- a curve u_1 in a class $B_1 \in \pi_2(\mathbf{x}, \mathbf{w})$ with $\operatorname{ind}(B_1) = 0$ holomorphic with respect to J followed by a curve u_2 in a class $B_2 \in \pi_2(\mathbf{w}, \mathbf{y})$ with $\operatorname{ind}(B_2) = 1$, holomorphic with respect to J_2, with $B_1 * B_2 = B$;
- a curve u_1 in a class $B_1 \in \pi_2(\mathbf{x}, \mathbf{w})$ with $\operatorname{ind}(B_1) = 1$ holomorphic with respect to J_1 followed by a curve u_2 in a class $B_2 \in \pi_2(\mathbf{w}, \mathbf{y})$ with $\operatorname{ind}(B_2) = 0$, holomorphic with respect to J, with $B_1 * B_2 = B$.

The first case corresponds to the $\mathbf{y}$ coefficient of $\partial_{J_2} \circ \Phi(\mathbf{x})$, while the second one to the $\mathbf{y}$ coefficient of $\Phi \circ \partial_{J_1}(\mathbf{x})$. As the ends of a 1-manifold always come in a even number, $\partial_{J_2} \circ \Phi + \Phi \circ \partial_{J_1} = 0$, i.e. Φ is a chain map. □

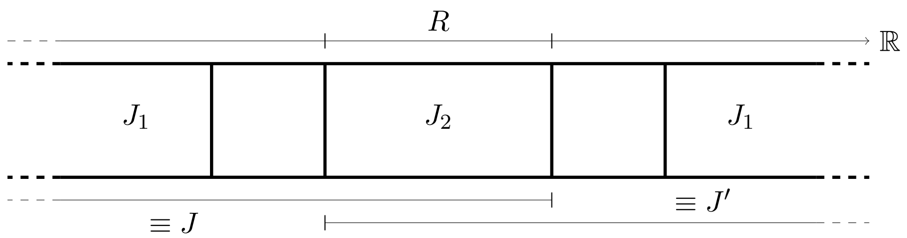
LEMMA 6.3. *Given two generic admissible almost complex structures J and J' connecting J_1 and J_2, the induced chain maps Φ, Φ' are chain homotopic.*

PROOF. Choose a generic path $\{J_t\}_{t\in[0,1]}$ connecting J and J'. Here the genericity of the path implies that for all but finitely many t the complex structure J_t achieves transversality, and for only finitely many $0 < t_1 < \cdots < t_k < 1$ there are non empty moduli spaces of J_t-holomorphic curves with index -1. Define the map

$$H : \widehat{CF}(\mathcal{H}; J_1) \to \widehat{CF}(\mathcal{H}; J_2) \qquad \mathbf{x} \mapsto \sum_{i=1}^{k} \sum_{\mathbf{y}\in\mathfrak{S}(\mathcal{H})} \sum_{\substack{B\in\pi_2(\mathbf{x},\mathbf{y}) \\ \operatorname{ind}(B)=-1}} \#\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; J_{t_k})\big) \cdot \mathbf{y}.$$

By considering the ends of the 1-manifolds $\bigcup_{t\in[0,1]} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; J_t)$ with $\operatorname{ind}(B) = 0$ (there are four different kinds of them), one gets as in the previous lemma $\partial_{J_2} \circ H + H \circ \partial_{J_1} = \Phi + \Phi'$, i.e. that H is a chain homotopy between Φ and Φ'. □

It is quite easy now to conclude. We get a map $\Psi : \widehat{CF}(\mathcal{H}; J_2) \to \widehat{CF}(\mathcal{H}; J_1)$ by counting curves in an almost complex structure J' connecting J_2 and J_1. Fix $R > 0$ and define the almost complex structure $J\natural_R J'$ connecting J_1 to J_1 defined by taking their 'connected sum', where the central part coinciding with J_2 has lenght R.



Then for R big enough the composite map $\Psi \circ \Phi$ coincides with the map F obtained by counting holomorphic curves in the almost complex structure $J\natural_R J'$. This is true because for a fixed $B \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\operatorname{ind}(B) = 0$ by gluing and compactness lemmas it follows that for $R > 0$ large enough

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; J\natural_R J') = \coprod \mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; J) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; J')$$

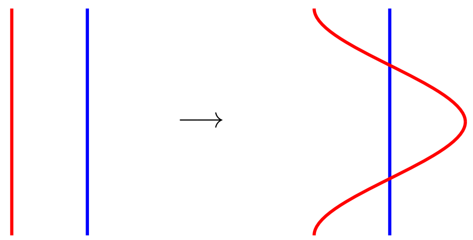
where the union is taken over $\mathbf{w} \in \mathfrak{S}(\mathcal{H})$ and index 0 classes $B_1 \in \pi_2(\mathbf{x}, \mathbf{w})$ and $B_2 \in \pi_2(\mathbf{w}, \mathbf{y})$.

For this reason, by the previous lemma $\Psi \circ \Phi$ is homotopy equivalent to the map induced by the almost complex structure J_1 (seen as an almost complex structure connecting J_1 to

J_1) which is clearly the identity. As the same also applies to $\Phi \circ \Psi$, we have that the two chain complexes are homotopy equivalent.

6.2. Isotopies. Define a *basic isotopy* to be an isotopy $\{\boldsymbol{\alpha}_t\}, \{\boldsymbol{\beta}_t\}$ of one of the two following forms:

(1) for each time t, $\boldsymbol{\alpha}_t$ and $\boldsymbol{\beta}_t$ are transverse;
(2) the isotopy introduces a single new pair of intersection points between $\boldsymbol{\alpha}_t$ and $\boldsymbol{\beta}_t$ (we call this a *finger move*).



It is clear that two isotopic Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ are isotopic through a sequence of basic isotopies, so in order to prove invariance under isotopy it suffices to address those simple cases separately.

The key observation for the first case is that it is equivalent to a deformation of the almost complex structure. In fact, there exists a diffeomorphism $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ between the Heegaard diagrams and computing $\widehat{HF}(\mathcal{H}_2; J)$ is exactly the same as computing $\widehat{HF}(\mathcal{H}_1; (\psi \times \mathrm{Id}_{\mathbb{D}})^* J)$, so invariance descends form the discussion of the previous subsection.

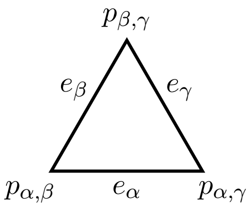
The proof in the second case is also similar in spirit to the complex structure change case. In this case one chooses a collection of lagrangian cylinders in $\Sigma \times [0,1] \times \mathbb{R}$ which agrees with $(\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})$ near $-\infty$ and $(\boldsymbol{\alpha}' \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta}' \times \{0\} \times \mathbb{R})$ near $+\infty$. As in the previous case, the absence of $\mathbb{R}$-invariance makes interesting to study index 0 moduli spaces of holomorphic curves with boundary in C in order to define maps $\widehat{CF}(\mathcal{H}_1) \to \widehat{CF}(\mathcal{H}_2)$. The structure at ∞ of C implies that the analogue maps are chain maps (by considering the ends of index 1 classes), and inverses are constructed in a similar way.

6.3. Handleslides. This case is surely the most subtle one. In fact, all the invariance proofs we have sketched before are fairly standard in the symplectic field theory context, while in order to prove handleslide invariance we will need to count special holomorphic curves into $\Sigma \times T$, where T is a holomorphic triangle. We briefly describe this construction (which is really useful in general to define 4-manifold invariants) before sketching the proof of invariance.

We start with a *pointed Heegaard triple* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ where we consider three families of pairwise disjoint homologically independent simple closed curves on Σ rather than two. This clearly determines three Heegaard diagrams $\mathcal{H}_{\alpha,\beta}, \mathcal{H}_{\beta,\gamma}, \mathcal{H}_{\alpha,\gamma}$. One can construct a map

$$\widehat{F}_{\alpha,\beta,\gamma} : \widehat{CF}(\mathcal{H}_{\alpha,\beta}; J_{\alpha,\beta}) \otimes \widehat{CF}(\mathcal{H}_{\beta,\gamma}; J_{\beta,\gamma}) \to \widehat{CF}(\mathcal{H}_{\alpha,\gamma}; J_{\alpha,\gamma})$$

in the following way. Consider a triangle T, i.e. a disk $\mathbb{D}$ with three punctures $p_{\alpha,\beta}, p_{\beta,\gamma}, p_{\alpha,\gamma}$ on the boundary



and endow $\Sigma \times T$ with an almost complex structure with properties analogue to those of definition 3.1 such that in a cylindrical neighborhood of $\Sigma \times p_{\alpha,\beta}$ it agrees with $J_{\alpha,\beta}$, and similarly for the other points. Then $\widehat{F}_{\alpha,\beta,\gamma}$ is constructed counting holomorphic maps

$$(S, \partial S) \to \big(\Sigma \times T, (\boldsymbol{\alpha} \times e_\alpha) \cup (\boldsymbol{\beta} \times e_\beta) \cup (\boldsymbol{\gamma} \times e_\gamma)\big)$$

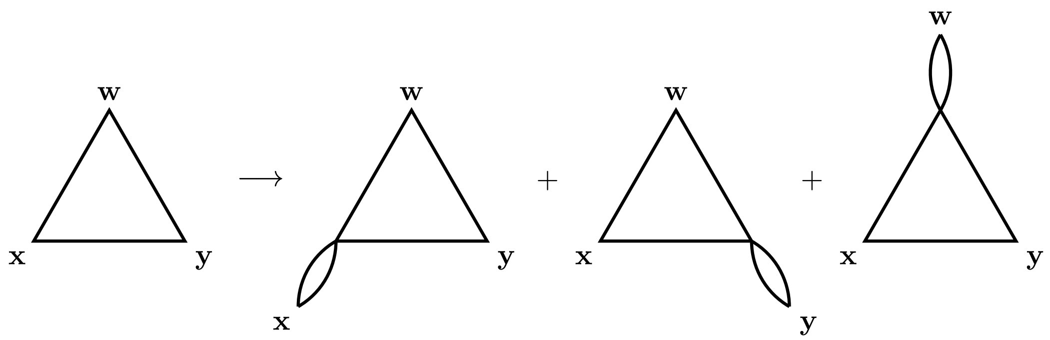
connecting $\mathbf{x} \in \mathfrak{S}(\mathcal{H}_{\alpha,\beta})$ at $p_{\alpha,\beta}$, $\mathbf{y} \in \mathfrak{S}(\mathcal{H}_{\beta,\gamma})$ at $p_{\beta,\gamma}$ and $\mathbf{w} \in \mathfrak{S}(\mathcal{H}_{\alpha,\gamma})$ at $p_{\alpha,\gamma}$, and satisfying some generalizations of the conditions required to define the Floer chain complex. In particular also indexes and homology classes of maps have generalizations, and the map is

$$\widehat{F}_{\alpha,\beta,\gamma}(\mathbf{x} \otimes \mathbf{y}) = \sum_{\mathbf{w} \in \mathfrak{S}(\mathcal{H}_{\alpha,\gamma})} \sum_{\substack{B \in \pi_2(\mathbf{x},\mathbf{y},\mathbf{w}) \\ \mathrm{ind}(B)=0}} \#(\mathcal{M}^B(\mathbf{x},\mathbf{y},\mathbf{w})) \cdot \mathbf{w}.$$

Studying the ends of 1-dimensional moduli spaces one proves that

$$\widehat{F}_{\alpha,\beta,\gamma} \circ \partial_{\alpha,\beta} + \widehat{F}_{\alpha,\beta,\gamma} \circ \partial_{\beta,\gamma} + \partial_{\alpha,\gamma} \circ \widehat{F}_{\alpha,\beta,\gamma} = 0$$

where here we use the compatibility of the structures near the punctures of T. Intuitively this relation can be drawn as follows.



In particular, $\hat{F}_{\alpha,\beta,\gamma}$ is a chain map, and so it induces the map in homology

$$\widehat{F}_{\alpha,\beta,\gamma} : \widehat{HF}(\mathcal{H}_{\alpha,\beta}) \otimes \widehat{HF}(\mathcal{H}_{\beta,\gamma}) \to \widehat{HF}(\mathcal{H}_{\alpha,\gamma})$$

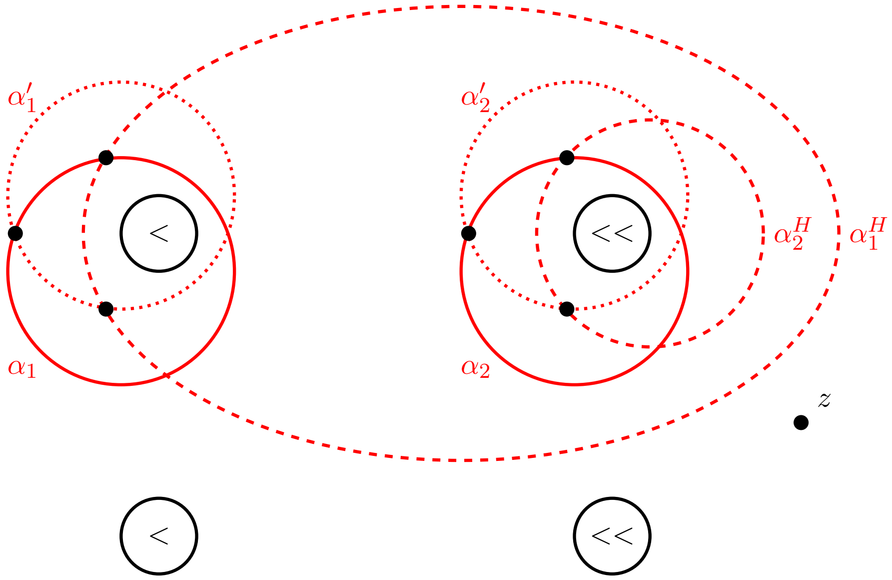
which can be seen to be independent of the complex structure and the isotopy class of the curves.

Those maps satisfy the following nice associativity property (which is proved by counting holomorphic curves in $\Sigma \times R$, where R is a rectangle).

PROPOSITION 6.4. *Given a Heegaard quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$ (whose definition is obvious) then for every $\mathbf{x} \in \widehat{HF}(\mathcal{H}_{\alpha,\beta})$, $\mathbf{y} \in \widehat{HF}(\mathcal{H}_{\beta,\gamma})$ and $\mathbf{w} \in \widehat{HF}(\mathcal{H}_{\gamma,\delta})$ we have*

$$\widehat{F}_{\alpha,\gamma,\delta}(\hat{F}_{\alpha,\beta,\gamma}(\mathbf{x} \otimes \mathbf{y}) \otimes \mathbf{w}) = \hat{F}_{\alpha,\beta,\delta}(\mathbf{x} \otimes \hat{F}_{\beta,\gamma,\delta}(\mathbf{y} \otimes \mathbf{w})) \in \widehat{HF}(\mathcal{H}_{\alpha,\delta}).$$

We are now ready to sketch the proof of the handleslide invariance. Given a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, we want to prove invariance with respect to a handleslide of α_1 over α_2. One can construct auxiliary sets of curves $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}^H$ as in the next figure (which represents the case $g = 2$).



In particular, $\boldsymbol{\alpha}'$ is obtained by $\boldsymbol{\alpha}$ by a small isotopy which introduces two new intersection points for each curve, and $\boldsymbol{\alpha}^H$ is a small isotopy of the handleslided set of curves, such that the intersection are as in the figure. The four sets of attaching curves $\boldsymbol{\alpha}, \boldsymbol{\alpha}', \boldsymbol{\alpha}^H, \boldsymbol{\beta}$ define a Heegaard quadruple. In particular there are special generators $\theta_{\alpha,\alpha'} \in \mathfrak{S}(\mathcal{H}_{\alpha,\alpha'})$, $\theta_{\alpha',\alpha^H} \in \mathfrak{S}(\mathcal{H}_{\alpha',\alpha^H})$ and $\theta_{\alpha,\alpha^H} \in \mathfrak{S}(\mathcal{H}_{\alpha,\alpha^H})$ determined by the marked intersection points.

REMARK 6.5. These are actually the generators with maximum grading, in the sense of subsection 7.2.

The following proposition proves handleslide invariance.

PROPOSITION 6.6. *The map*

$$\widehat{F}_{\beta,\alpha,\alpha^H}(\cdot \otimes \theta_{\alpha,\alpha^H}) : \widehat{HF}(\mathcal{H}_{\beta,\alpha}) \to \widehat{HF}(\mathcal{H}_{\beta,\alpha^H})$$

is an isomorphism.

Supersketchily, the proof follows from the chain of equalities

$$\begin{aligned}\widehat{F}_{\beta,\alpha,\alpha'}(\widehat{F}_{\beta,\alpha,\alpha^H}(\cdot \otimes \theta_{\alpha,\alpha^H}) \otimes \theta_{\alpha^H,\alpha'}) = \widehat{F}_{\beta,\alpha,\alpha'}(\cdot \otimes \widehat{F}_{\alpha,\alpha^H,\alpha'}(\theta_{\alpha,\alpha^H} \otimes \theta_{\alpha^H,\alpha'})) = \\ = \widehat{F}_{\beta,\alpha,\alpha'}(\cdot \otimes \theta_{\alpha,\alpha'}) = \Phi_{\alpha,\alpha'}(\cdot)\end{aligned}$$

where the first identity is the associativity formula for triangles, the second follows from a local computation and $\Phi_{\alpha,\alpha'}$ is the map induced by a isotopy as discussed in the previous subsection, which we know is an isomorphism. In particular, this implies that the map we are interested in is injective. Switching the roles of $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^H$, the same argument proves the surjectivity of our map.

6.4. Stabilizations. Here is the last Heegaard move to check. Because of the handleslide invariance, it is enough to prove the result in the case in which one makes the connected sum at the point z of $\mathcal{H}$ with the standard genus 1 Heegaard diagram $(\mathcal{T}, \alpha_{g+1}, \beta_{g+1}, w)$ for S^3 from example 1.3, obtaining a diagram $\mathcal{H}'$. This turns out to be quite easy.

In fact there is a clear correspondence between generators

$$\mathbf{x} = (x_1, \dots, x_g) \in \mathfrak{S}(\mathcal{H}) \mapsto (x_1, \dots, x_g, \alpha_{g+1} \cap \beta_{g+1}) \in \mathfrak{S}(\mathcal{H}').$$

Furthermore, as we are considering homology classes B with $n_z(B) = 0$, it is clear that disks involving the intersection point $\alpha_{g+1} \cap \beta_{g+1}$ have to be trivial, and so there is also a canonical correspondence between moduli spaces (for the right choice of the almost complex structure).

7. Additional structures

As already said in the introduction of this chapter, $\widehat{HF}(Y)$ is the simplest version of Heegaard Floer homology. In this last section we briefly discuss some of its variants and refinements.

7.1. Spin^c**-structures.** A Spin^c-structure on a 3-manifold Y is a lift of the $SO(3)$-structure of its tangent bundle TY to a Spin$^c(3)$ structure via the canonical projection Spin$^c(3) \to SO(3)$. A more practical way to see Spin^c structures on 3-manifolds is due to Turaev [**Tur97**]. Recall that as $\chi(Y) = 0$, there always exists a nowhere vanishing vector field on Y.

Definition 7.1. Define two nowhere vanishing vector fields to be *homologous* if they are homotopic outside a finite disjoint union of balls. A Spin^c-*structure* is a homology class of vector fields.

Given a pointed Heegard diagram $\mathcal{H}$, there is a natural way to assign to each generator $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ a Spin^c-structure $\mathfrak{s}_z(\mathbf{x})$, where the z stresses the dependence on the basepoint.

Construction 7.2. Consider a self-indexing Morse function $f : Y \to \mathbb{R}$ which induces the Heegaard diagram via construction 1.6. Then $\mathbf{x}$ determines g trajectories of the gradient vector field $-\nabla f$ connecting the index 1 and the index 2 critical points, and z determines a trajectory from the index 0 to the index 3 point.

Outside a tubular neighborhood of those $g + 1$ trajectories the vector field $-\nabla f$ is not vanishing, and as each trajectory connects critical points with opposite parity, this can be extended to a globally non-vanishing vector field. Then $\mathfrak{s}_z(\mathbf{x})$ is the well-defined homology class of this vector field.

The following purely topological lemma is the key of the introduction of Spin^c-structures.

Lemma 7.3. *For* $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$, $\pi_2(\mathbf{x}, \mathbf{y})$ *is non empty if and only if* $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_z(\mathbf{y})$.

So, our Floer chain complex intrinsically splits as a direct sum of chain complexes generated by intersection points within a Spin^c-structure $\mathfrak{s}$, which we denote by $\widehat{CF}(\mathcal{H}, \mathfrak{s}; J)$, and so does the homology.

$$\widehat{HF}(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s}).$$

7.2. Gradings. There is a relative grading on each $\widehat{HF}(Y, \mathfrak{s})$. Given two generators $\mathbf{x}, \mathbf{y}$ within the Spin^c-structure $\mathfrak{s}$, define $\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \operatorname{ind} B$ for $B \in \pi_2(\mathbf{x}, \mathbf{y})$ (which is non empty by lemma 7.3). This is well defined up to indexes of periodic classes, and one can prove ([**OS04c**]) that those are all multiples of

$$\mathfrak{d}(\mathfrak{s}) = \gcd_{\xi \in H_2(Y)} \langle c_1(\mathfrak{s}), \xi \rangle$$

where $c_1(\mathfrak{s})$ is the first Chern class of the Spin^c-structure, i.e. the first Chern class of the oriented 2-plane field complementary to a vector field representing $\mathfrak{s}$, and so one obtains a $\mathbb{Z}/\mathfrak{d}(\mathfrak{s})\mathbb{Z}$-valued relative grading.

In some cases, with more work one can extend those relative gradings to absolute $\mathbb{Q}$ gradings, see [**OS03a**].

7.3. Variants of $\widehat{HF}$. So far we have concentrated only on the hat version of Heegaard Floer homology, which will be the one we will be interested in for the rest of the work. This is defined by counting curves which do not intersect the basepoint $z \in \Sigma$, but actually there are more refined versions where one counts also curves intersecting it, taking account of the number of intersections.

We describe the infinity version of Heegaard Floer homology. Given a Heegaard diagram $\mathcal{H}$ and chosen a generic admissible almost complex structure J, the complex $CF^\infty(\mathcal{H}; J)$ is generated over $\mathbb{F}_2$ by pairs $[\mathbf{x}, i]$ with $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ and $i \in \mathbb{Z}$ and has boundary map

$$\partial^\infty [\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \operatorname{ind} B = 1}} \# \mathcal{M}^B(\mathbf{x}, \mathbf{y}) \cdot [\mathbf{y}, i - n_z(B)]$$

where here in the definition of the moduli spaces we consider also curves intersecting the basepoint. The homology of this complex is denoted by $HF^\infty(\mathcal{H}; J)$ and is in fact an invariant of the 3-manifold $HF^\infty(Y)$. Note that in this case our admissibility condition does not assure that the sum is well defined, as by the proof of proposition 4.7 we can only conclude that for fixed j there are only finitely homology classes B with $n_z(B) = j$ supporting holomorphic curves. The proof that this is effectively a chain complex and that the homology in an invariant of the 3-manifold is similar to that in the hat case, but technically much more complicated.

Other very interesting invariants can be constructed in an algebraic way. The subspace $CF^-(\mathcal{H}; J) \subset CF^\infty(\mathcal{H}; J)$ is the one generated by pairs $[\mathbf{x}, i]$ with $i < 0$. It is straightforward from lemma 3.3 that this is a subcomplex, and the homology $HF^-(\mathcal{H}; J)$ turns out to be an invariant of the manifold. In the same manner the quotient complex $CF^+(\mathcal{H}; J) = CF^\infty(\mathcal{H}; J)/CF^-(\mathcal{H}; J)$ gives an invariant of the manifold $HF^+(Y)$.

All these variants are really interesting because using them one can construct invariants of closed 4-manifolds [**OS06**], which are conjecturally equivalent to the Seiberg-Witten invariants.

7.4. Orientations. We have defined all our invariants with $\mathbb{F}_2$ coefficients, but actually with some more work one can lift all the theory to a construction with $\mathbb{Z}$ coefficients. To do this, one need to put an orientation on the moduli spaces, so one can define a differential that count the points with signs. This choice of orientations has to be coherent, so that counting the ends on 1-dimensional moduli spaces one finds that the boundary map is actually a differential.

It turns out (see [**OS04c**]) that there are $2^{b_1(Y)}$ different orientation conventions that may lead to different results. For example one gets that the homology of $S^2 \times S^1$ is $\mathbb{Z} \oplus \mathbb{Z}$ in one convention, and $\mathbb{F}_2$ in the other, depending if we count the two disks of example 5.3 as points with the same or the opposite orientation. In any case, for each 3-manifold one can construct a privileged one (see [**OS04b**]), which gives back for example the expected computation for $S^2 \times S^1$ with $\mathbb{F}_2$ coefficients.

CHAPTER 2

The bordered invariants

In this chapter we introduce the objects of bordered Heegaard Floer homology ([**LOT11b**]), which are invariants of a 3-manifold Y together with a suitably parametrized boundary component. In particular, bordered Heegaard Floer homology associates:

- to a closed connected and oriented surface F with a fixed handle decomposition (determined by a pointed matched circle $\mathcal{Z}$) a differential algebra $\mathcal{A}(\mathcal{Z})$;
- to a connected and oriented 3-manifold Y with one boundary component ∂Y together with an orientation preserving diffeomorphism $F \to \partial Y$ a differential module $\widehat{CFD}(Y)$ over $\mathcal{A}(-\mathcal{Z})$ (well defined up to homotopy equivalence) and an $\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$ over $\mathcal{A}(\mathcal{Z})$ (well defined up to $\mathcal{A}_\infty$ homotopy equivalence).

The basic idea behind the whole construction consists in the study of what happens to holomorphic curves in $\Sigma \times [0,1] \times \mathbb{R}$ when Σ is cut along a separating curve Z containing the basepoint z into two pieces Σ_1 and Σ_2. If Z does not intersect any attaching curve, this operation is the inverse of a connected sum of Heegaard diagrams, and calling $\mathcal{H}_1$ and $\mathcal{H}_2$ those Heegaard diagrams, one has immediately as in subsection 6.4 of chapter 1 that for appropriate choices of the almost complex structure there is the quasi-isomorphism

$$\widehat{CF}(\mathcal{H}) \cong \widehat{CF}(\mathcal{H}_1) \otimes \widehat{CF}(\mathcal{H}_2).$$

This operation corresponds to cut the 3-manifold along a separating sphere, and the aim of bordered Heegaard Floer homology is to generalize this to situations where the cutting surface is more complicated, or, similarly, the separating circle Z intersects some α-curves. The big complication here is that while in the previous case the two pieces did not interact at all, here one can have holomorphic curves crossing this circle. The key idea is to encode those curves as elements in an algebra $\mathcal{A}(\mathcal{Z})$ associated to Z, as differentials (which motivates the D) in the algebraic object $\widehat{CFD}$ associated to $\mathcal{H}_2$ and as actions (which motivates the A) of the elements of $\mathcal{A}(\mathcal{Z})$ on the algebraic object $\widehat{CFA}$ associated to $\mathcal{H}_1$.

This idea is very neat when one considers *planar grid diagrams* ([**LOT09**]). In these objects inspired by combinatorial knot Floer homology ([**MOS09**]) the differential is obtained by counting special rectangles in a grid, and they serve as a really nice toy model to get acquainted with the theory. In any case, the real world is much more complicated both from the algebraic and analytical point of view, so we will not enter the details of that construction, for which we refer the reader to the original paper.

This is the plan for the chapter. In section 1 we show how to describe surfaces via pointed matched circles and construct the algebras associated to them. Section 2 is dedicated to the topological preliminaries of the construction, namely bordered Heegaard diagrams and homology classes of curves. In section 3 we sketch the analytical aspects of bordered Heegaard Floer homology, discussing moduli spaces of holomorphic curves with asymptotics at east infinity and their compactifications. Finally we will be ready to define the main algebraic objects. In particular, in section 4 we will introduce the type D modules and after having recalled the basic definitions regarding $\mathcal{A}_\infty$ algebras in section 5, we will construct the type A modules in section 6. Finally we briefly discuss the invariance properties of such objects in section 7.

Many of the proofs are adaptations of the ones in the closed case we have already sketched in chapter 1, but the details of the whole construction are quite long and technical so we will not deepen them too much referring the reader to the original paper [**LOT11b**], preferring to explain the main ideas using some practical local examples.

1. Pointed matched circles and their algebras

First of all, we discuss the algebra associated to a surface parametrized by means of matched circles. This will be a subalgebra of the strands algebra $\mathcal{A}(n)$ generated by combinatorially determined sums of generators. We also discuss how this construction interacts with Reeb chords, which are combinatorial objects coming from the geometry of holomorphic curves.

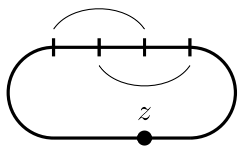
1.1. Pointed matched circles. We will denote the set $\{1, \ldots, n\}$ by $[n]$.

Definition 1.1. A *matched circle* is a triple $(Z, \mathbf{a}, M)$ consisting of an oriented circle Z, a subset $\mathbf{a} \subset Z$ of $4k$ points and a matching function $M : \mathbf{a} \to [2k]$ such that after performing surgery along the $2k$ pairs of matched points (which are copies of S^0) one obtains a connected 1-manifold. A *pointed matched circle* $\mathcal{Z}$ is a matched circle together with a basepoint $z \in Z \backslash \mathbf{a}$.

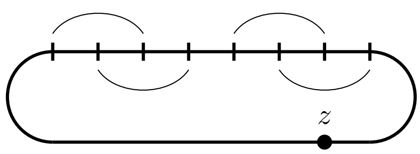
To each pointed matched circle $\mathcal{Z}$ with $4k$ points one can associate uniquely an oriented surface $F(\mathcal{Z})$ of genus k, the *surface associated to* $\mathcal{Z}$. Take a disk with boundary Z (with the correct orientation), and attach $2k$ 2-dimensional 1-handles as specified by the matching function. By hypothesis, the resulting manifold has connected boundary, which one can cap off in order to get a closed oriented surface $F(\mathcal{Z})$.

Note that the associated surface comes with a specified handle decomposition with only one 0-handle and one 2-handle, and we will think of this as a parametrization.

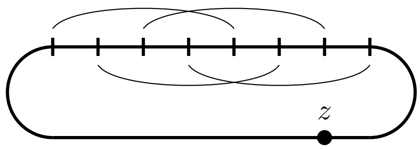
Example 1.2. The following is the simplest pointed matched circle, whose associated surface is the torus.



Example 1.3. A simple example of pointed matched circle with associated surface the genus 2 surface is obtained by taking the *connected sum* of two pointed matched circles as in the previous example at the basepoint.



The following is also a pointed matched circle with associated surface the genus 2 one, but with a different handle decomposition.



1.2. Reeb chords. We now give the basic definitions regarding Reeb chords, which are arcs in Z describing the asymptotics at east infinity of our holomorphic curves.

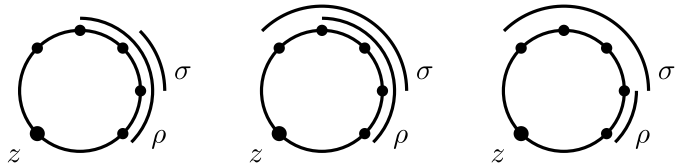
DEFINITION 1.4. Given an oriented circle Z with marked points $\mathbf{a}$ and a basepoint $z \in Z \setminus \mathbf{a}$, a *Reeb chord* ρ is an embedded arc in $Z \setminus \{z\}$ with endpoints in $\mathbf{a}$, with the orientation induced by the circle Z. We denote the initial and the end point of the Reeb chord ρ respectively by ρ^- and ρ^+.

NOTATION 1.5. The name Reeb chord is due to the fact that one can think an end Z of punctured surface as a contact 1-manifold. Then the points $\mathbf{a}$ are legendrian 0-submanifolds, and the arcs connecting them can be seen as the flow lines of the Reeb vector field (for the basic definitions in contact geometry see for example [**Gei08**]).

Clearly a Reeb chord is uniquely determined by its initial and endpoint ρ^- and ρ^+, so one can use the notation $\rho = [\rho^-, \rho^+]$. Here are the basic definitions and operations regarding Reeb chords. Note that the orientation of the circle together with the basepoint determines an ordering on the set of marked points $\mathbf{a}$.

DEFINITION 1.6. An ordered pair of Reeb chords (ρ, σ) is:
- *nested* if $\rho^- < \sigma^- < \sigma^+ < \rho^+$;
- *interleaved* if $\rho^- < \sigma^- < \rho^+ < \sigma^+$;
- *abutting* if $\rho^+ = \sigma^-$.

These definitions have a clear geometric meaning.



Note that these relations are all asymmetric. When ρ and σ are abutting, one can define their *join* as the Reeb chord $\rho \uplus \sigma = [\rho^-, \sigma^+]$. A *splitting* of a Reeb chord ρ_1 is a pair of abutting Reeb chords (ρ_2, ρ_3) such that $\rho_2 \uplus \rho_3 = \rho_1$.

For the definition of our invariants, we will be interested in sets of Reeb chords $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_j\}$, and the following operations between them will describe the degenerations of holomorphic curves. Let $\boldsymbol{\rho}^- = \{\rho_1^-, \dots, \rho_j^-\}$ and $\boldsymbol{\rho}^+ = \{\rho_1^+, \dots, \rho_j^+\}$ be the sets of initial and end points of the chords in $\boldsymbol{\rho}$.

DEFINITION 1.7. A set of j Reeb chords $\boldsymbol{\rho}$ is *consistent* if $\boldsymbol{\rho}^-$ and $\boldsymbol{\rho}^+$ are both sets of j points, i.e. no pair of Reeb chords have the same initial or endpoint.

Given two consistent sets of Reeb chords $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$, $\boldsymbol{\rho} \uplus \boldsymbol{\sigma}$ is obtained from the union $\boldsymbol{\rho} \cup \boldsymbol{\sigma}$ by replacing each abutting pair $\rho_k \in \boldsymbol{\rho}$ and $\sigma_l \in \boldsymbol{\sigma}$ by their join $\rho_k \uplus \sigma_l$. Given $\rho_k \in \boldsymbol{\rho}$, one can define

$$\rho_k^{++} = \begin{cases} \sigma_l^+ & \text{if for some } \sigma_l \in \boldsymbol{\sigma},\ \rho_k, \sigma_l \text{ abut} \\ \rho_k^+ & \text{otherwise.} \end{cases}$$

Note that this is well defined because of the consistency hypothesis. In the same manner one can define σ_j^{--} for a Reeb chord $\sigma_j \in \boldsymbol{\sigma}$.

DEFINITION 1.8. Two consistent sets of Reeb chords $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are *composable* if $\boldsymbol{\rho} \uplus \boldsymbol{\sigma}$ is consistent and has no *double crossings*, i.e. there are no $\rho_k \in \boldsymbol{\rho}$ and $\sigma_l \in \boldsymbol{\sigma}$ such that:

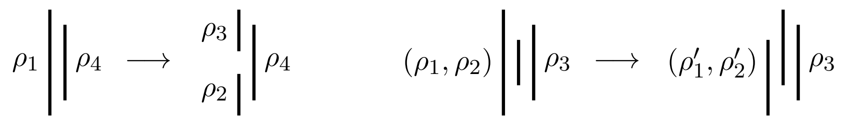
- $\rho_i^- < \sigma_j^{--}$;
- $\sigma_j^- < \rho_i^+$;
- $\rho_i^{++} < \sigma_j^+$.

This notion can be easily iterated to sequences of consistent sets of Reeb chords, i.e. $(\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n)$ is *composable* if for each $i = 1, \dots, n-1$ the consistent sets of Reeb chords $\biguplus_{j=1}^i \boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{i+1}$ are composable.

A *splitting* of $\boldsymbol{\rho}$ is a set of Reeb chords $\boldsymbol{\rho}'$ obtained by substituting to a chord $\rho_1 \in \boldsymbol{\rho}$ a splitting $\{\rho_2, \rho_3\}$ of it, such that $\boldsymbol{\rho}'$ is consistent and the operation does not introduce double crossings, i.e. there is no chord ρ_4 nested in ρ_1 such that $\rho_4^- < \rho_2^+ = \rho_3^- < \rho_4^+$.

A *shuffle* of $\boldsymbol{\rho}$ is a set of Reeb chords $\boldsymbol{\rho}'$ obtained by replacing a pair of nested chords (ρ_1, ρ_2) by the interleaved pair $(\rho_1', \rho_2') = ([\rho_1^-, \rho_2^+], [\rho_2^-, \rho_1^+])$, such that no double crossing is introduced, i.e. there is no $\rho_3 \in \boldsymbol{\rho}$ such that ρ_2 is nested in ρ_3 and ρ_3 is nested in ρ_1.

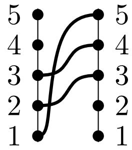
REMARK 1.9. One can also introduce the notion of *weak splitting* and *weak shuffle*, where we do not require the absence of double crossings (these are graphically explained in the next figure).



This further condition (which name will be justified in the next subsection) is anyway more interesting to describe the moduli spaces we will deal with.

1.3. The strands algebra. We now turn our attention to the strand algebra on n points $\mathcal{A}(n)$. As a module this will be freely generated over $\mathbb{F}_2$, and the generators, relations and differential will have a nice pictorial description in terms of strands diagrams (see [**LOT09**] for a motivation behind the definitions).

A *strand diagram* on n points consists of a diagram with n dots on the left and on the right (numbered from the bottom up) and a set of strands connecting points on the two columns such that for each strand the right end is greater or equal to the left end, and no point is the initial or end point of more than one strand.



These elements, considered up to homotopy, are the generators of $\mathcal{A}(n)$ over $\mathbb{F}_2$. Clearly every generator admits a representative such that each pair of strands crosses at most once, and is uniquely determined by the initial and end point of each strand.

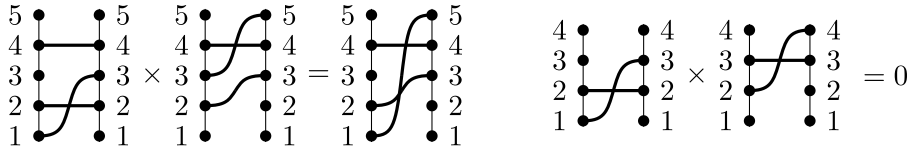
Remark 1.10. In fact, there is a less visual but completely algebraic construction of the algebra. In that case the generators are partial non decreasing permutations (S, T, ϕ) of $[n]$, i.e. two subsets $S, T \subset [n]$ together with a bijection $\phi : S \to T$ such that $\phi(i) \geq i$ for every $i \in S$. There is a straightforward correspondence between these maps and the generators of the strands algebra, for example the generator above corresponds to

$$\phi : \{1, 2, 3\} \to \{3, 4, 5\} \qquad \begin{cases} 1 \mapsto 5 \\ 2 \mapsto 3 \\ 3 \mapsto 4. \end{cases}$$

All what we will say can be formulated in this language, but we will adopt the strands approach as it is much more visual and intuitive.

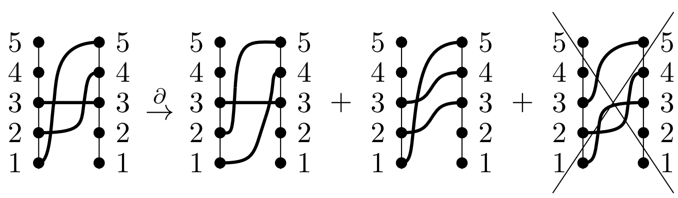
The algebra $\mathcal{A}(n, k)$ is the direct summand of $\mathcal{A}(n)$ generated by the strand diagrams with exactly k strands. To each generator a we may associate its number of *inversions* $\mathrm{inv}(a)$, which is number of pairs of strands such that the ordering of the end points is switched, or, equivalently, the minimal number of crossings in any strand diagram representing it. For example the generator pictured above has 2 inversions.

We now turn to the differential algebra structure on $\mathcal{A}(n)$. The product is defined as in the following figure.

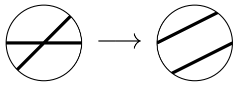


More in detail, given two generators a, b we set $a \cdot b = 0$ if the end points of a do not correspond to the initial points of b. If they correspond, one can obtain a new strand diagram $a * b$ placing the two diagrams next to each other and joining the ends. Then one defines $a \cdot b$ to be 0 if double crossings are introduced, i.e. $\mathrm{inv}(a * b) \neq \mathrm{inv}(a) + \mathrm{inv}(b)$ (on the right of the figure), and $a * b$ otherwise (on the left).

The differential comes from the smoothings of the strand diagram, as in the next figure.



More in detail a *smoothing* of a generator is a generator obtained by replacing a crossing with two parallel strands as in the next figure.



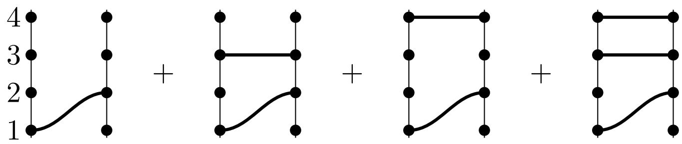
Then, the differential of a generator a is obtained by summing all the possible smoothings of the strand diagram with no double crossings, i.e. such that the number of inversions is decreased exactly by one.

LEMMA 1.11. *When equipped with these operations, $\mathcal{A}(n,k)$ is a differential algebra.*

PROOF. One can define an algebra $\overline{\mathcal{A}}(n,k)$ in the same manner as $\mathcal{A}(n,k)$ but considering as generators strand diagrams up to isotopies that preserve double crossings (and without setting them equal to 0 in the products ad differentials). This is clearly a differential algebra. The subspace $\overline{\mathcal{A}}_d(n,k) \subset \overline{\mathcal{A}}(n,k)$ generated by all strand diagrams with at least one double crossing is a differential ideal: in fact, if s has at least two double crossings then $\partial s \in \overline{\mathcal{A}}_d(n,k)$, while if it has only one it easy to check that the terms in its differential with no double crossings cancel in pairs. So $\mathcal{A}(n,k) = \overline{\mathcal{A}}(n,k)/\overline{\mathcal{A}}_d(n,k)$ is a differential algebra. $\square$

Each subset $S \subset [n]$ with k elements has an associated idempotent $I(S) \in \mathcal{A}(n,k)$ which is represented by the strand diagram with all horizontal strands starting from elements of S. The subalgebra of $\mathcal{A}(n,k)$ generated by those idempotents is denoted by $\mathcal{I}(n,k)$.

With this in mind, there is another way to describe the generators of the algebra $\mathcal{A}(n)$ related to the notion of Reeb chords introduced in the previous subsection. Given a pointed circle with n marked points, one can number the points from 1 to n in a canonical way thanks to the basepoint and the orientation of the circle. In this way, every Reeb chord ρ defines a strand, namely the one connecting ρ^- to ρ^+. So given a consistent set of Reeb chords $\boldsymbol{\rho}$, one can define the element $a_0(\boldsymbol{\rho}) \in \mathcal{A}(n)$ which is obtained by taking the sum over all strand diagrams obtained by the strand diagram for $\boldsymbol{\rho}$ (which is the one with one strand for each Reeb chord of $\boldsymbol{\rho}$) adding horizontal strands in a consistent way. For example if $n = 4$ and $\boldsymbol{\rho} = \{[1,2]\}$, $a_0(\boldsymbol{\rho})$ is the following element of $\mathcal{A}(4)$.



NOTATION 1.12. We define for convenience $a_0(\boldsymbol{\rho}) = 0$ if $\boldsymbol{\rho}$ is not consistent.

It is clear then that the basic additive generators for $\mathcal{A}(n,k)$ we have described are exactly the non zero products of the form $I(S)a_0(\boldsymbol{\rho})$, where $S \subset [n]$ is a subset with k elements and $\boldsymbol{\rho}$ is a consistent set of Reeb chords.

REMARK 1.13. This description will fit better in our construction, as we will always sum over all possible matchings at $e\infty$, and then get the right element multiplying by a suitable idempotent.

Here is a simple lemma, which proof descends straightforwardly from the definitions of the algebra and the operations between Reeb chords. The first part tells us that a_0 behaves well under composition, while the second gives us a nice description of its differential.

LEMMA 1.14. *Given two consistent sets of Reeb chords $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ and a subset $S \subset [n]$ such that $I(S)a_0(\boldsymbol{\rho}) \neq 0$, then*

$$I(S)a_0(\boldsymbol{\rho})a_0(\boldsymbol{\sigma}) = \begin{cases} I(S)a_0(\boldsymbol{\rho} \uplus \boldsymbol{\sigma}) & \text{if } \boldsymbol{\rho} \text{ and } \boldsymbol{\sigma} \text{ are composable;} \\ 0 & \text{otherwise.} \end{cases}$$

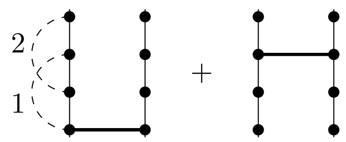
If $\boldsymbol{\rho}$ is consistent, then

$$\partial a_0(\boldsymbol{\rho}) = \sum_{\boldsymbol{\rho}' \text{ splitting of } \boldsymbol{\rho}} a_0(\boldsymbol{\rho}') + \sum_{\boldsymbol{\rho}' \text{ shuffle of } \boldsymbol{\rho}} a_0(\boldsymbol{\rho}').$$

REMARK 1.15. Note that the absence of double crossings condition of definition 1.8 for splittings and shuffles of Reeb chords corresponds exactly to the relation in the algebra that sets the product of two generators to zero if there is a double crossing in the product.

1.4. The algebra of a pointed matched circle. In the previous subsections we have seen how to associate to a pointed circle with n marked points an algebra $\mathcal{A}(n)$. We now use the further information given by a pointed matched circle $\mathcal{Z}$ with $4k$ marked points to construct a special subalgebra $\mathcal{A}(\mathcal{Z}) \subset \mathcal{A}(4k)$.

First of all, the matching permits to construct special idempotents in $\mathcal{A}(4k)$. To $\mathbf{s} \subset [2k]$ one associates $I(\mathbf{s}) \in \mathcal{I}(n, |\mathbf{s}|)$ which is obtained by summing all the $2^{|\mathbf{s}|}$ possible primitive idempotents in $\mathcal{I}(n, |\mathbf{s}|)$ with exactly one horizontal strand for each matching. For example, in the pointed matched circle of example 1.2, the idempotent associated to the subset $\{1\} \subset [2]$ is the following.



The subring of $\mathcal{A}(4k)$ generated by all the $I(\mathbf{s})$ is the *ring of idempotents associated to pointed matched circle*, denoted by $\mathcal{I}(\mathcal{Z})$. This has the unit element

$$\mathbf{I} = \sum_{\mathbf{s} \subset [2k]} I(\mathbf{s}).$$

DEFINITION 1.16. The *algebra associated to a pointed matched circle $\mathcal{Z}$* is the subalgebra $\mathcal{A}(\mathcal{Z}) \subset \mathcal{A}(4k)$ generated by $\mathcal{I}(\mathcal{Z})$ and the elements of the form $\mathbf{I}a_0(\boldsymbol{\rho})\mathbf{I}$. We will denote the latter as $a(\boldsymbol{\rho})$, and call it the *algebra element associated to $\boldsymbol{\rho}$*.

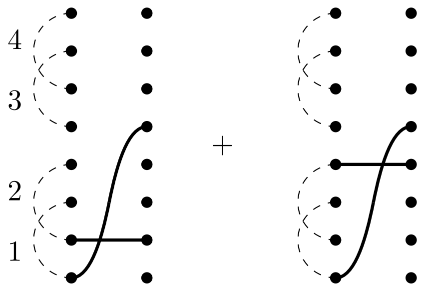
REMARK 1.17. It is easy to see that $\mathcal{A}(\mathcal{Z})$ is actually closed under multiplications and differentials, and so is effectively a subalgebra.

Note that $a(\boldsymbol{\rho})$ is obtained by taking the sum over all the strand diagrams obtained by the one for $\boldsymbol{\rho}$ adding consistently horizontal strands such that there are no matchings between any pair of initial points, and the same for end points.

As for the strands algebra, there is a convenient basis for $\mathcal{A}(\mathcal{Z})$ given by the non zero products of the form $I(\mathbf{s})a(\boldsymbol{\rho})$, where $\mathbf{s} \subset [2k]$ and $\boldsymbol{\rho}$ is a consistent set of Reeb chords. There is a simple notation for these kind of generators:

$$\left[\begin{smallmatrix} x_1,\dots,x_k & z_1,\dots,z_l \\ y_1,\dots,y_k & \end{smallmatrix}\right] = I(\{M(x_1),\dots,M(x_k),M(z_1),\dots,M(z_l)\})a(\{[x_1,y_1],\dots[x_k,y_k]\}).$$

For example, for the first pointed matched circle of example 1.3, the element $\left[\begin{smallmatrix} 1 & 2 \\ 5 & \end{smallmatrix}\right]$ is the following one.



Note that this notation is not unique, as the preceding element is also described by $\left[\begin{smallmatrix} 1 & 4 \\ 5 & \end{smallmatrix}\right]$.

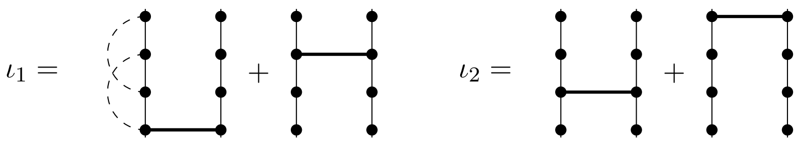
Actually $\mathcal{A}(\mathcal{Z})$ is a subalgebra of $\bigoplus_{i=0}^{2k} \mathcal{A}(4k,i)$. One can then define

$$\mathcal{A}(\mathcal{Z},i) = \mathcal{A}(\mathcal{Z}) \cap \mathcal{A}(4k,k+i).$$

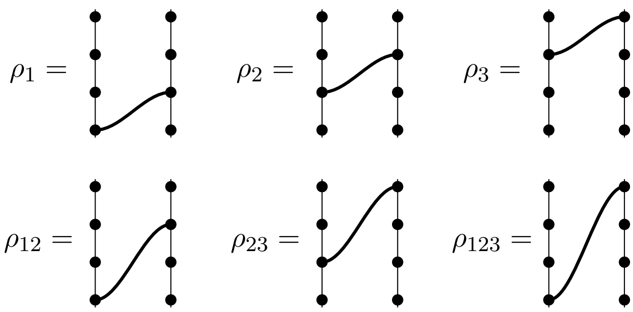
Remark 1.18. In the definition of our bordered invariants we will be interested only in the $\mathcal{A}(\mathcal{Z},0)$ algebra, as it will be the only one for which the defined action will be non trivial.

The following is a simple concrete example of the algebra.

Example 1.19. Let us analyze the algebra $\mathcal{A}(\mathcal{Z},0)$ where $\mathcal{Z}$ is the pointed matched circle of the torus as in example 1.2. Its algebra of idempotents is generated by the elements



while the basic generators are the following ones.



The non trivial products of generators are

$$\rho_1\rho_2 = \rho_{12} \qquad \rho_2\rho_3 = \rho_{23} \qquad \rho_1\rho_{23} = \rho_{123} \qquad \rho_{12}\rho_3 = \rho_{123}$$

and clearly, as there are no crossings, all the differentials are trivial.

1.5. A Maslov index on Reeb chords. We conclude this section with the definition of an index for sets of Reeb chords, to which we refer as the *Maslov index* as it will have a key role in the index formula for moduli spaces.

Given $\alpha \in H_1(Z, \mathbf{a})$ and $p \in \mathbf{a}$, define the *multiplicity* $m(\alpha, p)$ of α at p as the average multiplicity with which α covers the two regions adjacent to p. This can be extended to a bilinear map

$$m : H_1(Z, \mathbf{a}) \times H_0(\mathbf{a}) \to \frac{1}{2}\mathbb{Z}.$$

A consistent set of Reeb chords $\boldsymbol{\rho}$ determines in a natural way a homology class $[\boldsymbol{\rho}] \in H_1(Z, \mathbf{a})$. Then one can define its *Maslov index* to be

$$\iota(\boldsymbol{\rho}) = \mathrm{inv}(\boldsymbol{\rho}) - m([\boldsymbol{\rho}], \boldsymbol{\rho}^-)$$

where $\mathrm{inv}(\boldsymbol{\rho})$ is the number of inversions of the strands diagram associated to $\boldsymbol{\rho}$. Finally, one defines the Maslov index for a sequence of sets of Reeb chords $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n)$ as

$$\iota(\vec{\boldsymbol{\rho}}) = \sum_i \iota(\boldsymbol{\rho}_i) + \sum_{i<j} m([\boldsymbol{\rho}_j], \boldsymbol{\rho}_i^+ - \boldsymbol{\rho}_i^-).$$

REMARK 1.20. Given two Reeb chords ρ and σ the quantity $m(\sigma, \rho^+ - \rho^-)$ can be thought as the linking number between them.

REMARK 1.21. This Maslov grading can be used to define on $\mathcal{A}(\mathcal{Z})$ a grading with values in a non abelian group. In fact, the whole bordered theory can be made into a graded one (see [**LOT11b**]) which fits well with the notion of grading in the closed case (see subsection 7.2). A slight complication here is that the target of the grading has to be a non abelian group (in fact it is easy to see that $\mathcal{A}(\mathcal{Z})$ does not admit any non trivial abelian grading).

2. Topological preliminaries

This is the extension of the definitions and results of sections 1, 2 and 4 of the previous chapter to the bordered case. In particular, we define what a bordered 3-manifold is, how it can be represented by means of bordered Heegaard diagrams and discuss homology classes of curves and the admissibility conditions.

2.1. Bordered 3-manifolds and Heegaard diagrams.

DEFINITION 2.1. A *bordered* 3*-manifold* is a triple $(Y, \mathcal{Z}, \phi)$ where Y is a compact 3-manifold with one boundary component ∂Y, $\mathcal{Z}$ is a pointed matched circle and $\phi : F(\mathcal{Z}) \to \partial Y$ is a orientation preserving diffeomorphism.

We say that two bordered 3-manifolds $(Y, \mathcal{Z}, \phi)$ and $(Y', \mathcal{Z}', \phi')$ are *equivalent* if $\mathcal{Z} = \mathcal{Z}'$ and there exists an orientation preserving diffeomorphism $\psi : Y \to Y'$ such that $\psi \circ \phi = \phi'$.

NOTATION 2.2. We will usually suppress $\mathcal{Z}$ and ϕ from our notation, referring to a bordered 3-manifold simply by Y. We will also always consider equivalence classes of 3-manifols rather than bordered 3-manifolds themselves.

We now define the analogue of Heegaard diagrams for bordered 3-manifolds.

DEFINITION 2.3. A *bordered Heegaard diagram* is a quadruple $(\overline{\Sigma}, \overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}, z)$ consisting of

- a genus g compact oriented surface $\overline{\Sigma}$ with one boundary component;
- a collection of g disjoint simple closed curves $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ which are independent in $H_1(\overline{\Sigma})$;
- a collection of $g+k$ disjoint curves $\overline{\boldsymbol{\alpha}}$ which are independent in $H_1(\overline{\Sigma}, \partial\overline{\Sigma})$ with $g-k$ of them $\overline{\boldsymbol{\alpha}}^c = \{\overline{\alpha}_1^c, \dots, \overline{\alpha}_{g-k}^c\}$ simple closed curves in the interior of $\overline{\Sigma}$ and the other $2k$ curves $\overline{\boldsymbol{\alpha}}^a = \{\overline{\alpha}_1^a, \dots, \overline{\alpha}_{2k}^a\}$ arcs properly embedded in $(\overline{\Sigma}, \partial\overline{\Sigma})$.
- a basepoint $z \in (\partial\overline{\Sigma}) \setminus (\overline{\boldsymbol{\alpha}} \cap \partial\overline{\Sigma})$.

We will refer to the closed α-curves as α-circles, and to the other ones as α-arcs.

Definition 2.4. The *pointed matched circle associated to a bordered Heegaard diagram* $\mathcal{H}$ is $(\partial\overline{\Sigma}, \partial\overline{\Sigma} \cap \overline{\boldsymbol{\alpha}})$ together with the matching $M : \overline{\Sigma} \cap \overline{\boldsymbol{\alpha}} :\to [2k]$ defined by the α-arcs $M(\alpha_i^a \cap \partial\Sigma) = i$ and the basepoint $z \in \partial\overline{\Sigma}$. We will denote it by $\partial\mathcal{H}$.

Lemma 2.5. *This is a pointed matched circle.*

Proof. We just need to verify that the result Z' of performing surgery along the matched pairs is connected. To see this, we surger out all the α-circles (i.e. for each circle delete a tubular neighborhood and glue back two disks), obtaining a new connected surface $\overline{\Sigma}'$. Now by the homological hypothesis $\overline{\Sigma}' \setminus \mathrm{nbd}(\overline{\boldsymbol{\alpha}}^a)$ is a disk, and clearly Z' is its boundary. □

To a bordered Heegaard diagram $\mathcal{H}$ one can associate a bordered 3-manifold Y with boundary canonically identified with $F(\partial\mathcal{H})$, in a way analogue to construction 1.2 of chapter 1.

Construction 2.6. Consider the thickened Heegaard surface $\overline{\Sigma} \times [0, 1]$ and surger out the curves $\beta_i \times \{1\}$ and $\overline{\alpha}_i^c \times \{0\}$. The resulting manifold has exactly one boundary component which is naturally identified with $F(\mathcal{Z})$ thanks to the curves $\overline{\alpha}_i^a \times \{0\}$ which determine the 1-handles of the handle decomposition. Here the 0-handle is the annulus $\partial\overline{\Sigma} \times [0, 1]$ together with the disk obtained by $\overline{\Sigma} \times \{1\}$ after surgerying out all the β-curves.

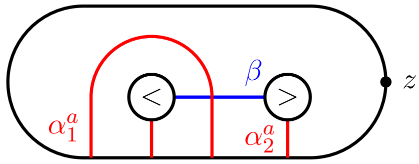
In analogy with the closed case (see proposition 1.9 of chapter 1), one has the following result.

Proposition 2.7. *Any bordered 3-manifold is represented by a bordered Heegaard diagram via the previous construction. Furthermore, any pair of bordered Heegaard diagrams for equivalent bordered 3-manifolds can be made diffeomorphic by a sequence of:*

- *isotopies of the α-curves and β-curves;*
- *handleslides of the β-curves and of α-curves over α-circles;*
- *stabilizations in the interior of $\overline{\Sigma}$.*

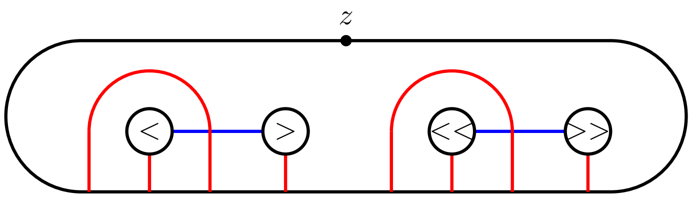
The proof has the same spirit of the original one, and involves a Morse theory construction. Actually one has to modify a little bit the construction of the associated 3-manifold (introducing some extra pairs of canceling handles) in order to make it fit well in that setting, see [**LOT11b**] for details.

Example 2.8. The following bordered Heegaard diagram represents the genus 1 handlebody (note that any diagram without α-circles is necessarily a handlebody).

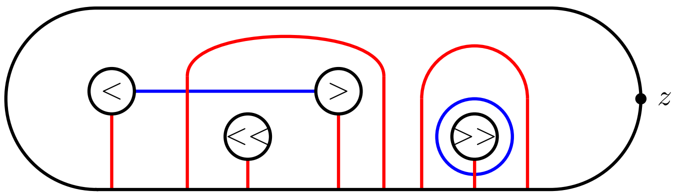


Here the parametrization of the boundary is given by the classical meridian and longitude.

Example 2.9. A bordered Heegaard diagram for the genus 2 handlebody can be obtained by taking the connected sum at the basepoint z of 2 copies of the previous diagram



and has as associated pointed matched circle the first one of example 1.3. The following bordered Heegaard diagram also represents the genus 2 handlebody.



Notice that this handlebody comes with a parametrization of the boundary different to the previous example, and indeed its pointed matched circle is none of those of example 1.3.

Remark 2.10. When talking about holomorphic curves we will be more kind to consider (improperly called) bordered Heegaard diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ with a cylindrical end, i.e. Σ is a punctured surface and near the puncture the couple $(\Sigma, \boldsymbol{\alpha}^a)$ has a product (translational invariant) structure. Clearly topologically one has that $\Sigma = \overline{\Sigma} \setminus \partial\overline{\Sigma}$. We will use one notion or the other depending on the setting.

We conclude this paragraph by discussing how the gluing and cutting operations work from the point of view of Heegaard diagrams. Recall that in order to get a well defined orientation in a manifold obtained by gluing one has to make the appropriate orientation reversal. Here by $-Y$ we denote Y with the orientation reversed, and given a Heegaard diagram $\mathcal{H} = (\overline{\Sigma}, \overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}, z)$ by $-\mathcal{H}$ we denote the Heegaard diagram with the orientation of Σ reversed. Note that if Y is the 3-manifold associated to $\mathcal{H}$, $-Y$ is the 3-manifold associated to $-\mathcal{H}$, and $F(-\mathcal{Z})$ is naturally identified with $F(\mathcal{Z})$ with the orientation reversed.

The following two lemmas, which simply state that cutting and gluing of 3-manifolds correspond to cutting and gluing of Heegaard diagrams, are immediate from construction 2.6.

Lemma 2.11. *Given bordered Heegaard diagrams* $(\overline{\Sigma}_1, \overline{\boldsymbol{\alpha}}_1, \boldsymbol{\beta}_1, z_1)$ *and* $(\overline{\Sigma}_2, \overline{\boldsymbol{\alpha}}_2, \boldsymbol{\beta}_2, z_2)$ *for the bordered* 3*-manifolds* $(Y_1, \mathcal{Z}, \phi_1)$ *and* $(Y_2, \mathcal{Z}, \phi_2)$*, one can obtain a closed Heegaard diagram*

$\mathcal{H} = -\mathcal{H}_1 \cup_\partial \mathcal{H}_2$ by gluing them along the boundaries according to the markings of their pointed matched circles:

$$-\mathcal{H}_1 \cup_\partial \mathcal{H}_2 = (-\overline{\Sigma}_1 \cup_\partial \overline{\Sigma}_2, \overline{\boldsymbol{\alpha}}_1 \cup \overline{\boldsymbol{\alpha}}_1, \boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2, z_1 = z_2).$$

Then $\mathcal{H}$ represents the manifold $-Y_1 \cup_\partial Y_2 = -Y_1 \cup_{\phi_2 \circ \phi_1^{-1}} Y_2$.

Lemma 2.12. *Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a Heegaard diagram for the closed 3-manifold Y, and let Z be a separating curve such that:*

- *$Z \cap \boldsymbol{\beta} = \emptyset$;*
- *$\Sigma \setminus (\boldsymbol{\alpha} \cup Z)$ has exactly two components;*
- *Z passes through the basepoint z.*

Let $\Sigma = \overline{\Sigma}_L \cup_Z \overline{\Sigma}_R$. Then

$$\mathcal{H}_L = (\overline{\Sigma}_L, \boldsymbol{\alpha} \cap \overline{\Sigma}_L, \boldsymbol{\beta} \cap \overline{\Sigma}_L, z) \qquad \text{and} \qquad \mathcal{H}_R = (\overline{\Sigma}_R, \boldsymbol{\alpha} \cap \overline{\Sigma}_R, \boldsymbol{\beta} \cap \overline{\Sigma}_R, z)$$

are bordered Heegaard diagrams, and there exists a separating surface $F \subset Y$ such that the two components of $Y \setminus F$ are exactly the 3-manifolds associated to $\mathcal{H}_L$ and $\mathcal{H}_R$ via construction 2.6.

2.2. Generators, homology classes and domains. Fix a bordered Heegaard diagram $\mathcal{H} = (\overline{\Sigma}, \overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}, z)$ of genus g.

Definition 2.13. A *generator* of $\mathcal{H}$ is a g-tuple $\mathbf{x} = \{x_1, \dots, x_g\}$ of points of $\overline{\Sigma}$ such that:

- each β-circle contains exactly one x_i;
- each α-circle contains exactly one x_i;
- each α-arc contains at most one x_i.

We denote the set of generators of $\mathcal{H}$ by $\mathfrak{S}(\mathcal{H})$. Given $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, we define $o(\mathbf{x})$ to be the set with k elements of α-arcs occupied by $\mathbf{x}$, i.e. $o(\mathbf{x}) = \{i | \mathbf{x} \cap \alpha_i^a \neq \emptyset\} \subset [2k]$.

Remark 2.14. Note that the generators of a Heegaard diagram $-\mathcal{H}_1 \cup_\partial \mathcal{H}_2$ are naturally identified with the pairs $(\mathbf{x}_1, \mathbf{x}_2) \in \mathfrak{S}(\mathcal{H}_1) \times \mathfrak{S}(\mathcal{H}_2)$ such that $o(\mathbf{x}_1) \cap o(\mathbf{x}_2) = \emptyset$. We say that such generators are *compatible.*

In the bordered setting, similarly to the closed case (section 2 of chapter 1), given a pair of generators $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ we are interested in topological curves $S \to (\overline{\Sigma} \setminus \{z\}) \times [0,1] \times \mathbb{R}$ from a compact surface S (with punctures on the boundary) with boundary in

$$C = [(\overline{\boldsymbol{\alpha}} \times \{1\}) \cup (\boldsymbol{\beta} \times \{0\}) \cup (\partial\overline{\Sigma} \times [0,1])] \times \mathbb{R}$$

and asymptotics at the punctures the chords $(\mathbf{x} \times [0,1] \times \{-\infty\})$ and $(\mathbf{y} \times [0,1] \times \{-\infty\})$. These carry a relative homology class in

$$H_2\big(\overline{\Sigma} \times [0,1] \times \overline{\mathbb{R}}, \overline{C} \cup (\mathbf{x} \times [0,1] \times \{-\infty\} \cup (\mathbf{y} \times [0,1] \times \{+\infty\})\big)$$

where $\overline{C}$ is the compactification of C in the $\mathbb{R}$ direction. We denote by $\pi_2(\mathbf{x}, \mathbf{y})$ the set of relative homology classes of such curves connecting $\mathbf{x}$ and $\mathbf{y}$.

As in the closed case, we have a natural concatenation product $* : \pi_2(\mathbf{x}, \mathbf{y}) \times \pi_2(\mathbf{y}, \mathbf{w}) \to \pi_2(\mathbf{x}, \mathbf{w})$, and each homology class has an associated *domain* (necessarily with $n_z = 0$).

Note that in this case the boundary of a domain D can be split into three parts, $\partial^\alpha D$ contained in $\overline{\boldsymbol{\alpha}}$, $\partial^\beta D$ contained in $\boldsymbol{\beta}$ and $\partial^\partial D$ contained in $\partial\overline{\Sigma}$. We can think the latter as an element of $H_1(\partial\overline{\Sigma}, \partial\overline{\Sigma} \cap \overline{\boldsymbol{\alpha}})$.

DEFINITION 2.15. Given two generators $\mathbf{x} = \{x_1, \dots, x_g\}$ and $\mathbf{y} = \{y_1, \dots, y_g\}$, we say that a domain D connects them if up to relabeling one has that:

- $\partial^{\alpha_i} D$ is a 1-chain with boundary $y_i - x_i$;
- $\partial^{\beta_i} D$ is a 1-chain with boundary $x_{\sigma(i)} - y_i$,

for some permutation $\sigma \in S_g$. We denote the set of domains connecting $\mathbf{x}$ to $\mathbf{y}$ by $P(\mathbf{x}, \mathbf{y})$.

A domain B is *periodic* if $\partial^\alpha B = \partial^\beta B = 0$. We say that B is *provincial* if $\partial^\partial B = 0$.

As in the closed case, the map that associates to each relative homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ its domain $\mathcal{D}(B)$ maps into $P(\mathbf{x}, \mathbf{y})$ and is a bijection with the subset of $P(\mathbf{x}, \mathbf{y})$ with $n_z = 0$, so we will always confuse a homology class with its domain.

Finally it is clear that domains can be glued and cut like the Heegaard diagrams in the previous section. The following lemma is immediate.

LEMMA 2.16. *Let $\mathcal{H}_1$ ad $\mathcal{H}_2$ be two bordered Heegaard diagrams with $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2 = \mathcal{Z}$. Then, given two pairs of compatible generators $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2)$ in $\mathfrak{S}(\mathcal{H}_1) \times \mathfrak{S}(\mathcal{H}_2)$, there is a bijection between pairs of homology classes $B_1 \in \pi_2(\mathbf{x}_1, \mathbf{y}_1)$ and $B_2 \in \pi_2(\mathbf{x}_2, \mathbf{y}_2)$ with $\partial^\partial B_1 = -\partial^\partial B_2 \in H_1(Z, \mathbf{a})$ and homology classes $B \in \pi_2\big((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2)\big)$.*

2.3. Admissibility. We discuss admissibility criteria for bordered Heegaard diagrams. As in the closed case (definition 2.6 of chapter 1), we will say that a domain is *positive* if every coefficient is non negative.

DEFINITION 2.17. A Heegaard diagram is called *admissible* if every non zero periodic domain D with $n_z(D) = 0$ has both positive and negative coefficients. A Heegaard diagram is called *provincially admissible* if every non zero provincial periodic domain D with $n_z(D) = 0$ has both positive and negative coefficients.

The following propositions are completely analogue to the closed case.

PROPOSITION 2.18. *Every bordered Heegaard diagram is isotopic to a (provincially) admissible one. Furthermore, any two (provincially) admissible Heegaard diagrams can be related by a sequence of Heegaard moves such that at each stage the diagram is (provincially) admissible.*

PROPOSITION 2.19. *Given a Heegaard diagram $\mathcal{H}$, fix generators $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Then:*

- *if $\mathcal{H}$ is admissible, there are only finitely many positive domains in $\pi_2(\mathbf{x}, \mathbf{y})$;*
- *if $\mathcal{H}$ is provincially admissible, for a fixed $h \in H_1(Z, \mathbf{a})$ there are only finitely many positive $B \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\partial^\partial B = h$.*

Finally, we discuss what happens when one glues admissible Heegaard diagrams.

LEMMA 2.20. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Heegaard diagrams with $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2$. If $\mathcal{H}_1$ is admissible and $\mathcal{H}_2$ is provincially admissible, then $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ is admissible.*

PROOF. A positive periodic domain B in $\mathcal{H}$ decomposes as a positive periodic domain B_1 of $\mathcal{H}_1$ and a positive periodic domain B_2 of $\mathcal{H}_2$. As $\mathcal{H}_1$ is admissible, $B_1 = 0$, but then $\partial^\partial B_2 = 0$, so as $\mathcal{H}_2$ is provincially admissible $B_2 = 0$, so $B = 0$. $\square$

2.4. Spin^c**-structures.** The notion of Spin^c-structure introduced and analyzed in subsection 7.1 of the previous chapter has a natural generalization in the bordered case. Furthermore, one can also define *relative* Spin^c*-structures* in this setting. We will not enter the details of the constructions, and we just point out that the fact that the Heegaard Floer homology decomposes as a direct sum of the groups within a fixed Spin^c-structure is actually true also for the bordered invariants we are going to define (see [**LOT11b**] for the details).

3. Moduli spaces of holomorphic curves

In this section we give an overview of the theory of holomorphic curves we will use in the construction of the bordered invariants. This will be much more complicated than the closed case because we will consider curves with asymptotics also in the Σ direction, as the latter is now a punctured surface. In particular, the compactification we will construct for 1-dimensional moduli spaces will not in general be a manifold, but we will be able to describe its 'ends' in a nice way. The aim of this section is to give an intuitive idea of the subject providing some key examples. In this sense, we will not bother the reader with many technical details (for which we refer as usual to [**LOT11b**]).

3.1. Holomorphic curves. Given a bordered Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ choose a complex structure on Σ with a cylindrical end $Z \times \mathbb{R}$ (or, equivalently, a puncture p). Then the manifold $\Sigma \times [0,1] \times \mathbb{R}$ has three kinds of 'ends', namely the usual ones at $+\infty$ and $-\infty$ in the $\mathbb{R}$ direction, and those at $p \times [0,1] \times \mathbb{R}$, to which we refer as *east infinity* $e\infty$.

We are interested in some special curves

$$u : (S, \partial S) \to \big((\Sigma \setminus \{z\}) \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})\big)$$

where S is a *decorated source*, i.e. a smooth Riemann surface with boundary and punctures on the boundary such that each puncture is labeled by $+, -$ or e and each e puncture is labeled by a Reeb chord in $(Z, \mathbf{a})$. In particular we want curves which have asymptotics as prescribed by the punctures and which are holomorphic with respect to an *admissible* almost complex structure on $\Sigma \times [0,1] \times \mathbb{R}$. Here the definition of admissible almost complex structure is totally analogue to the closed case (definition 3.1 of chapter 1) with the extra assumption that it is split near the $\{p\} \times [0,1] \times \mathbb{R}$. More in detail, we want holomorphic maps $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ such that (using the usual coordinates (s,t) on $[0,1] \times \mathbb{R}$):

1. at each $-$ puncture q of S, $\lim_{w \to q} t \circ u(w) = -\infty$;
2. at each $+$ puncture q of S, $\lim_{w \to q} t \circ u(w) = +\infty$;
3. at each e puncture q of S, $\lim_{w \to q} \pi_\Sigma \circ u(w)$ is the Reeb chord ρ labeling q;
4. $\pi_{\mathbb{D}} \circ u$ is a g-fold covering (in particular is non constant on every component of S);
5. some technical conditions, namely that u is proper and extends to a proper finite energy map to the east punctures;
6. for each $t \in \mathbb{R}$, $u^{-1}(\beta_i \times \{0\} \times \{t\})$ consists exactly of one point for all $i = 1, \ldots, g$, $u^{-1}(\alpha_i^c \times \{1\} \times \{t\})$ consists exactly of one point for all $i = 1, \ldots, g-k$.

Definition 3.1. We call the last condition *weak boundary monotonicity.*

The weak boundary monotonicity condition implies that the curve is asymptotic at $-\infty$ to a chord of the form $\{x_1, \ldots, x_g\} \times [0,1] \times \{-\infty\}$ such that each β-curve and α-circle contains exactly one of the x_i's, and similarly for the $\{+\infty\}$ asymptotics. Note that $\{x_1, \ldots, x_g\}$ is generally not an element of $\mathfrak{S}(\mathcal{H})$, as there might be some α-arcs containing more than one x_i. We refer to such a g-uple of points as a *generalized generator.*

Given a pair of generalized generators $\mathbf{x}$ and $\mathbf{y}$, a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ (defined in the same manner as for genuine generators) and a decorated source S, one can consider the space $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S)$ of holomorphic curves connecting (in the usual sense) $\mathbf{x}$ to $\mathbf{y}$ in the homology class B from a fixed decorated source. Note that each $e\infty$ puncture q of S comes with a *evaluation map*, which assigns it its $\mathbb{R}$ coordinate $\mathrm{ev}_q(u)$. Putting all those maps together, we get

$$ev = \prod_{q \in E(S)} ev_q : \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S) \to \mathbb{R}^{E(S)}$$

where $E(S)$ is the set of east punctures of S.

With this map, one can define certain subspaces of $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S)$ where asymptotics at east infinity have the same height. Fix a partition $P = \{P_i\}$ of $E(S)$, and let $\Delta_P \subset \mathbb{R}^{E(S)}$ be the subspace defined by $\{x_p = x_q |\ p, q \text{ are in the same } P_i\}$. We then define $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; P)$ to be $ev^{-1}(\Delta_P)$, i.e. the subspace of $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S)$ where two punctures have the same evaluation if they are in the same partition. A generalization of proposition 3.5 of chapter 1 is the following.

Proposition 3.2. *The space $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; P)$ has expected dimension*

$$\mathrm{ind}(B, S, P) = g - \chi(S) + 2e(B) + |P|$$

where $|P|$ denotes the number of parts of the partition P.

Remark 3.3. Here to determine the Euler measure of a region in $\overline{\Sigma}$ we have to choose a Riemannian metric for which $\partial\overline{\Sigma}$ is geodesic.

We will also be interested in the spaces of curves $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; \vec{P})$, where $\vec{P} = (P_1, \dots, P_n)$ is an ordered partition. This is the subset of $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; P)$ where we require that the evaluations are strictly increasing with respect to the partition, i.e. if $p \in P_i$, $q \in P_j$ and $i < j$ then $ev_p(u) < ev_q(u)$.

As every puncture of S is labeled by a Reeb chord, to the ordered partition $\vec{P}$ of $E(S)$ is associated a sequence of sets or Reeb chords, which we denote by $[\vec{P}]$.

3.2. Strong boundary monotonicity. As we have already pointed out, weak boundary monotonicity is a condition too wide to describe the moduli spaces we are interested in. The main reason of its introduction is that the study of moduli spaces of curves and the construction of their compactification is neater in this setting. The right condition to impose is the following one.

Definition 3.4. A holomorphic curve $u : (S, \partial S) \to (\Sigma \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R}))$ is *strongly boundary monotone* if for each $t \in \mathbb{R}$:

- $u^{-1}(\beta_i \times \{0\} \times \{t\})$ consists exactly of one point for all $i = 1, \dots, g$;
- $u^{-1}(\alpha_i^c \times \{1\} \times \{t\})$ consists exactly of one point for all $i = 1, \dots, g - k$;
- $u^{-1}(\alpha_i^a \times \{1\} \times \{t\})$ consists at most of one point for all $i = 1, \dots, 2k$.

This immediately implies that the curve is asymptotic to a pair of genuine generators. The nice thing is that this notion is completely combinatorial once one knows the asymptotics of a curve.

Given a multiset $\mathbf{s} \subset [2k]$ and a sequence of sets of Reeb chords $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n)$, define recursively

$$o(\mathbf{s}, \boldsymbol{\rho}_1) = \big(\mathbf{s} \cup M(\boldsymbol{\rho}_1^+)\big) \setminus M(\boldsymbol{\rho}_1^-);$$
$$o(\mathbf{s}, (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{i+1})) = \big(o(\mathbf{s}, (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_i)) \cup M(\boldsymbol{\rho}_{i+1}^+)\big) \setminus M(\boldsymbol{\rho}_{i+1}^-),$$

where the operations are to be intended in the sense of multisets.

DEFINITION 3.5. The pair $(\mathbf{s}, \vec{\boldsymbol{\rho}})$ is *strongly boundary monotone* if

(1) $\mathbf{s}$ is actually a set (i.e. a multiset with no repeated elements);
(2) $M(\boldsymbol{\rho}_{i+1}^-) \subset o(\mathbf{s}, (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_i))$;
(3) $M(\boldsymbol{\rho}_{i+1}^+)$ is disjoint $o(\mathbf{s}, (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_i)) \setminus M(\boldsymbol{\rho}_{i+1}^-)$.

It is not a coincidence that we have called this condition (which might look pretty ugly at a first sight) strong boundary monotonicity.

PROPOSITION 3.6. *A curve $u \in \mathcal{M}(\mathbf{x}, \mathbf{y}; S; \vec{P})$ is strongly boundary monotone if and only if $(\mathbf{x}, [\vec{P}])$ is strongly boundary monotone.*

The key observation is that as u is holomorphic, its restriction to arcs in the boundary is strictly monotone (when not constant).

With this in mind, it is clear that if $(\mathbf{x}, [\vec{P}])$ is strongly boundary monotone then also u is. In fact, $o(\mathbf{x}, [(P_1, \dots, P_i)])$ represents exactly the multiset of α-arcs occupied by the curve between the asymptotics P_i and P_{i+1}, and strong boundary monotonicity just says that this is actually a set for every $i = 1, \dots, n$.

The other implication follows from a similar (but a little more tricky) argument.

3.3. Moduli spaces of embedded curves.

As in the closed case, in order to define the invariants we will focus only on spaces of embedded curves. The reason that led us to consider the more general spaces $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; P)$ is that the compactification theory is more neatly described in that setting. In any case, as in the closed case the embeddedness or not of the curve will be determined only by the Euler characteristic of the source.

DEFINITION 3.7. We say that a sequence of sets of Reeb chords $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n)$ and a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ are *compatible* if $\partial^\partial B = [\vec{\boldsymbol{\rho}}] \in H_1(Z, \mathbf{a})$ and $(\mathbf{x}, \vec{\boldsymbol{\rho}})$ is strongly boundary monotone. In this case, define the *embedded Euler characteristic* and the *embedded index* as

$$\chi_{emb}(B, \vec{\boldsymbol{\rho}}) := g + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) - \iota(\vec{\boldsymbol{\rho}})$$
$$\operatorname{ind}(B, \vec{\boldsymbol{\rho}}) := e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + |\vec{\boldsymbol{\rho}}| + \iota(\vec{\boldsymbol{\rho}}).$$

where $\iota(\vec{\boldsymbol{\rho}})$ is the Maslov index of the sequence of sets Reeb chords defined in subsection 1.5.

This is motivated by the following bordered analogue of proposition 3.6 of chapter 1.

PROPOSITION 3.8. *Let $u \in \widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S; \vec{P})$. Then u is an embedding if and only if $\chi(S) = \chi_{emb}(B, [\vec{P}])$. Furthermore the expected dimension of $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S; \vec{P})$ is* $\operatorname{ind}(B, S, P) = \operatorname{ind}(B, [\vec{P}])$.

This lets us define our spaces of embedded curves connecting generators $\mathbf{x}, \mathbf{y}$ in the homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ with prescribed asymptotics $\vec{\boldsymbol{\rho}}$ at $e\infty$, which is simply

$$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}}) = \bigcup_{\substack{\chi(S)=\chi_{emb}(B,\vec{\boldsymbol{\rho}}) \\ [\vec{P}]=\vec{\boldsymbol{\rho}}}} \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; \vec{P}).$$

Finally, here is the transversality result.

Proposition 3.9. *For a generic choice of the admissible almost complex structure on* $\Sigma \times [0,1] \times \mathbb{R}$*,* $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})$ *is a smooth manifold of dimension* $\mathrm{ind}(B, \vec{\boldsymbol{\rho}})$*.*

The moduli space of curves is obtained as usual by factoring out the $\mathbb{R}$-action by translations

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}}) = \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})/\mathbb{R}.$$

In the moduli space the evaluations are not well-defined, but the difference of the evaluation of two punctures is, and so one obtains a map

$$ev : \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}}) \to \mathbb{R}^E/\mathbb{R}$$

where E is the set of east punctures of the curves in the moduli space. Note that this may change from source to source, but two such sets can always be identified in a natural way.

3.4. Holomorphic combs. As in the closed case, we will need to study the 'ends' of the 1-dimensional moduli spaces, and so we need to construct a suitable compactification for them. The further complication here is due to the fact that we are considering curves with asymptotics at $e\infty$, so there can also be degenerations in that direction. To take account of this phenomenon, one has to construct the huge space of holomorphic combs, which will be unfortunately not a manifold in general even in the best cases. We just sketch the construction without entering too much into the details.

First of all, one defines holomorphic curves at $e\infty$, i.e. curves in $Z \times \mathbb{R} \times [0,1] \times \mathbb{R}$ where $Z \times \mathbb{R}$ is the cylindrical end of Σ. Note that this space contains the $4k$ planes $\mathbf{a} \times \mathbb{R} \times \{1\} \times \mathbb{R}$, which are lagrangians with respect to any split symplectic structure.

We consider maps

$$v : (T, \partial T) \to ((Z \setminus \{z\}) \times \mathbb{R} \times [0,1] \times \mathbb{R}, \mathbf{a} \times \mathbb{R} \times \{1\} \times \mathbb{R})$$

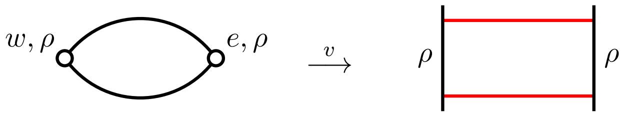
where T is a *bidecorated source*, i.e. a Riemann surface with boundary and punctures on the boundary where each puncture is assigned a Reeb chord and is labeled e or w (east or west), and v satisfies:

(1) v is holomorphic respect a fixed split almost complex structure on $Z \times \mathbb{R} \times [0,1] \times \mathbb{R}$;
(2) $s \circ v(\partial T) = 1$;
(3) v is proper;
(4) at each west puncture q of T labeled by ρ, $\lim_{z \to q} \pi_\Sigma \circ u(z) = \rho \subset Z \times \{-\infty\}$;
(5) at each east puncture q of T labeled by ρ, $\lim_{z \to q} \pi_\Sigma \circ u(z) = \rho \subset Z \times \{+\infty\}$.

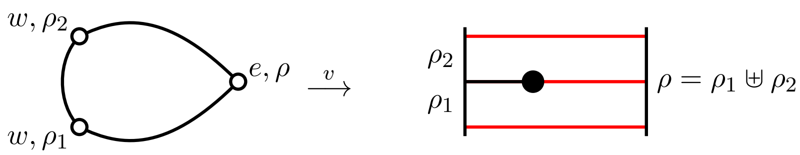
Note that the first condition implies via the maximum modulus principle that the projection on the $[0,1] \times \mathbb{R}$ factor is constant on each component of T. In this case there is a $\mathbb{R} \times \mathbb{R}$ action via translations on the space of curves at $e\infty$ with fixed source T, and we denote the resulting moduli space (generally not a manifold) by $\mathcal{N}(T)$. There are two evaluation maps ev_e and ev_w defined on this space, one for the east punctures and one for the west punctures.

There are some special kind of curves at east infinity with sources disjoint union of disks that we will be interested in.

EXAMPLE 3.10. The simplest example is the *trivial curve*, which consists of a single disk with two punctures both labeled with the same Reeb chord.

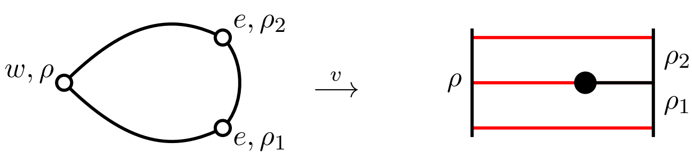


EXAMPLE 3.11. A *join component* has as source a disk with two west punctures and an east puncture, and the Reeb chords of the west punctures abut (in the right order), with their join the Reeb chord of the east puncture.



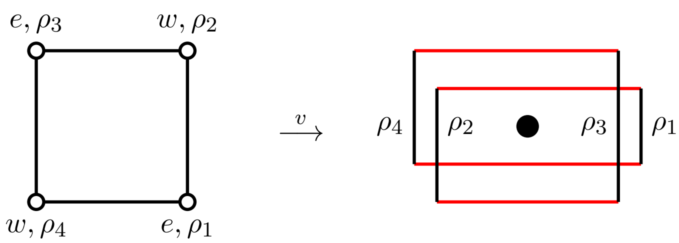
A *join curve* is a curve at $e\infty$ consisting of *one* join component and several trivial disks.

EXAMPLE 3.12. A *split component* is the reverse of a join curve. It has as source a disk with one west puncture and two east punctures, such that the Reeb chords of the east punctures abut (in the right order), with their join is the Reeb chord of the west puncture.

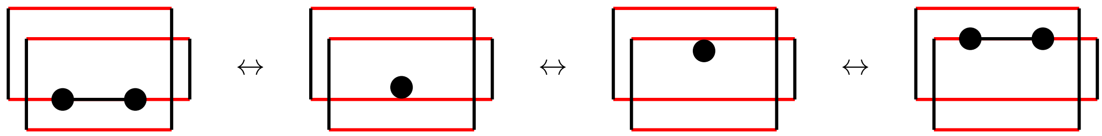


A *split curve* is a curve consisting of *some* split components and some trivial disks. Note that this definition is different from the one for the join curve.

EXAMPLE 3.13. An *odd shuffle component* is a curve with source a disk with two east punctures and two west punctures such that the Reeb chords associated to the east ones are the nested pair associated to the interleaved pair associated to the west pair.



Here the big central point represents the branching point. This can be in the interior, but also one can have two boundary branching points (in fact, the space of shuffle curves has 1-dimensional moduli).



An *even shuffle component* is defined in a similar manner, with the role of east and west punctures interchanged. A *(even or odd) shuffle curve* is a curve at $e\infty$ consisting of a (even or odd) shuffle curve and some trivial disks.

We are now ready to define the space of holomorphic combs, which is the compactification we are looking for. For every bidecorated source T denote by $W(T)$ and $E(T)$ respectively the sets of its west and east punctures.

Definition 3.14. A *holomorphic story* is a sequence $(u, v_1, \ldots, v_k)$ where $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S)$, $v_i \in \mathcal{N}(T_i)$ and such that there is a label preserving bijection $E(S) \leftrightarrow W(T_1)$ and $E(T_i) \leftrightarrow W(T_{i+1})$ for $i = 1, \ldots, k-1$ such that under this identification the evaluation maps coincide, i.e. $ev(u) = ev_w(v_1)$ in $\mathbb{R}^{E(S)}/\mathbb{R} \cong \mathbb{R}^{W(T_1)}/\mathbb{R}$ and $ev_e(v_i) = ev_w(v_{i+1})$ in $\mathbb{R}^{E(T)}/\mathbb{R} \cong \mathbb{R}^{W(T_1)}/\mathbb{R}$ for all $i = 1, \ldots, k-1$.

A *holomorphic comb* of height N is a sequence $\{(u_j, v_{1,j}, \ldots, v_{k_j,j})\}_{j=1,\ldots,N}$ of holomorphic stories such that $u_j \in \mathcal{M}^{B_j}(\mathbf{x}_j, \mathbf{x}_{j+1}; S_j)$.

Intuitively, this definition is made to take account of all possible degenerations at $\pm\infty$ and $e\infty$. In order to get a compact space, we have to consider singular holomorphic combs, i.e. we allows also singular Riemann surfaces (with suitable stability conditions) as sources.

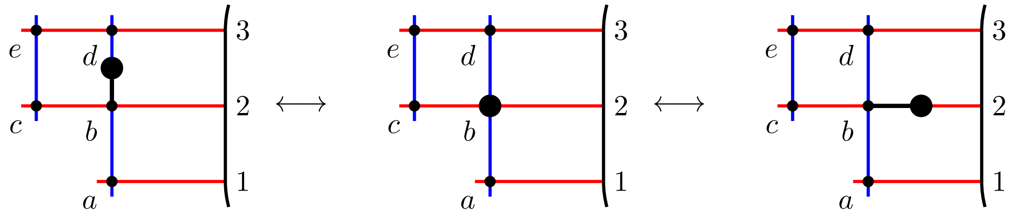
A singular holomorphic comb determines a smooth decorated source simply by gluing all the surfaces along the matching punctures and solving all the nodes and cusps (see remark 3.10 of chapter 1), and it connects two generators $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ in a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ in an obvious sense. One can then define the moduli space $\overline{\overline{\mathcal{M}}}^B(\mathbf{x}, \mathbf{y}; S)$ of such holomorphic combs with glued source S. As in the closed case, there is a complicated natural topology on such spaces, which turns out to be quite awkward: for instance, the space of curves at $e\infty$ is almost never a manifold. In any case, the fundamental result is the following one.

Theorem 3.15. *The moduli space of holomorphic combs* $\overline{\overline{\mathcal{M}}}^B(\mathbf{x}, \mathbf{y}; S)$ *is compact.*

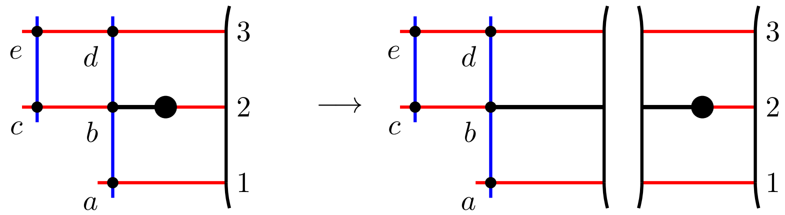
This space contains the (closed, and so compact) subspace $\overline{\overline{\mathcal{M}}}^B(\mathbf{x}, \mathbf{y}; S; P)$ of curves respecting the partition P of the eastmost punctures of S (in the usual case). Then one defines $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; P)$ to be the closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S; P)$ in $\overline{\overline{\mathcal{M}}}^B(\mathbf{x}, \mathbf{y}; S; P)$ and $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; \vec{P})$ to be the closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S; \vec{P})$ in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; P)$. By the previous theorem, all these spaces are compact, and are indeed the compactifications we will use subsequently.

3.5. Degenerations. We describe now how 1-dimensional moduli space of holomorphic curves may degenerate. Before stating the main result, we focus on some simple (but central) local examples.

Example 3.16. The following pictures show a 1-parameter family of curves connecting $\{a, c\}$ to $\{b, e\}$ with asymptotics at east infinity the Reeb chord $[1, 3]$.

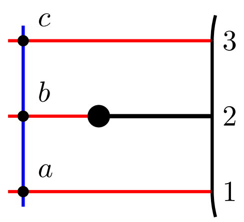


As the big dot approaches the intersection point d, we obtain a classical two-story holomorphic building, with a curve connecting $\{a, c\}$ to $\{c, d\}$ with east asymptotics the Reeb chord $[1, 3]$ followed by a curve connecting $\{c, d\}$ to $\{b, e\}$. On the other side, as the big dot approaches the 2, a join curve degenerates at east infinity. Namely, this end of the family is given by a curve connecting $\{a, c\}$ to $\{b, e\}$ with asymptotics at east infinity the set of Reeb chords $\{[1, 2], [2, 3]\}$, together with a join curve at east infinity with west labels $[1, 2]$ and $[2, 3]$ and east label $[1, 3]$.

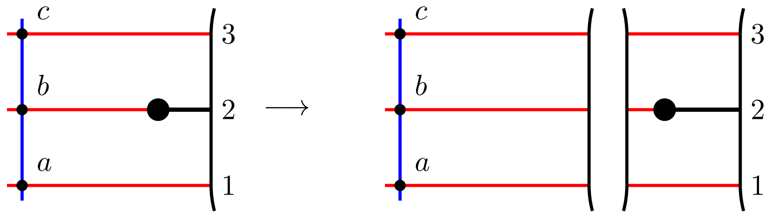


Notice that the curve connecting $\{a, c\}$ to $\{b, e\}$ has two components which are both disks, one connecting a to b with east asymptotic $[1, 2]$, one connecting c to e with east asymptotic $[2, 3]$, and those east asymptotics have the same height.

Example 3.17. This picture represents a 1-parameter family of holomorphic curves connecting $\{a\}$ to $\{c\}$ with east infinity asymptotics the sequence of Reeb chords $([1, 2], [2, 3])$.

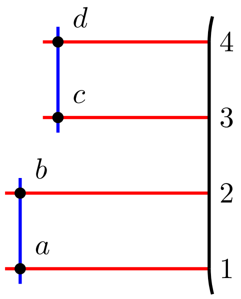


The end where the big dot reaches b corresponds to a two-story holomorphic building where the first curve connects $\{a\}$ to $\{b\}$ with east asymptotics the Reeb chord $[1, 2]$ and the second curve connects $\{b\}$ to $\{c\}$ with east asymptotics the Reeb chord $[2, 3]$. When approaching the other end, there is a split curve degenerating at $e\infty$.



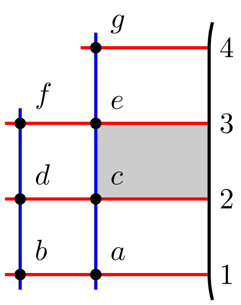
In particular this holomorphic story end consists of a holomorphic disk connecting $\{a\}$ to $\{c\}$ with east asymptotics $[1,3]$, and a split curve at $e\infty$ with west label $[1,3]$ and east labels $[1,2]$ and $[2,3]$.

EXAMPLE 3.18. The figure determines a 1-parameter family of holomorphic curves connecting $\{a,c\}$ to $\{b,d\}$ with asymptotics at $e\infty$ the sequence of Reeb chords $([1,2],[3,4])$. This family is parametrized by the difference of the evaluations of the two punctures $ev_{[3,4]} - ev_{[1,2]} \in \mathbb{R}^+$.



When this evaluation approaches $+\infty$, the curves break into a two story holomorphic building, the first curve connecting $\{a,c\}$ to $\{b,c\}$ with east asymptotics $[1,2]$, the second connecting $\{b,c\}$ to $\{b,d\}$ with east asymptotics $[3,4]$. On the other hand, when the evaluation approaches 0 the curves degenerate to a curve connecting $\{a,c\}$ to $\{b,d\}$ with asymptotics at $e\infty$ the set of Reeb chords $\{[1,2],[3,4]\}$. Notice that in this case the limit curve is actually a genuine one, but in another moduli space.

EXAMPLE 3.19. This example is more complicated to visualize. The following diagram represents a 1-parameter family of holomorphic disks connecting $\{b,c\}$ to $\{f,g\}$, with east infinity asymptotics the set of interleaved Reeb chords $\{[2,3],[1,4]\}$ (here the grey region is covered twice).



The moduli space looks somehow like that of example 3.13, and one can have curves with one branching point inside the grey region, curves with two boundary branching points on the α-arc from e to 3 and in the α-arc from c to 2 (in this case one branching point has to be c). Note that the type of such curves involved depends on the complex structure on the surface.

In any case, one end is reached for a branching point converging to c, and consists of a two story holomorphic building, with a single disk connecting $\{a, d\}$ to $\{b, c\}$ followed by a pair of disks one connecting a to g with Reeb chord $[1, 4]$ and the other d to f with Reeb chord $[2, 3]$, with the punctures at the same height. The other end is reached when a branching point approaches the boundary, and a odd shuffle curve from $\{[1, 3], [2, 4]\}$ to $\{[1, 4], [2, 3]\}$ is split out.

We now give names to these 'ends'. Recall the terminology on sets of Reeb chords of definition 1.8.

Definition 3.20. Fix a moduli space $\mathcal{M} = \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})$, with $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$, $B \in \pi_2(\mathbf{x}, \mathbf{y})$, and $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n)$ such that $\mathrm{ind}(B, \vec{\boldsymbol{\rho}}) = 2$ and $(B, \vec{\boldsymbol{\rho}})$ is compatible (definition 3.7).

A *two story end* of $\mathcal{M}$ is an element of $\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\boldsymbol{\rho}}_1) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\boldsymbol{\rho}}_2)$ where $B_1 * B_2 = B$, and $\vec{\boldsymbol{\rho}} = (\vec{\boldsymbol{\rho}}_1, \vec{\boldsymbol{\rho}}_2)$.

A *join curve end* of $\mathcal{M}$ at level i is an element of $\mathcal{M}^B\big(\mathbf{x}, \mathbf{y}; (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_{i-1}, \boldsymbol{\rho}'_i, \boldsymbol{\rho}_{i+1}, \ldots, \boldsymbol{\rho}_n)\big)$ and $\boldsymbol{\rho}'_i$ is a split of $\boldsymbol{\rho}_i$.

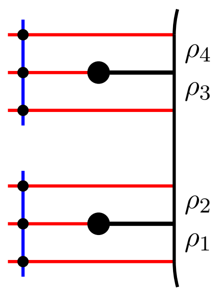
A *shuffle curve end* of $\mathcal{M}$ at level i is an element of $\mathcal{M}^B\big(\mathbf{x}, \mathbf{y}; (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_{i-1}, \boldsymbol{\rho}'_i, \boldsymbol{\rho}_{i+1}, \ldots, \boldsymbol{\rho}_n)\big)$ and $\boldsymbol{\rho}'_i$ is a shuffle of $\boldsymbol{\rho}_i$.

A *collision of levels* i and $i+1$ of $\mathcal{M}$ is an element of $\mathcal{M}^B\big(\mathbf{x}, \mathbf{y}; (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_i \uplus \boldsymbol{\rho}_{i+1}, \ldots, \boldsymbol{\rho}_n)\big)$ where the sets of Reeb chords $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_{i+1}$ are composable.

Remark 3.21. Notice that an end where a split curve is degenerated at east infinity (example 3.17) is indeed a special case of a collision of levels. Furthermore the definition of shuffle curve end contemplates only the degeneration of odd shuffle curves at east infinity (example 3.19).

These are morally the 'ends' of our moduli spaces. Here we say morally because the identification of those is not as neat as in the closed case, as generally $\overline{\mathcal{M}}$ does not have a manifold structure. In fact there are transversality issues at $e\infty$ where one has shuffle curves or split curves with more than one split component, because these kind of curves have moduli themselves. For the shuffle curves this was shown in example 3.13, while in what follows we show it for split curves.

Example 3.22. Consider the following situation, which is made taking two curves as in example 3.17.



Here we consider the curves with asymptotics the ordered partition $(\{\rho_3, \rho_1\}, \{\rho_4, \rho_2\})$, and this is a 1-dimensional space parametrized by the position of the upper branch point. As the upper branch point reaches the boundary, also the lower one does, so our moduli space degenerates at east infinity a split curve with two split components.

This end is not isolated (as it should be for the end of a 1-manifold), and has indeed one dimensional moduli (the parametrization is given by the difference in the $\mathbb{R}$ coordinate of $\mathbb{R} \times Z$ of the two branching points), and so the compactification does not achieve transversality at this end.

In order to circumvent these complications, when referring to the 'ends' of the moduli space $\mathcal{M}$ we will always intend the curves of the previous definition, and the weaker but central result we are interested in is the following one.

THEOREM 3.23. *Fix a generic admissible almost complex structure, suppose* $(\mathbf{x}, \vec{\boldsymbol{\rho}})$ *is strongly boundary monotone and consider a* 1*-dimensional moduli space* $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})$*. Then the total number of*

(1) two story ends;
(2) join curves;
(3) shuffle curves;
(4) collision of levels;

is even.

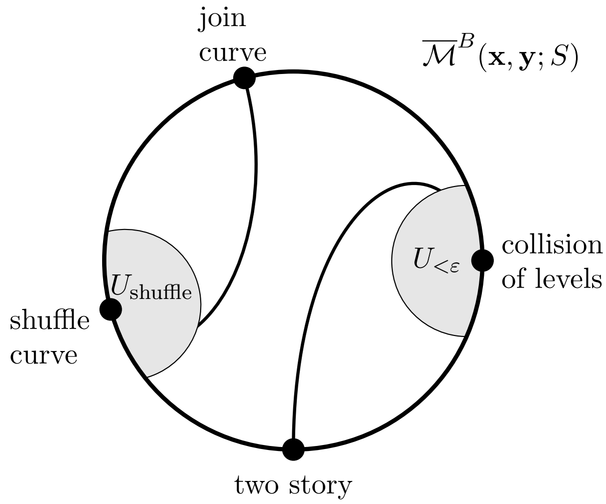
PROOF. We give a sketch of the proof. The big part is the study of the moduli spaces of non necessarily embedded curves $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S; P)$ we have introduced before. By dimensional and topological considerations analogue to those in the closed case (see proposition 3.12 of chapter 1), one obtains that the points of $\partial\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; P)$ are holomorphic combs of the following form:

- a two-story holomorphic building (u_1, u_2);
- a holomorphic story (u, v) with v is a join curve;
- a holomorphic story $(u, v_1, \ldots, v_k)$ where each v_i is a split curve, and the result of gluing all the sources of the v_i's is also a split curve;
- a holomorphic story (u, v) with v is a shuffle curve.

Furthermore, as we consider $\partial\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; \vec{P})$, there will be also a general case where two distinct levels collide (as in example 3.18). The main technical complication compared to the closed case is due to the fact that generally the closures $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; P)$ will not be manifolds at all, showing a bad behavior near the boundary where there are shuffle curves or two different levels of $\vec{P}$ collide. The idea is then to consider the space

$$\overline{\mathcal{M}}^B_{\text{cropped}} = \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; \vec{P}) \setminus (U_{<\varepsilon} \cup U_{\text{shuffle}})$$

where $U_{<\varepsilon}$ (for ε small enough) is the open subset in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S; \vec{P})$ where two levels are closer than ε and U_{shuffle} is a small neighborhood of the shuffle curves in the boundary.



The space $\overline{\mathcal{M}}^B_{\text{cropped}}$ is a compact 1-manifold, and its ends are by gluing and compactness results exactly the analogue of definition 3.20 for not necessarily embedded curves. Note that for not necessarily embedded curves the notions of split, shuffle and composable have to be intended only in a *weak* sense (meaning for example that in a split of Reeb chords there might be the creation of double crossings, see remark 1.9). Furthermore, our definition of shuffle curve end contemplates only odd shuffle curves at $e\infty$, and this is consistent because one proves by some local considerations (following example 3.13) that actually each odd shuffle curves at $e\infty$ contributes to $\partial\overline{\mathcal{M}}^B_{\text{cropped}}$ with an odd number of points, while even shuffle curves contributes with an even number (which also justifies the name).

Finally, in order to study the real moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})$, one has just to notice that from what we have said the statement of the theorem is reduced to an easy algebraic problem as by proposition 3.8 embeddedness is a numerical condition and also for a weak split (shuffle, composable) to be genuinely split (shuffle, composable) is an algebraic condition on the Maslov index of the induced sequences of sets of Reeb chords. ☐

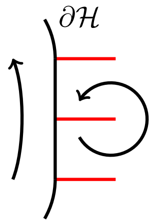
Similar (but simpler) arguments imply the compactness of 0-dimensional moduli spaces as in the closed case (corollary 3.13 of chapter 1):

PROPOSITION 3.24. *For a generic choice of the admissible almost complex structure, given a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and a sequence of sets of Reeb chords $\vec{\boldsymbol{\rho}}$ with $\operatorname{ind}(B, \vec{\boldsymbol{\rho}}) = 1$, the moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})$ is a compact 0-manifold, i.e. a finite set of points.*

4. Type D modules

We turn now our attention to the definition of the type D module $\widehat{CFD}(\mathcal{H}; J)$ for a bordered Heegaard diagram $\mathcal{H}$ and a generic admissible almost complex structure J. The module structure of this object will be straightforward, while as we have said in the introduction of this chapter, the differential will be defined by counting some special kind holomorphic curves.

4.1. Definition of the type D module. Fix a provincially admissible bordered Heegaard diagram $\mathcal{H}$, and let $\mathcal{Z}$ be the pointed matched circle $-\partial\mathcal{H}$, that is the pointed matched circled determined by $\mathcal{H}$ with its orientation reversed. We perform this orientation reversal because we think of $\mathcal{H}$ on the right side, so the orientation induced on the boundary is the opposite of the usual one.



Fix also a generic admissible almost complex structure J on $\Sigma \times [0,1] \times \mathbb{R}$. Consider the vector space $X(\mathcal{H})$ generated by the set of generators $\mathfrak{S}(\mathcal{H})$ over $\mathbb{F}_2$. We define a left action of the subring of idempotents $\mathcal{I}(\mathcal{Z}) \subset \mathcal{A}(\mathcal{Z})$ as follows.

Given $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, $o(\mathbf{x}) \subset [2k]$ denotes as usual the set of arcs occupied by $\mathbf{x}$. We then define

$$I(\mathbf{s}) \cdot \mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \mathbf{s} = [2k] \setminus o(\mathbf{x}), \\ 0 & \text{otherwise.} \end{cases}$$

i.e. an idempotent acts as the identity on $\mathbf{x}$ if it corresponds to the complement of the arcs occupied by $\mathbf{x}$, and acts trivially otherwise.

We define $\widehat{CFD}(\mathcal{H}; J)$ to be the left $\mathcal{A}(\mathcal{Z})$-module

$$\widehat{CFD}(\mathcal{H}; J) = \mathcal{A}(\mathcal{Z}) \otimes_{\mathcal{I}(\mathcal{Z})} X(\mathcal{H})$$

where the module structure is the one induced by the tensor product.

Remark 4.1. Note that only the $\mathcal{A}(\mathcal{Z}, 0)$ summand of $\mathcal{A}(\mathcal{Z})$ acts non trivially.

The count of holomorphic curves comes in the definition of the differential as follows. By $\vec{\rho}$ we will denote a sequence of Reeb chords $\vec{\rho} = (\rho_1, \ldots, \rho_n)$ in $\partial\mathcal{H}$. To this we may associate an element

$$a(-\vec{\rho}) = a(-\rho_1) \ldots a(-\rho_n) \in \mathcal{A}(\mathcal{Z}).$$

Notice that because of the orientation reversing, if ρ is a Reeb chord in $\partial\mathcal{H}$ that $-\rho$ is a Reeb chord in $\mathcal{Z}$, and also that the notation here is a little misleading as this algebra element is usually different from the algebra element associated to the set of Reeb chords. Then we define

$$\partial(\mathbf{I} \otimes \mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mathrm{ind}(B, \boldsymbol{\rho}) = 1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})) \cdot a(-\vec{\rho}) \otimes \mathbf{y}$$

where here we implicitly imposed the compatibility condition for $(B, \vec{\rho})$ (definition 3.7), and extend this map to all $\widehat{CFD}(\mathcal{H}; J)$ by the Leibniz rule

$$\partial(a \otimes \mathbf{x}) = \partial a \otimes \mathbf{x} + a \cdot \partial(\mathbf{I} \otimes \mathbf{x}).$$

Notation 4.2. We will now on use the notation $a\mathbf{x}$ for $a \otimes \mathbf{x}$. In particular $\mathbf{I} \otimes \mathbf{x}$ will simply denoted by $\mathbf{x}$.

As usual, the admissibility condition assures that the map is well defined.

LEMMA 4.3. *Suppose $\mathcal{H}$ is provincially admissible. Then the boundary map ∂ is well defined, i.e. the sum involved in the definition of $\partial\mathbf{x}$ is finite for every $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$.*

PROOF. First of all, notice then all that each element $a \in \mathcal{A}(\mathcal{Z})$ can be written as $a(-\vec{\rho})$ for only finitely many sequences of Reeb chords $\vec{\rho}$. Then, lemma 2.20 tells us that there are only finitely many $B \in \pi_2(\mathbf{x}, \mathbf{y})$ such that B is positive and $\partial^\partial B = [\vec{\rho}]$, so that $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ is not empty only for finitely many pairs $(B, \vec{\rho})$, and by proposition 3.13 in these cases it consists only of finitely many points. $\square$

4.2. $\partial^2 = 0$. We now want to prove that the boundary map ∂ is effectively a differential, i.e. $\partial^2 = 0$. Observe that the boundary map can be written in the form $\partial\mathbf{x} = \sum_{\mathbf{y}} a_{\mathbf{x},\mathbf{y}}\mathbf{y}$ where $a_{\mathbf{x},\mathbf{y}} \in \mathcal{A}(\mathcal{Z})$, and as

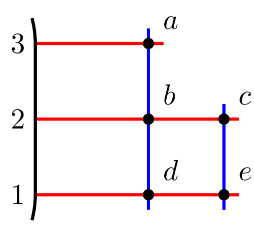
$$\begin{aligned}\partial^2(a\mathbf{x}) &= \partial\big[(\partial(a)\mathbf{x} + a\big(\sum_{\mathbf{w}} a_{\mathbf{x},\mathbf{w}}\mathbf{w}\big)\big]\\ &= (\partial^2 a)\mathbf{x} + 2(\partial a)(\partial\mathbf{x}) + a\big[\big(\sum_{\mathbf{w}}(\partial a_{\mathbf{x},\mathbf{w}})\mathbf{w}\big) + \big(\sum_{\mathbf{w}}\sum_{\mathbf{y}} a_{\mathbf{x},\mathbf{w}}a_{\mathbf{w},\mathbf{y}}\mathbf{y}\big)\big]\end{aligned}$$

in order to prove $\partial^2 = 0$ one has just to prove that for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathfrak{S}(\mathcal{H})$

$$\partial a_{\mathbf{x},\mathbf{y}} + \sum_{\mathbf{w}} a_{\mathbf{x},\mathbf{w}}a_{\mathbf{w},\mathbf{y}} = 0. \tag{2.1}$$

Before sketching the details of the proof, we expose some simple local examples which illustrate how the whole thing works.

EXAMPLE 4.4. This local example is based on example 3.16, with the orientation reversal needed to define type D structures.



This diagram has 4 generators $\{a, c\}, \{a, e\}, \{b, e\}$ and $\{c, d\}$, and the non trivial boundary maps are (using the notations of subsection 1.4 of chapter 1):

$$\begin{aligned}\partial\{a, c\} &= \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\{a, e\} + \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\{c, d\}\\ \partial\{c, d\} &= \{b, e\}\\ \partial\{a, e\} &= \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\{b, e\}.\end{aligned}$$

The fact that $\partial^2 = 0$ is then exactly the relation in the algebra that

$$\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right].$$

This has the following geometric interpretation. There are two 1-dimensional moduli spaces connecting $\{a, c\}$ to $\{b, e\}$, the one as in example 3.16 which has $e\infty$ asymptotics $[1, 3]$ and another with $e\infty$ asymptotics $([1, 2], [2, 3])$, given by two holomorphic disks, one connecting $\{c\}$ to $\{e\}$ and one connecting $\{a\}$ to $\{b\}$, and parametrized by the positive difference of their evaluations (notice that in this situation the moduli space connecting $\{a,c\}$ to $\{b,e\}$ with asymptotics $([2,3],[1,2])$ is empty as it not boundary monotone).

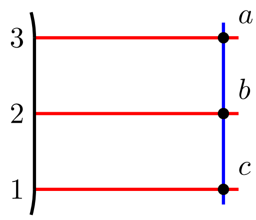
Then the join end of the first family and the collision end of the second family are clearly the same curve connecting $\{a,c\}$ to $\{b,e\}$ with asymptotics $\{[1,2],[2,3]\}$. On the other hand the two-story end of the first family corresponds to the double boundary

$$\{a,c\} \xrightarrow{\partial} \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\{c,d\} \xrightarrow{\partial} \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\{b,e\},$$

while the two-story end of the second family corresponds to the double boundary

$$\{a,c\} \xrightarrow{\partial} \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\{a,e\} \xrightarrow{\partial} \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\{b,e\}.$$

Example 4.5. This is the type D interpretation of example 3.17.



Here we have 3 generators $\{a\},\{b\}$ and $\{c\}$ with non trivial boundary maps

$$\partial\{a\} = \left[\begin{smallmatrix}2&1\\3&\end{smallmatrix}\right]\{b\} + \left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right]\{c\}$$
$$\partial\{b\} = \left[\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right]\{c\}$$

where here in the algebra element we also add the strand of the element not involved in the differential to get the right idempotent (this operation was trivial in the previous example). The relation $\partial^2 = 0$ comes from the relation in the algebra

$$\partial\left[\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&2\\2&3\end{smallmatrix}\right] = \left[\begin{smallmatrix}2&3\\1&\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right].$$

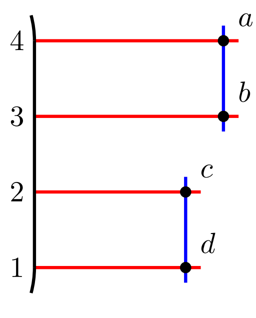
This is explained in a geometric fashion by considering the 1-parameter family of holomorphic curves in example 3.17 connecting $\{a\}$ to $\{c\}$ with $e\infty$ asymptotics $([2,3],[1,2])$. The two-story end corresponds to the chain of boundary maps

$$\{a\} \xrightarrow{\partial} \left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right]\{b\} \xrightarrow{\partial} \left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right]\left[\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right]\{c\}.$$

while the split curve end corresponds via the Leibniz rule and lemma 1.14 to

$$\{a\} \xrightarrow{\partial} \left[\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right]\{c\} \xrightarrow{\partial} \left[\begin{smallmatrix}1&2\\2&3\end{smallmatrix}\right]\{c\}.$$

Example 4.6. This corresponds to example 3.18.



There are four generators $\{a,c\},\{a,d\},\{b,c\}$ and $\{b,d\}$, and non trivial boundary maps

$$\begin{aligned}\partial\{a,c\} &= \left[\begin{smallmatrix}3&1\\4&\end{smallmatrix}\right]\{b,c\} + \left[\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right]\{a,d\}\\ \partial\{b,c\} &= \left[\begin{smallmatrix}1&4\\2&\end{smallmatrix}\right]\{b,d\}\\ \partial\{a,d\} &= \left[\begin{smallmatrix}3&2\\4&\end{smallmatrix}\right]\{b,d\}\end{aligned}$$

and the relation $\partial^2 = 0$ corresponds to the identity in the algebra

$$\left[\begin{smallmatrix}3&1\\4&\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&4\\2&\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}3&2\\4&\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\right]$$

i.e. the fact that the algebra elements associated to the Reeb chords $[1,2]$ and $[3,4]$ commute.

The geometrical intuition behind this is given by considering the two 1-dimensional moduli spaces connecting $\{a,c\}$ to $\{b,d\}$, the first with east asymptotics $([1,2],[3,4])$ and the second with east asymptotics $([3,4],[1,2])$. In particular, both have a collision of levels end, which is precisely the same curve connecting $\{a,c\}$ to $\{b,d\}$ with east asymptotics $\{[1,2],[3,4]\}$. The two story end of the first family, which is reached for $ev_{[3,4]} - ev_{[1,2]} \to +\infty$ corresponds to the double boundary

$$\{a,c\} \xrightarrow{\partial} \left[\begin{smallmatrix}3&4\\2&\end{smallmatrix}\right]\{b,c\} \xrightarrow{\partial} \left[\begin{smallmatrix}3&4\\2&\end{smallmatrix}\right]\left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right]\{b,d\},$$

while the second, which is reached for $ev_{[3,4]} - ev_{[1,2]} \to -\infty$, corresponds to

$$\{a,c\} \xrightarrow{\partial} \left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right]\{a,d\} \xrightarrow{\partial} \left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right]\left[\begin{smallmatrix}3&4\\2&\end{smallmatrix}\right]\{b,d\}.$$

THEOREM 4.7. *The boundary operator ∂ is a differential, i.e. $\partial^2 = 0$.*

PROOF. We give a sketch of the proof. As usual, this is proved by considering the ends of 1-dimensional moduli spaces connecting two fixed generators $\mathbf{x},\mathbf{y} \in \mathfrak{S}(\mathcal{H})$. Given $B \in \pi_2(\mathbf{x},\mathbf{y})$ and a compatible $\vec{\rho}$ such that $\operatorname{ind}(B,\vec{\rho}) = 2$, by theorem 3.23 we have that the sum of the following quantitites is equal to 0:

(1) the number of two-story ends, i.e. the number of elements of the form

$$\mathcal{M}^{B_1}(\mathbf{x},\mathbf{w};\vec{\rho}_1) \times \mathcal{M}^{B_2}(\mathbf{w},\mathbf{y};\vec{\rho}_2)$$

where $B = B_1 * B_2$ and $\vec{\rho} = (\vec{\rho}_1,\vec{\rho}_2)$;

(2) the number of split curve ends (with necessarily one split component), i.e. the number of elements of

$$\mathcal{M}^B\big(\mathbf{x},\mathbf{y};(\rho_1,\dots,\rho_{i-1},\rho_i \uplus \rho_{i+1},\rho_{i+2},\dots,\rho_n)\big)$$

where ρ_i and ρ_{i+1} abut;

(3) the number of collision of levels, i.e. the number of curves in

$$\mathcal{M}^B\big(\mathbf{x},\mathbf{y};(\rho_1,(\rho_1,\dots,\rho_{i-1},\{\rho_i,\rho_{i+1}\},\rho_{i+2},\dots,\rho_n)\big);$$

where ρ_i and ρ_{i+1} do not abut;

(4) th number of join curve ends, i.e. the number of curves in

$$\mathcal{M}^B\big(\mathbf{x},\mathbf{y};(\rho_1,(\rho_1,\dots,\rho_{i-1},\{\rho_j,\rho_k\},\rho_{i+1},\dots,\rho_n)\big)$$

where $\rho_i = \rho_j \uplus \rho_k$.

Note that curves of type (2) and (3) all come from collision of level curves of definition 3.20 (depending whether the colliding Reeb chords abut or not), and that shuffle curves do not appear as every set of the partition has only one element. Fix an element $a \in \mathcal{A}(\mathcal{Z})$, and sum the previous objects over all $\vec{\rho}$ such that $a = a(-\vec{\rho})$. If we define

$$a^B_{\mathbf{x},\mathbf{y}} = \sum_{\operatorname{ind}(B,\vec{\rho})=1} \#\big(\mathcal{M}^B(\mathbf{x},\mathbf{y};\vec{\rho})\big) \cdot a(-\vec{\rho})$$

then the first type of curves corresponds straightforwardly to the coefficient of a in the sum

$$\sum_{\mathbf{w}} \sum_{B_1 * B_2 = B} a^{B_1}_{\mathbf{x},\mathbf{w}} a^{B_2}_{\mathbf{w},\mathbf{y}}.$$

as in the two-story ends of our examples. The second kind of curves corresponds to $\partial a^B_{\mathbf{x},\mathbf{y}}$ through the observation that

$$\partial(a(-\rho_i)) = \sum_{\{\rho_j,\rho_k | \rho_i = \rho_j \uplus \rho_k\}} a(\{-\rho_j, -\rho_k\}) = \sum_{\{\rho_j,\rho_k | \rho_i = \rho_j \uplus \rho_k\}} a(-\rho_j)a(-\rho_k)$$

so that summing all up and relabeling we obtain that

$$\partial a^B_{\mathbf{x},\mathbf{y}} = \sum_{\operatorname{ind}(B,\vec{\rho})=2} \sum_{\left\{\substack{\rho_i,\rho_{i+1} \\ \text{abut}}\right\}} \#\big(\mathcal{M}^B(\mathbf{x},\mathbf{y};(\rho_1,\dots,\rho_i \uplus \rho_{i+1},\dots,\rho_n)\big)a(-\vec{\rho})$$

which is exactly what happens in the split curve end of example 4.5. For the third case, there are many cases that do not contribute because of combinatorial or algebraic reasons. For example, if ρ_i and ρ_{i+1} are interleaved, $a(\rho_i)a(\rho_{i+1}) = 0$. At the end, after having analyzed many cases, one remains only with two possibilities that may happen:

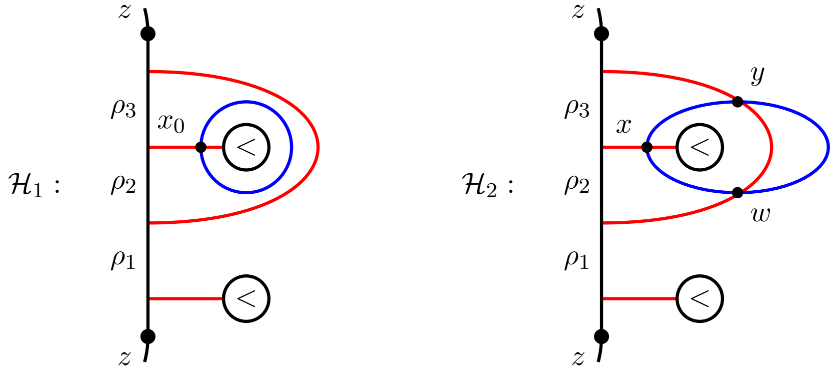
- ρ_{i+1} and ρ_i abut. Then this end is exactly the same end of case for where one considers the east asymptotics $(\rho_1,\dots,\rho_{i+1} \uplus \rho_i,\dots,\rho_n)$, as

  $$a(-\rho_1)\dots a(-\rho_i)a(-\rho_{i+1})\dots a(-\rho_n) = a(-\rho_1)\dots a(-(\rho_{i+1} \uplus \rho_i))\dots a(-\rho_n).$$

  This situation is what happens in example 4.4 where a join curve end and a collision of levels cancel with each other.
- ρ_i and ρ_{i+1} are nested or disjoint (in either order), and all the matchings are disjoint. Then as $a(-\rho_i)$ and $a(-\rho_{i+1})$ commute in $\mathcal{A}(\mathcal{Z})$, one gets the same degeneration considering the asymptotics $(\rho_1,\dots,\rho_{i+1},\rho_i,\dots,\rho_n)$. This is exactly the case of example 4.6.

So the third and fourth case all cancel with each other. Summing then over all possible $B \in \pi_(\mathbf{x},\mathbf{y})$, one proves the relation 2.1. □

4.3. The torus. Here we analyze a simple global example, namely the genus 1 handlebody. We will use the following bordered Heegaard diagrams.



The diagram $\mathcal{H}_1$ is exactly the one of example 1.2 drawn from a type D point of view and is provincially admissible, while $\mathcal{H}_2$ is obtained from $\mathcal{H}_1$ by isotopy of the β-curve and is admissible.

Then $\widehat{CFD}(\mathcal{H}_1; J)$ has only one generator x, and a differential

$$\partial x_0 = \rho_{23} x_0$$

where we use the notation of the torus algebra of example 1.19. Notice that there are infinitely many curves connecting x to x, respectively with asymptotics $(\rho_2, \rho_{23}, \rho_3)$, $(\rho_2, \rho_{23}, \rho_{23}, \rho_3)$ and so on, but the algebra elements associated to these curves is trivial (this is possible because of the non admissibility of the diagram). On the other hand $\widehat{CFD}(\mathcal{H}_2; J)$ has three generators x, y, w and differentials

$$\begin{aligned} \partial x &= \rho_2 w \\ \partial y &= \rho_3 x + w + \rho_{23} w \\ \partial w &= 0, \end{aligned}$$

and it is straightforward to verify that in both cases $\partial^2 = 0$. Notice that $\widehat{CFD}(\mathcal{H}_1; J)$ and $\widehat{CFD}(\mathcal{H}_2; J)$ are not isomorphic, and they define an invariant of the bordered 3-manifold in a sense that will be more precise later.

5. Something about $\mathcal{A}_\infty$ structures

In this section we briefly introduce the basic definitions regarding $\mathcal{A}_\infty$ algebras and modules which will be used in the rest of the work. In fact, while the type D module $\widehat{CFD}(\mathcal{H}; J)$ is a genuine differential module, its left counterpart $\widehat{CFA}(\mathcal{H}; J)$ has a much more complicated structure, where associativity holds only up to homotopy (in a suitable sense). This is neatly described in the framework of $\mathcal{A}_\infty$ structures, which have recently become a common topic in symplectic geometry (see [**Sei08**]).

The $\mathcal{A}_\infty$ modules we will consider subsequently will be defined over the differential algebra $\mathcal{A}(\mathcal{Z})$ rather than a general $\mathcal{A}_\infty$ algebra, and so the structure will be much more simple. Anyway for the sake of clarity we will describe the situation in his full generality. We give the definition for right $\mathcal{A}_\infty$ modules, but clearly all the definitions have a left counterpart.

5.1. $\mathcal{A}_\infty$ algebras. Let $\mathbf{k}$ be a characteristic 2 ring (but everything can be defined with the right sign conventions for every ring).

DEFINITION 5.1. An $\mathcal{A}_\infty$ *algebra* $\mathcal{A}$ over $\mathbf{k}$ is a $\mathbf{k}$-module A together with a collection of $\mathbf{k}$-linear maps

$$\mu_i : A^{\otimes i} \to A, \qquad i \geq 1$$

(where $A^{\otimes i}$ denotes the i-fold tensor product $A \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} A$) that satisfy the compatibility relation

$$\sum_{i+j=n+1} \sum_{l=1}^{n-j+1} \mu_i(a_1 \otimes \cdots \otimes a_{l-1} \otimes \mu_j(a_l \otimes \cdots \otimes a_{l+j-1}) \otimes a_{l+j} \otimes \cdots \otimes a_n) = 0$$

for every $n \geq 1$ and $a_1, \ldots, a_n \in A$.

NOTATION 5.2. We will denote with $\mathcal{A}$ the $\mathcal{A}_\infty$ algebra and with A the underlying $\mathbf{k}$-module.

The notion of $\mathcal{A}_\infty$ algebra is a generalization of the differential algebra one. In fact, the compatibility relation for $n = 1$ reads $\mu_1^2 = 0$, so (A, μ_1) is a chain complex over $\mathbf{k}$. Furthermore, if $\mu_i = 0$ for all $i > 2$, $\mathcal{A}$ is just a genuine differential algebra with associative multiplication μ_2. Indeed the compatibility relation with $n = 2$, is exactly the Leibniz relation

$$\mu_1(\mu_2(a_1 \otimes a_2)) + \mu_2(\mu_1(a_1) \otimes a_2) + \mu_2(a_1 \otimes \mu_1(a_2)) = 0$$

while for $n = 3$, as $\mu_3 = 0$, we get

$$\mu_2(\mu_2(a_1 \otimes a_2) \otimes a_3) + \mu_2(a_1 \otimes \mu_2(a_2 \otimes a_3)) = 0$$

which is exactly the associativity of the multiplication μ_2.

REMARK 5.3. In the case $\mu_3 \neq 0$, the compatibility relation for $n = 3$ tells that the maps

$$a_1 \otimes a_2 \otimes a_3 \mapsto \mu_2(\mu_2(a_1 \otimes a_2) \otimes a_3) \qquad \text{and} \qquad a_1 \otimes a_2 \otimes a_3 \mapsto \mu_2(a_1 \otimes \mu_2(a_2 \otimes a_3))$$

are homotopic as chain maps $A^{\otimes 3} \to A$. In this sense associativity holds up to homotopy.

There are some basic additional properties one usually requires on $\mathcal{A}_\infty$ algebras.

DEFINITION 5.4. An $\mathcal{A}_\infty$ algebra $\mathcal{A}$ is said to be *strictly unital* if there exists a *unit* $1 \in A$ such that $\mu_2(a, 1) = \mu_2(1, a) = a$ for every $a \in A$, and $\mu_i(a_1, \ldots, a_i) = 0$ if $i \neq 2$ and at least one of the a_j's is equal to 1.

$\mathcal{A}$ is *operationally bounded* if $\mu_i = 0$ for i sufficiently big.

All the relations we will deal with while working with $\mathcal{A}_\infty$ structures are quite ugly, but there is a nicer way to treat them by means of the tensor algebra $\mathcal{T}^*A = \bigoplus_{n=0}^{\infty} A^{\otimes n}$. Define the endomorphism $\overline{D} : \mathcal{T}^*A \to \mathcal{T}^*A$ as

$$\overline{D}(a_1 \otimes \cdots \otimes a_n) = \sum_{j_1}^{n} \sum_{l=1}^{n-j+1} a_1 \otimes \cdots \otimes \mu_j(a_l \otimes \cdots \otimes a_{l+j-1}) \otimes \cdots \otimes a_n.$$

Then the compatibility relation is then simply stated as $\overline{D} \circ \overline{D} = 0$ or $\mu \circ \overline{D} = 0$, which can be drawn in a graphical way

$$\begin{array}{ccc} \Downarrow & \qquad\qquad & \Downarrow \\ \overline{D} & & \overline{D} \\ \Downarrow & & \Downarrow \\ \overline{D} & & \mu \\ \Downarrow & & \downarrow \end{array}$$

where the normal arrows mean that we are dealing with elements of $\mathcal{A}$, while the thick arrows are for elements of $\mathcal{T}^*A$.

5.2. $\mathcal{A}_\infty$ modules. We now define $\mathcal{A}_\infty$ modules.

Definition 5.5. A *(right)* $\mathcal{A}_\infty$ *module* $\mathcal{M}$ over $\mathcal{A}$ is a right $\mathbf{k}$-module M together with $\mathbf{k}$-linear operations defined for $i \geq 1$

$$m_i : M \otimes A^{\otimes(i-1)} \to M$$

satisfying the compatibility conditions

$$\begin{aligned} 0 = &\sum_{i+j=n+1} m_i(m_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) + \\ &+ \sum_{i+j=n+1} \sum_{l=1}^{n-j} m_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{l-1} \otimes \mu_j(a_l \otimes \cdots \otimes a_{l+j-1}) \otimes \cdots \otimes a_{n-1}) \end{aligned}$$

for every $n \geq 1$, $\mathbf{x} \in M$ and $a_1, \ldots, a_{n-1} \in A$.

If $\mathcal{A}$ is strictly unital, $\mathcal{M}$ is said to be *strictly unital* if for every $\mathbf{x} \in \mathcal{M}$, $m_2(\mathbf{x}, 1) = \mathbf{x}$ and $m_i(\mathbf{x}, a_1, \ldots, a_{i-1}) = 0$ if $i > 2$ and at least one of the a_j's is equal to 1.

The module $\mathcal{M}$ is *bounded* if $m_i = 0$ for i sufficiently big.

Notation 5.6. As in the $\mathcal{A}_\infty$ algebra case, we refer to the $\mathcal{A}_\infty$ module as $\mathcal{M}$ while M will denote the underlying $\mathbf{k}$-module.

Like the notion of $\mathcal{A}_\infty$ algebra generalizes differential algebras, $\mathcal{A}_\infty$ modules generalize the notion of differential module over a differential algebra, which is exactly the case where $m_i = 0$ and $\mu_i = 0$ for all $i > 2$.

As before, the condition has a nicer interpretation in terms of the tensor algebra. One can promote the operations $\{m_i\}$ to a map $\overline{m} : M \otimes \mathcal{T}^*A \to M \otimes \mathcal{T}^*A$

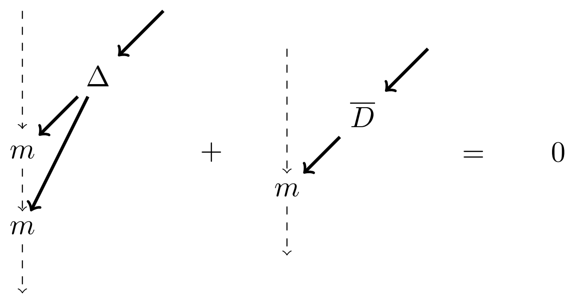
$$\begin{aligned} \overline{m}(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{n-1}) = &\sum_{l=1}^{n-1} m_l(\mathbf{x} \otimes a_2 \otimes \cdots \otimes a_l) \otimes \cdots \otimes a_{n-1} + \\ &+ \sum_{j=2}^{n-1} \sum_{l=1}^{n-j+1} \mathbf{x} \otimes a_1 \otimes \cdots \otimes \mu_j(a_l \otimes \cdots \otimes a_{l+j-1}) \otimes \cdots \otimes a_{n-1} \end{aligned}$$

and the $\mathcal{A}_\infty$ conditions are equivalent to $\overline{m} \circ \overline{m} = 0$, i.e to the fact that $(M \otimes \mathcal{T}^*A, \overline{m})$ is a chain complex.

One can also give a nice graphical representation of these relations. Define the canonical *diagonal map* $\Delta : \mathcal{T}^*A \to \mathcal{T}^*A \otimes \mathcal{T}^*A$

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{l=0}^{n} (a_1 \otimes \cdots \otimes a_l) \otimes (a_{l+1} \otimes \cdots \otimes a_n).$$

Then the compatibility condition can be represented as



where the dashed arrows refer to elements of M.

5.3. $\mathcal{A}_\infty$ homomorphisms. We define now homomorphisms between $\mathcal{A}_\infty$ modules and homotopies between homomorphisms.

DEFINITION 5.7. Given two strictly unital $\mathcal{A}_\infty$ modules $\mathcal{M}$ and $\mathcal{M}'$ over the $\mathcal{A}_\infty$ algebra $\mathcal{A}$ a *strictly unital homomorphism* $f = \{f_i\}$ *of* $\mathcal{A}_\infty$ *modules* (or simply an $\mathcal{A}_\infty$ homomorphism) is a collection of $\mathbf{k}$-linear maps

$$f_i : M \otimes A^{\otimes(i-1)} \to M'$$

with $i \geq 1$ satisfying for each n the following compatibility condition

$$\begin{aligned} 0 = & \sum_{i+j=n+1} m'_i(f_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_j) \otimes \cdots \otimes a_{n-1}) + \\ & + \sum_{i+j=n+1} f_i(m_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_j) \otimes \cdots \otimes a_{n-1}) + \\ & + \sum_{i+j=n+1} \sum_{l=1}^{n-j} f_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{l-1} \otimes \mu_j(a_l \otimes \cdots \otimes a_{l+j-1}) \otimes \cdots \otimes a_{n-1}) \end{aligned}$$

and the unital condition that $f_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$ if $i > 1$ and some $a_j = 1$.

The homomorphism f is *bounded* if $f_i = 0$ for i sufficiently large.

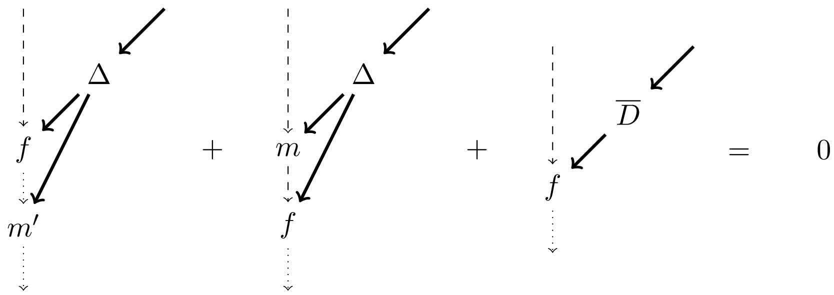
This also has a nice interpretation in terms of tensor algebras. Promote the maps $\{f_i\}$ to $\overline{f} : M \otimes \mathcal{T}^*A \to M' \otimes \mathcal{T}^*A$ by the formulas

$$\overline{f}(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{l=0}^{n} f_{l+1}(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_l) \otimes \cdots \otimes a_n. \tag{2.2}$$

Then the compatibility relation simply states that $\overline{f}$ is a chain map between chain complexes $(M \otimes \mathcal{T}^*A, \overline{m})$ and $(M' \otimes \mathcal{T}^*A, \overline{m}')$, i.e.

$$\overline{m}' \circ \overline{f} + \overline{f} \circ \overline{m} = 0.$$

Furthermore the compatibility condition can be drawn as



where the dotted arrows refer to elements of $\mathcal{M}'$.

EXAMPLE 5.8. For any $\mathcal{A}_\infty$ module $\mathcal{M}$, the *identity homomorphism* $\mathbb{I}$ is the map

$$\mathbb{I}_1(\mathbf{x}) := \mathbf{x}$$
$$\mathbb{I}_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) := 0 \qquad (i > 0).$$

We now discuss the notions of composition and homotopy of maps. At this point the definitions are forced to be as they are, and even if they involve a lot of indices, they are quite intuitive.

DEFINITION 5.9. Given two $\mathcal{A}_\infty$ homomorphisms $f : \mathcal{M} \to \mathcal{M}'$ and $g : \mathcal{M} \to \mathcal{M}''$, one can form their *composite* $g \circ f : \mathcal{M} \to \mathcal{M}''$ as the $\mathcal{A}_\infty$ homomorphism with nth component

$$(g \circ f)_n(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{n-1}) := \sum_{i+j=n+1} g_j(f_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) \otimes \cdots \otimes a_{n-1}).$$

DEFINITION 5.10. An $\mathcal{A}_\infty$ homomorphisms $f : \mathcal{M} \to \mathcal{M}'$ is *nullhomotopic* if there exists a nullhomotopy, i.e. a collection of $\mathbf{k}$-linear maps

$$h_i : M \otimes A^{\otimes(i-1)} \to M'$$

which are unital (i.e. higher maps are trivial if at least one entry is the identity) such that

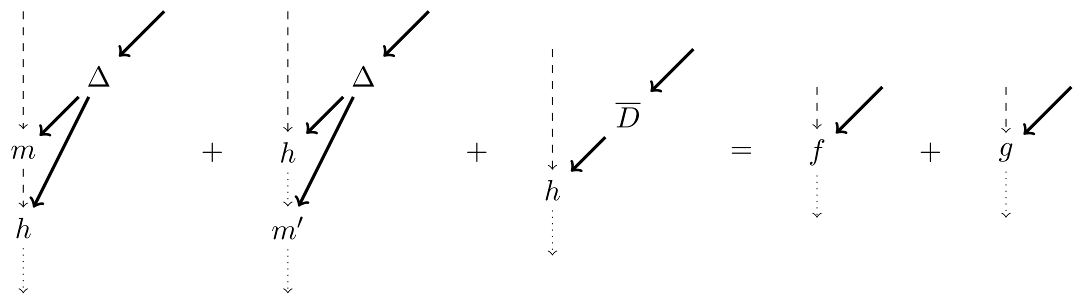
$$\begin{aligned} f_n(\mathbf{x} \otimes a_1 \otimes \ldots \otimes a_{n-1}) = \\ &= \sum_{i+j=n+1} m'_i(h_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_j) \otimes \cdots \otimes a_{n-1}) + \\ &+ \sum_{i+j=n+1} h_i(m'_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_j) \otimes \cdots \otimes a_{n-1}) + \\ &+ \sum_{i+j=n+1} \sum_{l=1}^{n-j} h_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{l-1} \otimes \mu_j(a_l \otimes \cdots \otimes a_{l+j-1}) \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

Two homomorphisms $f, g : \mathcal{M} \to \mathcal{M}'$ are *homotopic* if their sum $f + g$ is nullhomotopic. Two $\mathcal{A}_\infty$ modules $\mathcal{M}$ and $\mathcal{M}'$ are *homotopy equivalent* if there exist homomorphisms $f : \mathcal{M} \to \mathcal{M}'$ and $g : \mathcal{M}' \to \mathcal{M}$ such that their compositions $f \circ g$ and $g \circ f$ are homotopic to the respective identities (example 5.8).

A homotopy h defines a map $\overline{h} : M \otimes \mathcal{T}^*A \to M' \otimes \mathcal{T}^*A$ as for functions (equation 2.2). Then the relation of being homotopic can be rephrased as the more familiar

$$\overline{h} \circ \overline{m} + \overline{m}' \circ \overline{h} = \overline{f} + \overline{g}.$$

We can also express these relations in a graphical way as follows.



6. Type A modules

We now define the type A modules $\widehat{CFA}(\mathcal{H}; J)$ associated to a bordered Heegaard diagram $\mathcal{H}$ together with a generic admissible almost complex structure J on $\Sigma \times [0,1] \times \mathbb{R}$. Here we will use the count of holomorphic curves to define the right $\mathcal{A}_\infty$ module structure over the pointed matched circle algebra $\mathcal{A}(\mathcal{Z})$, with ground ring $\mathcal{I}(\mathcal{Z})$. In particular, unlike the definition of the type D module, here we will consider as $e\infty$ asymptotics sequences of sets of Reeb chords not necessarily consisting of a single element.

6.1. Definition of the type A modules. Fix as usual a bordered Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for the bordered 3-manifold Y, and let $\mathcal{Z} = \partial \mathcal{H}$ be the pointed matched circle associated to its boundary. Fix also a generic admissible almost complex structure J on $\Sigma \times [0,1] \times \mathbb{R}$. Then $\widehat{CFA}(\mathcal{H}; J)$ is generated as a $\mathbb{F}_2$-module by the set of generators of the Heegaard diagram $\mathfrak{S}(\mathcal{H})$, and one defines the right action of the subring of idempotents $\mathcal{I}(\mathcal{Z}) \subset \mathcal{A}(\mathcal{Z})$ as

$$\mathbf{x} \cdot I(\mathbf{s}) = \begin{cases} \mathbf{x} & \text{if } \mathbf{s} = o(\mathbf{x}), \\ 0 & \text{otherwise.} \end{cases}$$

where as usual $o(\mathbf{x}) \subset [2k]$ denotes the set of arcs occupied by the generator $\mathbf{x}$ (so in particular this action is opposite to the one defining $\widehat{CFD}$). This makes $\widehat{CFA}(\mathcal{H}; J)$ a right $\mathcal{I}(\mathcal{Z})$-module. Observe that as in the type D case, only the summand $\mathcal{I}(\mathcal{Z}, 0)$ acts non trivially.

We now define the $\mathcal{A}_\infty$ module structure over the $\mathcal{A}_\infty$ algebra $\mathcal{A}(\mathcal{Z})$, considering as ground ring $\mathcal{I}(\mathcal{Z})$ (so all the following tensor products will be intended to be over this ring of idempotents). In order to do so, one has to define the multiplications

$$m_{n+1} : \widehat{CFA}(\mathcal{H}; J) \otimes \mathcal{A}(\mathcal{Z})^{\otimes n} \to \widehat{CFA}(\mathcal{H}; J)$$

and as $\mathcal{A}(\mathcal{Z})$ is generated by the products of the form $\mathcal{I}(\mathbf{s})a(\boldsymbol{\rho})$ with $\mathbf{s} \subset [2k]$ and $\boldsymbol{\rho}$ a set of Reeb chords, one has just to define the products of the form $m_{n+1}\big(\mathbf{x} \otimes a(\boldsymbol{\rho}_1) \otimes \cdots \otimes a(\boldsymbol{\rho}_n)\big)$ (which we denote by $m_{n+1}\big(\mathbf{x}, a(\boldsymbol{\rho}_1), \ldots, a(\boldsymbol{\rho}_n)\big)$ for notational convenience). It follows immediately from the definitions of the algebra that $\mathbf{x} \otimes a(\boldsymbol{\rho}_1) \otimes \cdots \otimes a(\boldsymbol{\rho}_n)$ is non zero if and only if $(\mathbf{x}, (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n))$ is strongly boundary monotone (subsection 3.2), so we can restrict to that case.

Fix a generic admissible almost complex structure on $\Sigma \times [0,1] \times \mathbb{R}$. Define then

$$\begin{aligned} m_{n+1}\big(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_n)\big) &:= \sum_{\mathbf{y}} \sum_{\substack{B \in \pi_2(\mathbf{x},\mathbf{y}) \\ \operatorname{ind}(B,\vec{\boldsymbol{\rho}})=1}} \#(\mathcal{M}^B(\mathbf{x},\mathbf{y};\vec{\boldsymbol{\rho}})) \cdot \mathbf{y} \\ m_2(\mathbf{x}, \mathbf{I}) &:= \mathbf{x} \\ m_{n+1}(\mathbf{x}, \dots, \mathbf{I}, \dots) &:= 0, \qquad n > 1. \end{aligned}$$

where we implicitly suppose the pair $(B, \vec{\boldsymbol{\rho}})$ to be compatible, and the last two conditions simply state that the $\mathcal{A}_\infty$ structure will be strictly unital.

As usual, admissibility conditions on $\mathcal{H}$ will assure that the sum is actually well defined.

LEMMA 6.1. *If $\mathcal{H}$ is provincially admissible, then all the m_k are well defined. Furthermore, if $\mathcal{H}$ is admissible, there are only finitely many non zero m_k's.*

PROOF. The proof of the first part is straightforward from the positivity of the domains. For the second statement, note that if for a domain B we define $|B|$ to be the sum of all the local multiplicities of the regions, then m_{n+1} involves the count of domains with $|B| \geq n$ (just consider the regions adjacent to the east asymptotics). As in an admissible Heegaard diagram for each pair of generators $\mathbf{x}, \mathbf{y}$ there are only finitely many positive domains connecting them, this implies that $m_i = 0$ for i sufficiently large. □

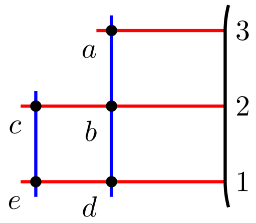
REMARK 6.2. Notice that the m_1 operation is obtained by counting curves not approaching $e\infty$.

6.2. $\mathcal{A}_\infty$ relations. In this subsection we prove that the previously introduced operations actually define a right $\mathcal{A}_\infty$ module structure on $\widehat{CFA}(\mathcal{H}; J)$. This is done as usual by considering the 'ends' of 1-dimensional moduli spaces of curves. Before doing this, we illustrate some local examples which contain the key ideas of the proof. They deal with the same geometric situations as in the type D case (just considering them as rotated), but the degeneration phenomena will be here interpreted in a different fashion.

As $\mathcal{A}(\mathcal{Z})$ is a genuine differential algebra, in this case the $\mathcal{A}_\infty$ relations are simply

$$\begin{aligned} 0 = &\sum_{i+j=n+1} m_i(m_j(\mathbf{x}, a_1, \dots, a_{j-i}), a_j, \dots, a_{n-1}) + \\ &+ \sum_{l=1}^{n-1} m_n(\mathbf{x}, a_1, \dots, \partial a_l, \dots a_{n-1}) \\ &+ \sum_{l=1}^{n-2} m_n(\mathbf{x}, a_1, \dots, a_l a_{l+1}, \dots a_{n-1}). \end{aligned} \tag{2.3}$$

EXAMPLE 6.3. This is the type A interpretation of example 4.4.



Here the generators are $\{a,c\},\{a,e\},\{b,e\}$ and $\{c,d\}$, and the non trivial $\mathcal{A}_\infty$ multiplications are

$$\begin{aligned}
\partial\{b,e\} &= \{c,d\}\\
\{c,d\}\cdot\left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right] &= \{a,c\}\\
\{a,e\}\cdot\left[\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right] &= \{a,c\}\\
\{b,e\}\cdot\left[\begin{smallmatrix}1&2\\2&3\end{smallmatrix}\right] &= \{a,c\}\\
\{b,e\}\cdot\left[\begin{smallmatrix}2&1\\3&\end{smallmatrix}\right] &= \{a,e\}
\end{aligned}$$

so in this case $\widehat{CFA}$ is a genuine differential module over $\mathcal{A}$, as one can verify that the only non trivial $\mathcal{A}_\infty$ relation

$$(\partial\{b,e\})\cdot\left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right]+\{b,e\}\cdot\partial\left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right]=0$$

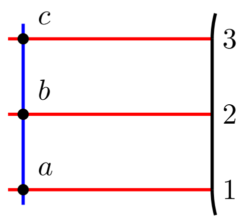
is true because $\partial\left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right]=\left[\begin{smallmatrix}1&2\\2&3\end{smallmatrix}\right]$. The geometric meaning of this relation comes from the usual 1-parameter moduli space connecting $\{b,e\}$ to $\{a,c\}$ with Reeb chord $[1,3]$. The two story end corresponds to the chain of multiplications

$$\{e,b\}\xrightarrow{m_1}\{c,d\}\xrightarrow{\cdot\left[\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right]}\{a,c\}$$

while the join curve end corresponds to

$$\{e,b\}\xrightarrow{\cdot\left[\begin{smallmatrix}1&2\\2&3\end{smallmatrix}\right]}\{a,c\}.$$

Example 6.4. This comes from example 4.5.



Here we have 3 generators $\{a\},\{b\}$ and $\{c\}$ and non trivial $\mathcal{A}_\infty$ operations

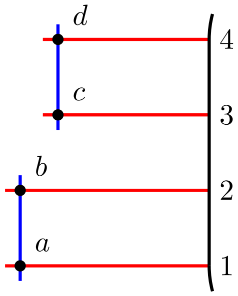
$$\begin{aligned}
\{a\}\cdot\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] &= \{b\}\\
\{a\}\cdot\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right] &= \{c\}\\
\{b\}\cdot\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right] &= \{c\}
\end{aligned}$$

and the only $\mathcal{A}_\infty$ relation to verify is that

$$(\{a\}\cdot\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right])\cdot\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]=\{a\}\cdot(\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right])$$

which is satisfied as $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$ in the algebra. The geometric intuition of this identity relies in the ends of the 1-dimensional moduli space connecting $\{a\}$ to $\{c\}$ with Reeb chords $([1,2],[2,3])$ as in example 3.17, where the two-story end corresponds to the left hand side of the identity, while the split curve end corresponds to the right hand side.

Example 6.5. Here we see example 4.6 from the type A point of view.



There are 4 generators $\{a,c\}$, $\{a,d\}$, $\{b,c\}$ and $\{b,d\}$, and non trivial operations

$$\begin{aligned}
\{a,c\} \cdot \left[\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right] &= \{b,c\}\\
\{a,c\} \cdot \left[\begin{smallmatrix}3&1\\4&\end{smallmatrix}\right] &= \{a,d\}\\
\{a,c\} \cdot \left[\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\right] &= \{b,d\}\\
\{b,c\} \cdot \left[\begin{smallmatrix}3&2\\4&\end{smallmatrix}\right] &= \{b,d\}\\
\{a,d\} \cdot \left[\begin{smallmatrix}1&4\\2&\end{smallmatrix}\right] &= \{b,d\}
\end{aligned}$$

and here the non trivial $\mathcal{A}_\infty$ relations are

$$(\{a,c\} \cdot \left[\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right]) \cdot \left[\begin{smallmatrix}3&2\\4&\end{smallmatrix}\right] = \{a,c\} \cdot \left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right] = (\{a,c\} \cdot \left[\begin{smallmatrix}3&1\\4&\end{smallmatrix}\right]) \cdot \left[\begin{smallmatrix}1&4\\2&\end{smallmatrix}\right].$$

Geometrically, the first identity comes from the moduli spaces of curves connecting $\{a,c\}$ to $\{b,d\}$ with Reeb chords $([1,2],[3,4])$, while the second from the moduli spaces of curves connecting $\{a,c\}$ to $\{b,d\}$ with Reeb chords $([3,4],[1,2])$.

Theorem 6.6. *Given a provincially admissible bordered Heegaard diagram and a generic admissible almost complex structure J, $(\widehat{CFA}(\mathcal{H};J),\{m_n\})$ is a $\mathcal{A}_\infty$ module over $\mathcal{A}(\mathcal{Z})$. Furthermore, if the bordered Heegaard diagram is admissible, the $\mathcal{A}_\infty$ structure is bounded.*

Proof. Fix generators $\mathbf{x},\mathbf{y} \in \mathfrak{S}(\mathcal{H})$, a domain $B \in \pi_2(\mathbf{x},\mathbf{y})$ and a sequence of sets of Reeb chords $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1,\dots,\boldsymbol{\rho}_n)$ with $(\mathbf{x},\vec{\boldsymbol{\rho}})$ strongly boundary monotone, $(B,\vec{\boldsymbol{\rho}})$ compatible and $\mathrm{ind}(B,\vec{\boldsymbol{\rho}}) = 2$. Theorem 3.23 implies that the sum of the following quantities is zero:

(1) the number of two-story ends, i.e. the number of curves in
$$\mathcal{M}^{B_1}(\mathbf{x},\mathbf{w};\vec{\boldsymbol{\rho}}_1) \times \mathcal{M}^{B_2}(\mathbf{w},\mathbf{y};\vec{\boldsymbol{\rho}}_2)$$
with $\mathbf{w} \in \mathfrak{S}(\mathcal{H})$, $B_1 * B_2 = B$ and $\vec{\boldsymbol{\rho}} = (\vec{\boldsymbol{\rho}}_1,\vec{\boldsymbol{\rho}}_2)$;

(2) the number of join curve ends, i.e. the numbers of curves in
$$\mathcal{M}^B\big(\mathbf{x},\mathbf{y};(\boldsymbol{\rho}_1,\dots,\boldsymbol{\rho}_i',\dots,\boldsymbol{\rho}_n)\big)$$
where $\boldsymbol{\rho}_i'$ is a splitting of $\boldsymbol{\rho}_i$;

(3) the number of shuffle curve ends, i.e. the number of elements of

$$\mathcal{M}^B\big(\mathbf{x}, \mathbf{y}; (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}'_i, \ldots, \boldsymbol{\rho}_n)\big)$$

where $\boldsymbol{\rho}'_i$ is a shuffle of $\boldsymbol{\rho}_i$;

(4) the number of collision of levels, i.e. the number of curves in

$$\mathcal{M}^B\big(\mathbf{x}, \mathbf{y}; (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_i \uplus \boldsymbol{\rho}_{i+1}, \ldots, \boldsymbol{\rho}_n)\big)$$

with $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_{i+1}$ composable.

Clearly the first elements correspond to the first term in the $\mathcal{A}_\infty$ relation 2.3. By lemma 1.14 one has that

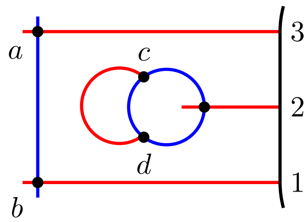
$$\partial a(\boldsymbol{\rho}) = \sum_{\substack{\boldsymbol{\rho}' \text{ a splitting} \\ \text{of } \boldsymbol{\rho}}} a(\boldsymbol{\rho}) + \sum_{\substack{\boldsymbol{\rho}' \text{ a shuffle} \\ \text{of } \boldsymbol{\rho}}} a(\boldsymbol{\rho})$$

so the second and the third quantities correspond to the second term in the $\mathcal{A}_\infty$ relation. Finally, as for composable sets of Reeb chords $a(\boldsymbol{\rho} \uplus \boldsymbol{\rho}') = a(\boldsymbol{\rho})a(\boldsymbol{\rho}')$ (lemma 1.14), the third term of the $\mathcal{A}_\infty$ relation corresponds exactly to the fourth type of ends.

Boundedness in the admissible case is exactly the second part of lemma 6.1. □

6.3. A remark about associativity. In all the local examples we have encountered so far $\widehat{CFA}$ is a differential module. Anyway, as the following example will illustrate, it is generally a genuine $\mathcal{A}_\infty$ module, in the sense that the multiplication m_2 is really associative only up to homotopy, and there are non trivial higher multiplications. Indeed, intuitively associativity holds only when there are only two-story ends (the terms of the form $m_2\big(m_2(\mathbf{x}, a(\boldsymbol{\rho})), a(\boldsymbol{\rho}')\big)$) and collision of levels ends (the terms of the form $m_2\big(\mathbf{x}, a(\boldsymbol{\rho})a(\boldsymbol{\rho}')\big)$).

EXAMPLE 6.7. The following diagram has four generators $\{a, c\}$, $\{a, d\}$, $\{b, c\}$ and $\{b, d\}$.



There are two nontrivial $m_1 = \partial$ operations obtained by counting the disk connecting $\{c\}$ to $\{d\}$

$$\partial\{a, c\} = \{a, d\}$$
$$\partial\{b, c\} = \{b, d\}.$$

For some choices of the generic almost complex structure J (see example 5.4 of chapter 1) there is a holomorphic anulus connecting $\{b, c\}$ to $\{a, c\}$ with Reeb chord $[1, 3]$, and so a non trivial operation

$$\{b, c\} \cdot \left[\tfrac{1}{3}\right] = \{a, c\}.$$

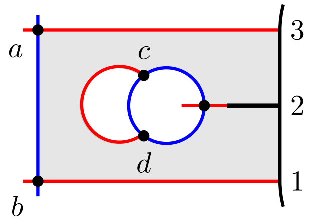
In particular, m_2 cannot be associative as

$$\{a, c\} = \{b, c\} \cdot \left[\tfrac{1}{3}\right] = \{b, c\} \cdot \left(\left[\tfrac{1}{2}\right] \cdot \left[\tfrac{2}{3}\right]\right) \neq \left(\{b, c\} \cdot \left[\tfrac{1}{2}\right]\right) \cdot \left[\tfrac{2}{3}\right] = 0$$

as $\{b,c\} \cdot \left[\frac{1}{2}\right] = 0$ (there are no topological curves indeed), so there should be some non trivial higher multiplications. In particular one has that

$$m_3(\{b,d\}, \left[\tfrac{1}{2}\right], \left[\tfrac{2}{3}\right]) = \{a,c\}$$

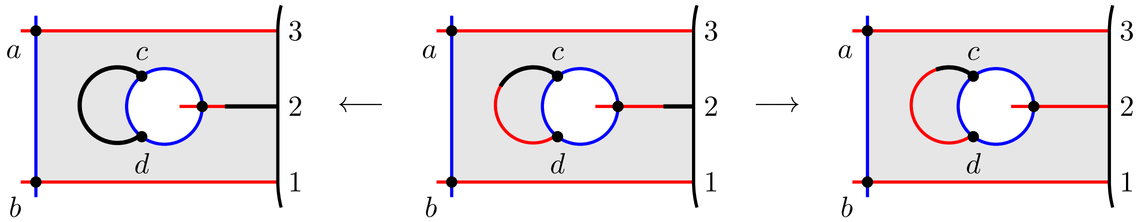
by counting the following annulus (for an appropriare cutting parameter).



The only non trivial associativity relation is then

$$m_3(\partial\{b,c\}, \left[\tfrac{1}{2}\right], \left[\tfrac{2}{3}\right]) + \{b,c\} \cdot \left[\tfrac{1}{3}\right] = 0$$

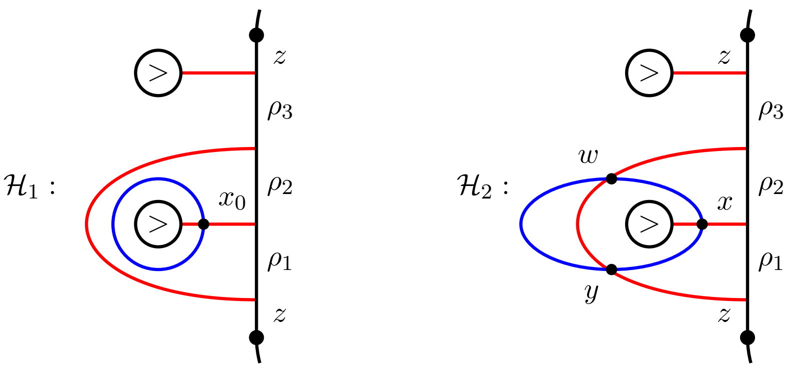
which is clearly true. Geometrically, this comes from the following 1-parameter family of curves connecting $\{b,c\}$ to $\{a,c\}$ with Reeb chords $([1,2],[2,3])$.



Here the end on the left is a two-story holomorphic building which corresponds to the first term of the associativity relation, while the one on the right is a split curve end which corresponds to the second term.

There are some special circumstances all the higher multiplications may vanish, making $\widehat{CFA}(\mathcal{H};J)$ a genuine differential module, see for example section 4 of chapter 3.

6.4. The torus. Here we analyze the bordered Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ for the genus 1 handlebody of example 4.3 from the type A point of view.



The generators are clearly the same as the associated type D modules. The non trivial multiplications for $\widehat{CFA}(\mathcal{H}_2; J)$ are

$$\begin{aligned} m_2(x, \rho_2) &= w \\ \partial y &= w \\ m_2(y, \rho_1) &= x \\ m_2(y, \rho_{12}) &= w \end{aligned}$$

and the $\mathcal{A}_\infty$ relations are readily verified. While this is a differential module, $\widehat{CFA}(\mathcal{H}_1; J)$ is a real $\mathcal{A}_\infty$ module, and furthermore it is not bounded, as

$$\begin{aligned} m_3(x_0, \rho_2, \rho_1) &= x_0 \\ m_4(x_0, \rho_2, \rho_{12}, \rho_1) &= x_0 \\ m_5(x_0, \rho_2, \rho_{12}, \rho_{12}, \rho_1) &= x_0 \\ &\vdots \end{aligned}$$

so there are infinitely many non trivial multiplications (which is possible because the diagram is only provincially admissible). As in example 4.3 the two $\mathcal{A}_\infty$ modules are not isomorphic, and they define an invariant of the bordered 3-manifold in a sense which will be made precise in the next section.

7. Invariance

This section is devoted to the discussion of the following invariance result.

THEOREM 7.1. *Fix a bordered 3-manifold Y. The modules $\widehat{CFD}(\mathcal{H}; J)$ and $\widehat{CFA}(\mathcal{H}; J)$ are, up to homotopy equivalence and $\mathcal{A}_\infty$ homotopy equivalence respectively, independent of the provincially admissible Heegaard diagram $\mathcal{H}$ for Y and the generic admissible almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$.*

In this sense they define invariants of a bordered 3-manifold Y, which we denote by $\widehat{CFD}(Y)$ and $\widehat{CFA}(Y)$.

As in the closed case (section 6 of chapter 1), in light of proposition 2.7 this result is proved by analyzing the effect of the following four types of moves:

- change of the generic admissible almost-complex structure;
- isotopy of the α and β-curves;
- handleslides of the β-curves and of the α-curves over the α-circles;
- stabilizations in the interior of Σ.

The spirit of the proofs of invariance under these moves is identic to the closed case, but the details are clearly more complicated because of the more involved algebra and analysis of our construction. Here we discuss some special cases to illustrate how these difficulties can be circumvented.

7.1. Almost complex structure change for type D modules. As in the closed case (see subsection 6.1 of chapter 1), given two generic admissible almost complex structures J_0 and J_1 one can construct a map $\Phi : (\widehat{CFD}(\mathcal{H}; J_0), \partial_0) \to (\widehat{CFD}(\mathcal{H}; J_1), \partial_1)$ by counting curves holomorphic with respect to a (non translational invariant) almost complex structure J connecting them. In particular the chain map is given on a generator $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ by

$$\Phi(\mathbf{x}) = \sum_{\mathbf{y}\in\mathfrak{S}(\mathcal{H})} \sum_{\substack{B\in\pi_2(\mathbf{x},\mathbf{y})\\ \mathrm{ind}(B,\vec{\rho})=0}} \#(\mathcal{M}^B(\mathbf{x},\mathbf{y};\vec{\rho};J)) \cdot a(-\vec{\rho}) \cdot \mathbf{y}$$

and can then be extended to the whole $\widehat{CFD}$ by $\Phi(a\mathbf{x}) = a\Phi(\mathbf{x})$. First, we need to prove that this is a chain map, i.e. that

$$\partial_1 \circ \Phi + \Phi \circ \partial_0 = 0.$$

As for all $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, $a \in \mathcal{A}(\mathcal{Z})$ one has that

$$\partial_1(\Phi(a\mathbf{x})) + \Phi(\partial_0(a\mathbf{x})) = (\partial a)\Phi(\mathbf{x}) + a\partial_1(\Phi(\mathbf{x})) + a\Phi(\partial_0(\mathbf{x})) + (\partial a)\Phi(\mathbf{x}),$$

one has just to prove that for every $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$

$$\Phi(\partial_0(\mathbf{x})) + \partial_1\Phi(\mathbf{x}) = 0.$$

As in the closed case we consider the ends of 1-dimensional moduli spaces, and the analogue of theorem 3.23 states that the total number of:

(1) two-story ends with a J_0 holomorphic curve followed by a J-holomorphic curve;
(2) two-story ends with a J-holomorphic curve followed by a J_1-holomorphic curve;
(3) join curve ends;
(4) odd shuffle curves;
(5) collision of levels;

is equal to zero. Now, the first term corresponds to $\Phi(\partial_0(\mathbf{x}))$, while the second and the third correspond to $\partial_1\Phi(\mathbf{x})$, respectively when one considers the differential applied to the generators or to the algebra elements. The fourth and the fifth terms cancel as in the proof of theorem 4.7.

The rest of the proof, with some adaptations, follows in the same way as in the closed case.

7.2. Almost complex structure change for type A modules. This is very similar to the previous paragraph, only with extra algebraic complications due to $\mathcal{A}_\infty$ structures.

One defines the $\mathcal{A}_\infty$ homomorphism

$$\Phi = \{\phi_i\} : (\widehat{CFA}(\mathcal{H}; J_0), \{m^0_n\}) \to (\widehat{CFA}(\mathcal{H}; J_1), \{m^1_n\})$$

by letting for a strongly boundary monotone pair $(\mathbf{x}, \vec{\boldsymbol{\rho}})$ with $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n)$

$$\phi_{n+1}(\mathbf{x}, a(\boldsymbol{\rho}_1), \ldots, a(\boldsymbol{\rho}_n)) := \sum_{\mathbf{y}\in\mathfrak{S}(\mathcal{H})} \sum_{\substack{B\in\pi_2(\mathbf{x},\mathbf{y})\\ \mathrm{ind}(B,\vec{\boldsymbol{\rho}})=0}} \#(\mathcal{M}^B(\mathbf{x},\mathbf{y};\vec{\boldsymbol{\rho}};J)) \cdot \mathbf{y}$$

and extending by multilinearity (and imposing strict unitarity) to a map

$$\widehat{CFA}(\mathcal{H}; J_0) \otimes A^{\otimes n} \to \widehat{CFA}(\mathcal{H}; J_1).$$

To show that this is effectively an $\mathcal{A}_\infty$ homomorphism we have to prove that

$$\begin{aligned}
0 = &\sum_{i+j=n+1} m_i^1(\phi_j(\mathbf{x}, a_1, \dots, a_{j-1}), a_j, \dots, a_{n-1}) + \\
= &\sum_{i+j=n+1} \phi_i(m_j^0(\mathbf{x}, a_1, \dots, a_{j-1}), a_j, \dots, a_{n-1}) + \\
&+ \sum_{l=1}^{n-1} \phi_n(\mathbf{x}, a_1, \dots, \partial a_l, \dots a_{n-1}) \\
&+ \sum_{l=1}^{n-2} \phi_{n-1}(\mathbf{x}, a_1, \dots, a_l a_{l+1}, \dots a_{n-1}).
\end{aligned}$$

which is done as usual by considering the ends of 1-dimensional moduli spaces. In particular, similarly to the proof of theorem 6.6 the first two terms correspond to two-story holomorphic buildings, the third to join curve ends and odd shuffle curve ends, and the last one to collision of levels.

With this in mind, the proof proceeds identically as in the closed case. One wants to prove, with the notations of subsection 6.1 of chapter 1, that Φ and Ψ are homotopy inverses. Here we have more $\mathcal{A}_\infty$ complications and we address one of these as an example.

In the closed case we used the fact that the composition $\Psi \circ \Phi$ was the same as the map F induced by the almost complex structure $J \natural_R J'$ for sufficiently big $R > 0$. In this case, the composition has to be intended in an $\mathcal{A}_\infty$ sense, and so one has to prove that:

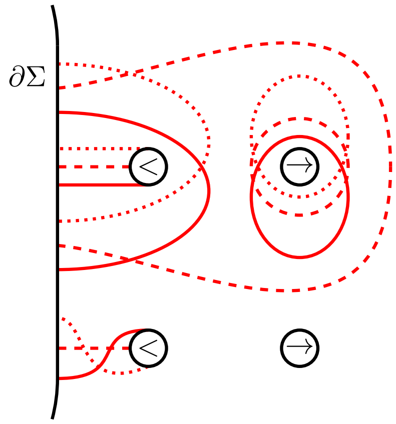
$$F_n(\mathbf{x}, a_1, \dots, a_{n-1}) = (\Psi \circ \Phi)_n(\mathbf{x}, a_1, \dots, a_{n-1}) = \sum_{i+j=n+1} \psi_i(\phi_j(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}).$$

The $\mathbf{y}$ coefficient of $\psi_i(\phi_j(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_{j-1})), \dots, a(\boldsymbol{\rho}_{n-1}))$ is the count of rigid curves of the form

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_j); J) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; (\boldsymbol{\rho}_j, \dots, \boldsymbol{\rho}_{n-1}); J')$$

which, by gluing and compactness results corresponds exactly for R sufficiently large to the count of curves in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{n-1}); J \natural_R J')$.

7.3. Handleslides. Here is the case where the biggest analytical complications come out, and in particular we focus on the handleslide of an α-arc over an α-circle. As in the closed case (subsection 6.3 of chapter 1) we introduce some auxiliary sets of attaching curves, $\boldsymbol{\alpha}^H$ (the dashed ones) and $\boldsymbol{\alpha}'$ (the dotted ones), which are constructed in a similar way as in the closed case.



This is made in order to reduce to a standard computation, and as in the closed case, the homotopy equivalence

$$F_{\boldsymbol{\alpha},\boldsymbol{\alpha}^H,\boldsymbol{\beta}}:\widehat{CFD}(\Sigma,\boldsymbol{\alpha}^H,\boldsymbol{\beta},z)\to\widehat{CFD}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z)$$

is constructed by counting holomorphic curves

$$(T,\partial T)\to\big(\Sigma\times\Delta,(\boldsymbol{\alpha}\times e_\alpha)\cup(\boldsymbol{\beta}\times e_\beta)\cup(\boldsymbol{\alpha}^H\times e_\gamma)\big)$$

with particular boundary conditions and asymptotics. Those curves may have asymptotics at Reeb chords along the e_α and e_β edges, but the main technical issues come from the asymptotics at $p_{\alpha,\gamma}$, as punctures of T mapped there may have as asymptotics both a point in $\boldsymbol{\alpha}\cap\boldsymbol{\alpha}^H$ or a Reeb chord in $\partial\overline{\Sigma}$ connecting points of $\mathbf{a}$ to points of $\mathbf{a}^H$. One has to introduce the concept of curves at $p_{\alpha,\gamma}$ in order to construct a compactification of such moduli spaces. The details are long and technical, but the general spirit is exactly as the closed case.

7.4. The type D genus 1 handlebody. Here we verify directly the invariance of the type D structures of the genus one handlebodies of example 4.3, which diagrams differ by an isotopy of the β-curve. Recall that the type D modules are respectively

$$\widehat{CFD}(\mathcal{H}_1):\ x_0\xrightarrow{\rho_{23}}x_0\qquad\text{and}\qquad\widehat{CFD}(\mathcal{H}_2):\ \begin{tikzcd} y \arrow[d, "\rho_3"'] \arrow[dr, "1+\rho_{23}"] & \\ x \arrow[r, "\rho_2"] & w \end{tikzcd}$$

One can define the maps

$$\Phi:\widehat{CFD}(\mathcal{H}_1)\to\widehat{CFD}(\mathcal{H}_2)\qquad x_0\mapsto x+\rho_2y+\rho_2w$$

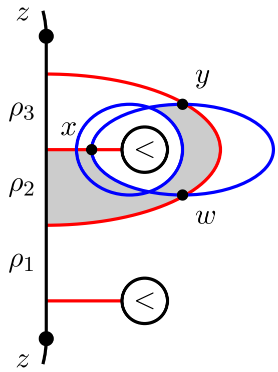
and

$$\Psi:\widehat{CFD}(\mathcal{H}_2)\to\widehat{CFD}(\mathcal{H}_1)\qquad\begin{cases}x&\mapsto&x_0\\ y&\mapsto&\rho_3x_0\\ w&\mapsto&\rho_3x_0\end{cases}$$

which are readily verified to be chain maps. Clearly $\Psi \circ \Phi$ is the identity map of $\widehat{CFD}(\mathcal{H}_1)$, while $\Phi \circ \Psi$ is homotopy equivalent to the identity of $\widehat{CFD}(\mathcal{H}_2)$ through the chain homotopy

$$H : \widehat{CFD}(\mathcal{H}_2) \to \widehat{CFD}(\mathcal{H}_2) \qquad \begin{cases} x & \mapsto & \rho_{23}x + \rho_2 w + \rho_2 y \\ y & \mapsto & \rho_3 x + w \\ w & \mapsto & \rho_3 x + y. \end{cases}$$

There is a geometric motivation behind these maps, analogue to the closed case (subsection 6.2 of chapter 1). Here, in order to prove invariance under an isotopy of the β-curves that moves $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$ one constructs chain maps considering holomorphic curves in $\Sigma \times [0,1] \times \mathbb{R}$ with boundary in a generic collection of lagrangian cylinders which coincides with $\boldsymbol{\beta} \times \{0\} \times \mathbb{R}$ near $-\infty$ and with $\boldsymbol{\beta}' \times \{0\} \times \mathbb{R}$ near $+\infty$. Then the chain maps are constructed by counting curves in index 0 moduli spaces. On the other hand, in order to construct homotopies between the chain maps induced by two such collection of cylinders C and C' one chooses a generic family of lagrangian cylinders C_t connecting C to C' and considers the union over t of index -1 moduli spaces of curves with boundary in C_t (which are non empty only for finitely many t's). For example, the y coefficient of $H(x)$ is guessed considering the following curve



and similarly all the other differentials have such graphical interpretation.

CHAPTER 3

The pairing theorem

In this chapter we study how to construct a pairing of the type A and type D modules in order to recover the information regarding the Heegaard Floer homology of a closed 3-manifold $Y = Y_1 \cup Y_2$ from the knowledge of $\widehat{CFA}(Y_1)$ and $\widehat{CFD}(Y_2)$. In particular, we will define the $\mathcal{A}_\infty$ tensor product $\widetilde{\otimes}$ between $\mathcal{A}_\infty$ modules (which returns a chain complex) and prove the following pairing theorem.

THEOREM 0.2. *$\widehat{CF}(Y)$ is homotopy equivalent to $\widehat{CFA}(Y_1)\widetilde{\otimes}\widehat{CFD}(Y_2)$. In particular*

$$\widehat{HF}(Y) \cong H_*\big(\widehat{CFA}(Y_1)\widetilde{\otimes}\widehat{CFD}(Y_2)\big).$$

We will discuss two proofs of this key result.

The first one, via *time dilation*, is the more geometric one, and involves the study of pairs of holomorphic curves with matching asymptotics which arise when one stretches to infinity the complex structure on the closed surface along a separating curve. This proof, which is quite complicated from an analytical point of view, has the advantage of being more intuitive and to motivate the definition of our modules, which has been quite mysterious by far. For example, it is not really clear at the moment why in the definition of the type A modules we consider curves with asymptotics sequences of sets Reeb chords, while in the type D case we simply consider curves with sequences of Reeb chords.

The second one, via *nice Heegaard diagrams*, is the more combinatorial one. Nice Heegaard diagrams, first introduced by Sarkar and Wang in [**SW10**], are a special class of Heegaard diagrams where the differential of the Floer complex can be computed in a combinatorial way simply by counting some specific domains (while, as we noticed in example 5.4, the Floer complex is generally far from being a combinatorial object). When dealing with such diagrams, the result becomes really neat but this proof in some sense hides the idea behind the whole construction. For example, in nice diagrams there is no need to deal with $\mathcal{A}_\infty$ structures as type A modules are easily shown to be honest differential modules (lemma 4.7)

This is the plan for the rest of the chapter. In section 1 we explain two naïve ways (one geometrical, one algebraic) to make two bordered Heegaard diagrams interact. In particular we will describe how holomorphic curves behave with respect to cutting of the surface, and how to construct a tensor product of $\mathcal{A}_\infty$ modules. The latter turns out to be always infinite dimensional, and so is unlikely to fit well in the count of holomorphic curves. For this reason in section 2 we study type D structures (which generalize our type D modules) and construct a special pairing between them and $\mathcal{A}_\infty$ modules. In section 3 we sketch the proof of the pairing theorem via time dilation, while in section 4, after having discussed the basic results regarding nice Heegaard diagrams, we will prove it using them. Finally in section 5 we show an application of the pairing theorem.

1. Towards the pairing theorem

In this section we describe two ideas which are a first step to prove the pairing theorem. In fact at a first sight it is not obvious how the algebraic objects we have defined on the two sides can describe properly the global invariant. Here we present two constructions, one to decouple the global geometry into the geometry of the two parts, the other to pair the algebraic objects of the two parts.

1.1. The naïve geometric idea. Here we describe a first naïve way to break the chain complex associated to a Heegaard diagram $\mathcal{H} = \mathcal{H}_1 \cup_Z \mathcal{H}_2$ into some algebraic objects depending on the two bordered Heegaard diagrams. The idea here is that given a holomorphic curve $u : S \to \Sigma \times [0,1] \times \mathbb{R}$, if we stretch to infinity the almost complex structure along the separating circle Z this curve will break up into a pair of holomorphic curves $u_1 : S_1 \to \Sigma_1 \times [0,1] \times \mathbb{R}$ and $u_2 : S_2 \to \Sigma_2 \times [0,1] \times \mathbb{R}$ with $S_1 \natural S_2 = S$ and matching asymptotics at east infinity.

This is made more precise as follows. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be provincially admissible Heegaard diagrams with compatible boundaries, i.e. $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2$, and suppose one of them is admissible. Then by lemma 2.20 we have that $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ is an admissible Heegaard diagram. Recall (remark 2.14 of chapter 2) that the set of generators $\mathfrak{S}(\mathcal{H})$ can be identified with the set of compatible pairs $(\mathbf{x}_1, \mathbf{x}_2) \in \mathfrak{S}(\mathcal{H}_1) \times \mathfrak{S}(\mathcal{H}_1)$ (i.e. such that $o(\mathbf{x}_1) \cap o(\mathbf{x}_2) = \emptyset$), which we denote by $\mathfrak{S}(\mathcal{H}_1, \mathcal{H}_2)$.

Define a pair of decorated surfaces (S_1, S_2) to be *compatible* if there exists a bijection $\phi : E(S_1) \to E(S_2)$ such that the Reeb chord labeling $\phi(q)$ is the Reeb chord labeling q with the orientation reversed. Given such a pair, one can construct the surface $S = S_1 \natural S_2$ by gluing along the corresponding punctures.

Fix two pairs of compatible generators $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2)$ in $\mathfrak{S}(\mathcal{H}_1, \mathcal{H}_2)$ inducing $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$, homology classes $B_i \in \pi_2(\mathbf{x}_i, \mathbf{y}_i)$ inducing $B \in \pi_2(\mathbf{x}, \mathbf{y})$ (see lemma 2.16 of chapter 2), and a compatible pair of sources (S_1, S_2) connecting $\mathbf{x}_1$ to $\mathbf{y}_1$ and $\mathbf{x}_2$ to $\mathbf{y}_2$ respectively. We then define the *moduli space of matching pairs* as the fibered product over $\mathbb{R}^{E(S_1)}/\mathbb{R}$

$$\widetilde{\mathcal{M}\mathcal{M}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1; \mathbf{x}_2, \mathbf{y}_2, S_2) = \widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1) \times_{\mathrm{ev}_1 = \mathrm{ev}_2} \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2)$$

i.e. the space of pairs of curves $(u_1, u_2) \in \widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1) \times \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2)$ such that $\mathrm{ev}_1(u_1) = \mathrm{ev}_2(u_2)$ under the correspondence of the punctures induced by the bijection ϕ.

There is the usual $\mathbb{R}$-action by translation on this space, and we will consider the quotient space indicated as usual by dropping the tilde. Actually, we will only be interested in the union of spaces of embedded such pairs inducing a fixed homology class B, which we denote by $\mathcal{M}\mathcal{M}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$.

The transversality results and index theory for embedded curves are totally analogue (with small complications) as the treatment in the previous chapter (see [**LOT11b**]), and in particular embeddedness turns out to be a purely numerical condition on the Euler characteristic of the sources.

We are now ready to define the chain complex $\widehat{CF}(\mathcal{H}_1;\mathcal{H}_2)$. This is generated as a vector space over $\mathbb{F}_2$ by $\mathfrak{S}(\mathcal{H}_1;\mathcal{H}_2)$, and the differential is defined as

$$\partial(\mathbf{x}_1,\mathbf{x}_2) = \sum_{(\mathbf{y}_1,\mathbf{y}_2)\in\mathfrak{S}(\mathcal{H}_1;\mathcal{H}_2)} \sum_{\substack{B\in\pi_2(\mathbf{x},\mathbf{y})\\ \mathrm{ind}(B)=1}} \#(\mathcal{M}\mathcal{M}^B(\mathbf{x}_1,\mathbf{x}_2;\mathbf{y}_1,\mathbf{y}_2))\cdot(\mathbf{y}_1,\mathbf{y}_2)$$

where the sum is finite because of the admissibility hypothesis we imposed on the two bordered Heegaard diagrams.

PROPOSITION 1.1. *For a generic choice of the almost complex structures, $(\widehat{CF}(\mathcal{H}_1;\mathcal{H}_2),\partial)$ is a chain complex and is isomorphic to $\widehat{CF}(\mathcal{H}_1\cup_Z\mathcal{H}_2)$.*

PROOF. The two objects agree as vector spaces by definition, so we just have to prove that the boundary maps coincide for a suitable choice of the almost complex structures. This follows from compactness ad gluing techniques totally analogue to those of the previous chapters, and in particular generic almost complex structures (for which the Z neck is sufficiently long) there is an identification for 0-dimensional moduli spaces

$$\mathcal{M}^B(\mathbf{x},\mathbf{y};S) \qquad \text{and} \qquad \bigcup_{S_1\natural S_2=S}\mathcal{M}\mathcal{M}^B(\mathbf{x}_1,\mathbf{y}_1,S_1;\mathbf{x}_2,\mathbf{y}_2,S_2)$$

and by index considerations this identification respects embeddedness. So the differential of $\widehat{CF}(\mathcal{H}_1;\mathcal{H}_2)$ is exactly the differential of $\widehat{CF}(\mathcal{H}_1\cup\mathcal{H}_2)$. $\square$

1.2. The naïve algebraic idea. As $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ are respectively a right and left $\mathcal{A}_\infty$ algebra over the differential algebra $\mathcal{A}(\mathcal{Z})$, there is a natural 'good' way to pair them via a tensor product $\widetilde{\otimes}$ in order to get a chain complex. The notion of 'good', which motivated by the invariance results we have discussed previously (section 7 of chapter 2), is that the pairing operation should respect quasi-isomorphisms, i.e. that if $\mathcal{M}\cong\mathcal{M}'$ and $\mathcal{N}\cong\mathcal{N}'$ then $\mathcal{M}\otimes\mathcal{N}\cong\mathcal{M}'\otimes\mathcal{N}'$ in a natural way.

The construction of such a pairing is not immediate. For example, even if we are talking of genuine differential modules over a differential algebra, the classical tensor product has not the required properties. A solution to this problem is the tensor product in the $\mathcal{A}_\infty$ setting we are going to define now.

DEFINITION 1.2. Fix a ground ring $\mathbf{k}$ and an $\mathcal{A}_\infty$ algebra over it. Given $\mathcal{M}$ and $\mathcal{N}$, respectively right and left $\mathcal{A}_\infty$ modules, their $\mathcal{A}_\infty$ *tensor product* $\mathcal{M}\widetilde{\otimes}\mathcal{N}$ is the chain complex with underlying module $M\otimes\mathcal{T}^*A\otimes N$ and differential defined by

$$\begin{aligned}\partial(\mathbf{x}\otimes a_1\otimes\cdots\otimes a_n\otimes\mathbf{y}) = &\sum_{i=1}^{n+1} m_i(\mathbf{x}\otimes a_1\otimes\cdots\otimes a_i)\otimes\cdots\otimes a_n\otimes\mathbf{y}\\ &+\sum_{i=1}^{n}\sum_{l=1}^{n-i+1}\mathbf{x}\otimes\cdots\otimes\mu_i(a_l\otimes\cdots\otimes a_{l+i-1})\otimes\cdots\otimes a_n\otimes\mathbf{y}\\ &+\sum_{i=1}^{n+1}\mathbf{x}\otimes a_1\otimes\cdots\otimes m_i(a_{n-i+2}\otimes\cdots\otimes a_n\otimes\mathbf{y}).\end{aligned}$$

The fact that ∂ is effectively a differential is an immediate consequence of the $\mathcal{A}_\infty$ relations. This is a 'good' tensor product as the following holds.

PROPOSITION 1.3. *An $\mathcal{A}_\infty$ homomorphism $f : \mathcal{M} \to \mathcal{M}'$ induces a chain map $f\widetilde{\otimes}\mathbb{I}_{\mathcal{N}} : \mathcal{M}\widetilde{\otimes}\mathcal{N} \to \mathcal{M}'\widetilde{\otimes}\mathcal{N}$. Furthermore, if f is nullhomotopic, then also $f\widetilde{\otimes}\mathbb{I}_{\mathcal{N}}$ is. In particular if $\mathcal{M} \cong \mathcal{M}'$, then $\mathcal{M}\widetilde{\otimes}\mathcal{N} \cong \mathcal{M}'\widetilde{\otimes}\mathcal{N}$.*

Clearly the result holds symmetrically for $\mathcal{N}$.

PROOF. One just defines the induced map by

$$(f\widetilde{\otimes}\mathbb{I}_{\mathcal{N}})(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes \mathbf{y}) = \sum_{i=1}^{n+1} f_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) \otimes \cdots \otimes a_n \otimes \mathbf{y},$$

and a homotopy $h : \mathcal{M} \to \mathcal{M}'$ induces a homotopy $h\widetilde{\otimes}\mathbb{I}_{\mathcal{N}}$ is the same way. □

One important construction in when one considers $\mathcal{N} = \mathcal{A}$ as a right $\mathcal{A}_\infty$ module. This allows in certain situations to construct a honest differential module homotopy equivalent to a given $\mathcal{A}_\infty$ module $\mathcal{M}$.

DEFINITION 1.4. The *bar resolution* $\overline{\mathcal{M}}$ of $\mathcal{M}$ is the chain complex $\mathcal{M}\widetilde{\otimes}\mathcal{A}$ together with the higher operations

$$m_i((\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n) \otimes b_1 \otimes \cdots \otimes b_{i-1}) = \sum_{l=1}^{n} \mathbf{x} \otimes a_1 \otimes \cdots \otimes \mu_{i+l-1}(a_{n-l+1} \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_{i-1}).$$

PROPOSITION 1.5. *If $\mathcal{A}$ and $\mathcal{M}$ are strictly unital then $\overline{\mathcal{M}}$ is an $\mathcal{A}_\infty$ algebra homotopy equivalent to $\mathcal{M}$. Furthermore, if $\mathcal{A}$ is a differential algebra, then $\overline{\mathcal{M}}$ is a genuine differential module.*

PROOF. The only non trivial part is the homotopy equivalence, which we construct concretely. We define $\varphi : \overline{\mathcal{M}} \to \mathcal{M}$ by

$$\varphi_i((\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n) \otimes b_1 \otimes \cdots \otimes b_{i-1}) := m_{i+n}(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_{i-1})$$

and $\psi : \mathcal{M} \to \overline{\mathcal{M}}$ by

$$\psi_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) = \mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes 1.$$

Then $\varphi \circ \psi$ is the $\mathcal{A}_\infty$ identity map (example 5.8), while $h : \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ defined as

$$h_i((\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n) \otimes b_1 \otimes \cdots \otimes b_{i-1}) = \mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_{i-1}.$$

is a homotopy between $\psi \circ \varphi$ and the identity. □

2. A simpler model for the algebraic pairing

Even though the $\mathcal{A}_\infty$ tensor product we have introduced in the previous section has a nice behavior with respect to homotopy equivalences, it is unlikely to fit well in our context. Indeed, it is always infinite dimensional, while our chain complexes are always finite dimensional. For this reason we will introduce a simpler model for the tensor product of $\widehat{CFA}(Y_1)$ and $\widehat{CFD}(Y_2)$, denoted by $\boxtimes$. This exploits the fact that $\widehat{CFD}(\mathcal{H}_2; J) = \mathcal{A}(\mathcal{Z}) \otimes X(\mathcal{H}_2)$ is indeed a genuine differential module over $\mathcal{A}(\mathcal{Z})$, and its differential is defined by a map

$$\partial : X(\mathcal{H}_2) \to \mathcal{A}(\mathcal{Z}) \otimes X(\mathcal{H}_2)$$

satisfying certain compatibility properties. This is a special case of the type D structures we are now going to define.

2.1. Type D structures. The $\widehat{CFD}(\mathcal{H}; J)$ modules will be a special case of the structures we are going to define, but as usual we give a more general definition because it is neater to state and work with.

Fix a characteristic 2 ground ring $\mathbf{k}$ an $\mathcal{A}_\infty$ algebra $\mathcal{A}$ over it. We will always suppose for simplicity that $\mathcal{A}$ is operationally bounded (see definition 5.4 of chapter 2). Consider pairs (N, δ^1) where N is a left $\mathbf{k}$-module and δ^1 is a map $N \to A \otimes N$, with the tensor product taken over $\mathbf{k}$. The latter may be iterated to construct maps $\delta^k : N \to A^{\otimes k} \otimes N$ by

$$\begin{aligned} &\delta^0 = \mathbb{I}_N \\ &\delta^i = (\mathbb{I}_{A^{\otimes(i-1)}} \otimes \delta^1) \circ \delta^{i-1}. \end{aligned}$$

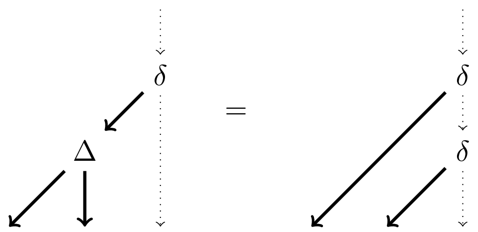
We say that (N, δ^1) is *bounded* if $\delta^n = 0$ for sufficiently large n. In this case usual one can promote the δ^k maps to a map $\delta : N \to \mathcal{T}^* A \otimes N$ by

$$\delta(\mathbf{x}) = \sum_{i=0}^{\infty} \delta^i(\mathbf{x}).$$

By definition we have the relations

$$(I_{A^{\otimes j}} \otimes \delta^i) \circ \delta^j = \delta^{i+j}$$

or, graphically



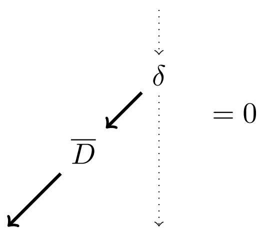
where the dotted arrows indicate elements of N.

Notice that all of this makes sense even if (N, δ^1) is not bounded if we consider instead of $\mathcal{T}^* A$ the complete tensor algebra $\overline{\mathcal{T}}^* A = \prod_{i=0}^{\infty} A^{\otimes i}$.

Definition 2.1. We say that the pair (N, δ^1) is a *type D structure over $\mathcal{A}$ with base ring* $\mathbf{k}$ if

$$(\overline{D} \circ \mathbb{I}_N) \circ \delta = 0$$

or, graphically,



We say that the type D structure is *bounded* if the pair (N, δ^1) is.

LEMMA 2.2. *Given a provincially admissible Heegaard diagram $\mathcal{H}$ and a generic admissible almost complex structure J, the pair $(\widehat{CFD}(\mathcal{H}; J), \partial)$ defines a type D structure over $\mathcal{A}(\mathcal{Z})$, where as usual $\mathcal{Z} = -\partial\mathcal{H}$. If $\mathcal{H}$ is also admissible, then the type D structure is bounded.*

NOTATION 2.3. There is a slight abuse of notation here, as the module N in this case should be the left $\mathcal{I}(\mathcal{Z})$-module $X(\mathcal{H})$ generated over $\mathbb{F}_2$ by $\mathfrak{S}(\mathcal{H})$, while $\widehat{CFD}(\mathcal{H}; J)$ is the associated $\mathcal{A}_\infty$ module (in a sense that will be made precise later).

PROOF. As $\mathcal{A}(\mathcal{Z})$ is a genuine differential algebra, the type D compatibility condition is simply

$$(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \partial) \circ \partial + (\mu_1 \otimes \mathbb{I}_N) \circ \partial = 0$$

which is exactly the fact that ∂ is a differential. Then we prove that the admissibility condition implies that $(\widehat{CFD}(\mathcal{H}), \partial)$ is bounded. That is because the $\mathbf{y}$ coefficient in $\delta^k(\mathbf{x})$ counts points in the product spaces $\prod_{i=1}^k \mathcal{M}^{B_i}(\mathbf{x}_i, \mathbf{x}_{i+1}; \vec{\rho}_i)$ where $\mathbf{x}_1 = \mathbf{x}$, $\mathbf{x}_{k+1} = \mathbf{y}$ and $B_1 * \cdots * B_k = B \in \pi_2(\mathbf{x}, \mathbf{y})$. Now, if as in the proof of the lemma 6.1 we define $|B|$ to be the sum of its local multiplicities, because of the positivity of the domains we have that for such a homology class $|B| \geq k$, and so we can conclude by lemma 2.19 of chapter 2. □

DEFINITION 2.4. Given a type D structure (N, δ^1) over $\mathcal{A}_\infty$, the *associated $\mathcal{A}_\infty$ module* $\mathcal{N}$ has underlying module $A \otimes N$ and operations defined by the sum

$$m_n = \sum_{k=0}^{\infty} (\mu_{i+k} \otimes \mathbb{I}_N) \circ (\mathbb{I}_{A^{\otimes i}} \otimes \delta^k).$$

(which is always finite because of the boundedness condition on $\mathcal{A}$).

It is straightforward to prove that such operations actually define an $\mathcal{A}_\infty$ structure over $\mathcal{A}$, and that this is exactly the way we constructed the differential module $\widehat{CFD}(\mathcal{H}; J)$ starting from $X(\mathcal{H})$.

We next define homomorphism between type D structures $(N_1, \delta^1_{N_1})$ and $(N_2, \delta^1_{N_2})$. Given a map $\psi^1 : N_1 \to A \otimes N_2$, one can construct

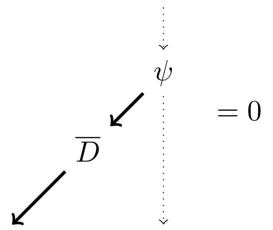
$$\psi^k : N_1 \to A^{\otimes k} \otimes N_2 \qquad \mathbf{x} \mapsto \sum_{i+j=k-1} (\mathbb{I}_{A^{\otimes(i+1)}} \otimes \delta^j_{N_2}) \circ (\mathbb{I}_{A^{\otimes i}} \otimes \psi^1) \circ \delta^i_{N_1}(\mathbf{x})$$

As usual one can put everything together in order to get a map $\psi : N_1 \to \overline{\mathcal{T}}^* \mathcal{A} \otimes N_2$ defined as

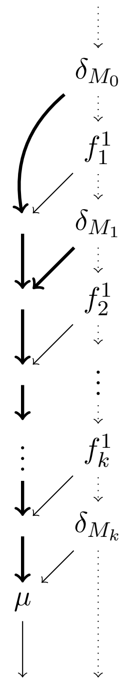
$$\psi(\mathbf{x}) = \sum_{i=0}^{\infty} \psi^i(\mathbf{x}).$$

We say that ψ^1 is bounded if $\psi^k = 0$ for k sufficiently big, which holds for example if N_1 and N_2 are both bounded. In particular in this case the map ψ has image in $\mathcal{T}^* A \otimes N_2$.

DEFINITION 2.5. ψ^1 is a *type D homomorphism* if $(\overline{D} \otimes \mathbb{I}_{N_2}) \circ \psi = 0$, or, graphically,



In a totally similar way one defines homotopies between type D homomorphisms. The composition of type D homomorphisms has a pretty complicated definition, and given $f_i : N_i \to A \otimes N_{i+1}$ for $i = 1, \ldots, k$, their composition $\circ_k(f_1, \ldots, f_k) : N_1 \to A \otimes N_{k+1}$ is given graphically by



Remark that the condition $\circ_1(f_1) = 0$ says precisely that f_1 defines a type D homomorphism.

Finally, we can construct from a type D homomorphism $\psi^1 : N_1 \to A \otimes N_2$ an associated homomorphism of $\mathcal{A}_\infty$ algebras between the associated type D modules.

Lemma 2.6. *Fix a bounded $\mathcal{A}_\infty$ algebra $\mathcal{A}$. A type D homomorphism $\psi^1 : N_1 \to A \otimes N_2$, induces an $\mathcal{A}_\infty$ homomorphism $\mathcal{N}_1 \to \mathcal{N}_2$ between the associated $\mathcal{A}_\infty$ structures whose ith component is given by*

$$\sum_{k=1}^{\infty} (\mu_{i+k+1} \otimes \mathbb{I}_{N_1}) \circ (\mathbb{I}_A^{\otimes i} \otimes \psi^k)$$

Similarly, a homotopy of type D structures induces a homotopy between the associated maps.

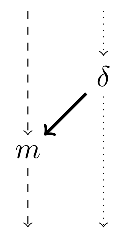
Remark 2.7. In light of these definitions, the invariance theorem (section 7 of chapter 2) says $\widehat{CFD}(\mathcal{H}; J)$ is homotopy invariant not only as a differential module, but also as a type D structure, in the sense that the homotopy equivalences are induced by type D homomorphisms. Furthermore by lemma 2.2 $\widehat{CFD}(\mathcal{H}; J)$ is always equivalent to a bounded type D structure.

2.2. The box tensor product. We are now ready to define our new version of the tensor product which gives us a pairing between a right $\mathcal{A}_\infty$ module and a left type D module. As we will see, this will fit nicely in the geometric context we are studying and will produce small sized chain complexes.

Definition 2.8. Given a operationally bounded $\mathcal{A}_\infty$ algebra $\mathcal{A}$, a right $\mathcal{A}_\infty$ module $\mathcal{M}$ over it and a left type D module (N, δ^1) over it such that at least one of $\mathcal{M}$ and (N, δ^1) is bounded, one can form the *box tensor product* $\mathcal{M} \boxtimes N$ which is the $\mathbf{k}$-module $M \otimes_{\mathbf{k}} N$ together with the boundary map

$$\partial^{\boxtimes}(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathbb{I}_N)(\mathbf{x} \otimes \delta^k(\mathbf{y}))$$

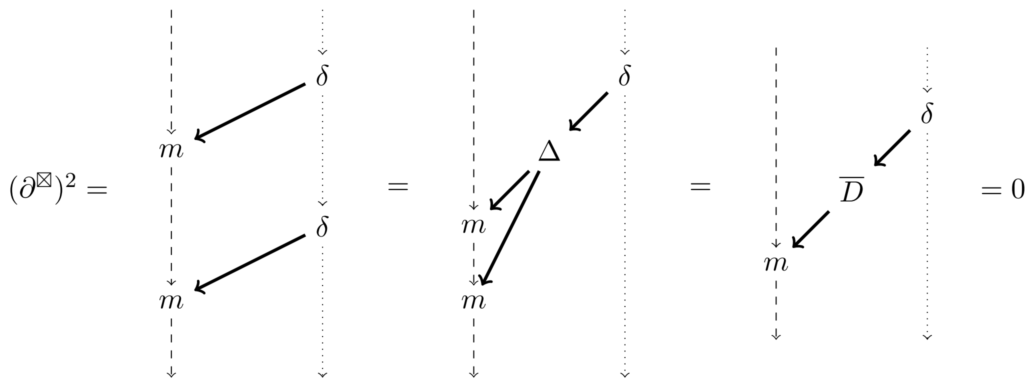
or, graphically,



Remark 2.9. The boundedness hypothesis for at least one of $\widehat{CFA}$ or $\widehat{CFD}$ is quite natural in view of lemmas 2.20 and 6.1 of chapter 2 and lemma 2.2 of this chapter.

Lemma 2.10. *The pair $(M \otimes N, \partial^{\boxtimes})$ is a chain complex.*

Proof. This fact the following nice pictorial proof



where the first identity comes from the definition of δ, the second from the $\mathcal{A}_\infty$ relations and the third them the definition of type D structure. □

This is indeed a 'good' pairing.

PROPOSITION 2.11. *Given an $\mathcal{A}_\infty$ homomorphism $f : \mathcal{M} \to \mathcal{M}'$, for every type D structure (N, δ^1) there is an induced a chain map*

$$f \boxtimes \mathbb{I}_N : \mathcal{M} \boxtimes N \to \mathcal{M}' \boxtimes N$$

if $\mathcal{M}, \mathcal{M}'$ and f are bounded or (N, δ^1) is bounded. Furthermore, under boundedness conditions such that all $\boxtimes$ are defined, homotopic maps induce homotopic maps and given another $\mathcal{A}_\infty$ homomorphism $g : \mathcal{M} \to \mathcal{M}'$ we have that $(g \circ f) \boxtimes \mathbb{I}_N$ is homotopic to $(g \boxtimes \mathbb{I}_N) \circ (f \boxtimes \mathbb{I}_N)$. Symmetrically, everything holds also for maps on the N side of the tensor product.

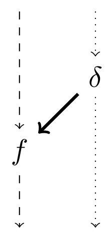
PROOF. The first claim follows from the fact that an $\mathcal{A}_\infty$ homomorphism $f : \mathcal{M} \to \mathcal{M}'$ induces a chain map $f \boxtimes \mathbb{I}_N : \mathcal{M} \boxtimes N \to \mathcal{M}' \boxtimes N$ defined as

$$(f \boxtimes \mathbb{I}_N)(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=0}^{\infty} (f_{k+1} \otimes \mathbb{I}_N) \circ (\mathbf{x} \otimes \delta^k(\mathbf{y}))$$

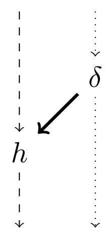
and, in the same spirit, an $\mathcal{A}_\infty$ homotopy h between maps f_1 and f_2 determines a chain homotopy $h \boxtimes \mathbb{I}_N$ between $f_1 \boxtimes \mathbb{I}_N$ and $f_2 \boxtimes \mathbb{I}_N$ by the formula

$$(h \boxtimes \mathbb{I}_N)(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=0}^{\infty} (h_{k+1} \otimes \mathbb{I}_N) \circ (\mathbf{x} \otimes \delta^k(\mathbf{y})).$$

Graphically, these maps are



and

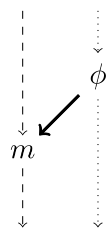


and the proof of their properties is an easy adaptation of that of $(\partial^{\boxtimes})^2 = 0$.

The second claim is completely analogous, as for a type D homomorphism $\phi^1 : N_1 \to A \otimes N_2$ one can define a chain map $\mathbb{I}_{\mathcal{M}} \boxtimes \phi^1 : \mathcal{M} \boxtimes N_1 \to \mathcal{M} \boxtimes N_2$ by

$$(\mathbb{I}_{\mathcal{M}} \boxtimes \phi^1)(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=1}^{\infty} (m_{k+1} \otimes \mathbb{I}_{N_2}) \circ (\mathbf{x} \otimes \phi^k(\mathbf{y}))$$

or, graphically



and similarly for homotopies of type D homomorphisms. ☐

Here is the main algebraic result we are interested in.

PROPOSITION 2.12. *In the previous setting, suppose the type D structure (N, δ^1) is bounded or $\mathcal{M}$ is bounded and (N, δ^1) is equivalent to a bounded one. Then the box tensor product $\mathcal{M} \boxtimes N$ is homotopy equivalent to the $\mathcal{A}_\infty$ tensor product $\mathcal{M} \tilde{\otimes} \mathcal{N}$.*

PROOF. By the previous proposition, the boundedness hypotheses imply that we can suppose N to be bounded. Then one has that $\mathcal{M} \boxtimes N \cong \overline{\mathcal{M}} \boxtimes N$. This follows from the fact that $\mathcal{M}$ and $\overline{\mathcal{M}}$ are homotopy equivalent (proposition 1.5) and the fact that the box pairing is 'good' (here we need the boundedness hypothesis on N).

Then one just notices that $\overline{\mathcal{M}} \boxtimes N$ and $\mathcal{M} \tilde{\otimes} \mathcal{N}$ are exactly the same chain complex with different names. For instance, the underlying $\mathbf{k}$-modules are

$$\overline{\mathcal{M}} \boxtimes N = (M \otimes \mathcal{T}^* A \otimes A) \otimes N = M \otimes \mathcal{T}^* A \otimes (A \otimes N) = \mathcal{M} \tilde{\otimes} \mathcal{N}.$$

and it is straightforward to check that also the differentials coincide. □

REMARK 2.13. In our case the hypoteses of this result are always satisfied, as every $\widehat{CFD}$ module is equivalent to a bounded one (see remark 2.7) and we will always suppose that one of the two Heegaard diagrams is admissible.

3. Time dilation

We are now ready to sketch the proof of the pairing theorem via time dilation. We have already seen in subsection 1.1 how to reformulate the chain complex $\widehat{CF}(\mathcal{H}_1 \cup \mathcal{H}_2)$ in terms of a chain complex $\widehat{CF}(\mathcal{H}_1; \mathcal{H}_2)$ counting pairs of holomorphic curves with matching asymptotics. Even if this is an interesting interpretation, it is far from our aim as the two Heegaard diagrams still interact in a complicated analytical way (i.e. via fibered products), which does not fit in a clear way in our algebraic construction. The ingenious idea is to consider more general fibered products where the matching condition is rescaled, in particular rather than imposing $\mathrm{ev}(u_1) = \mathrm{ev}(u_2)$ we want $T \cdot \mathrm{ev}(u_1) = \mathrm{ev}(u_2)$ for a real parameter $T > 0$. When letting $T \to \infty$, process we call *time dilation*, the two parts of the diagram will decouple, and we will recover exactly the algebraic definition of the box tensor product $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$.

We now introduce the family $\widehat{CF}(\mathcal{H}_1; \mathcal{H}_2; T)$ of chain complexes depending on the real parameter $T > 0$ (note that for $T = 1$ we will recover the previously constructed chain complex $\widehat{CF}(\mathcal{H}_1; \mathcal{H}_2)$). As a vector space this is generated by $\mathfrak{S}(\mathcal{H}_1; \mathcal{H}_2)$. In the differentials we will count pairs of holomorphic curves where the evaluations coincide after a T-dilation. In particular, given two pairs of compatible generators $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2)$ in $\mathfrak{S}(\mathcal{H}_1; \mathcal{H}_2)$ inducing $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$, homology classes $B_i \in \pi_2(\mathbf{x}_i, \mathbf{y}_i)$ inducing $B \in \pi_2(\mathbf{x}, \mathbf{y})$, and a compatible pair of sources (S_1, S_2) connecting $\mathbf{x}_1$ to $\mathbf{y}_1$ and $\mathbf{x}_2$ to $\mathbf{y}_2$ respectively we define the *space of T-matching pairs* as the fibered product

$$\widetilde{\mathcal{M}\mathcal{M}}^B(\mathbf{x}_1, \mathbf{y}_1; S_1; \mathbf{x}_2, \mathbf{y}_2; S_2; T) = \widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1) \times_{T \cdot \mathrm{ev}_1 = \mathrm{ev}_2} \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2)$$

i.e. the moduli space of pair of curves (u_1, u_2), $u_i \in \widetilde{\mathcal{M}}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i)$ with $T \cdot \mathrm{ev}(u_1) = \mathrm{ev}(u_2)$. This space has a natural $\mathbb{R}$-action by scaled translations, i.e. the translation on the first curve is amplified by a factor T, and we consider the quotient space of this action (denoted as usual by dropping the tildes).

We refer to the moduli space of such embedded pair of curves (the one we are interested in) as $\mathcal{M}\mathcal{M}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2; T)$. Again, transversality results and index theory are analogue to the closed case so for a generic choice of the admissible almost complex structure on $\Sigma_i \times [0,1] \times \mathbb{R}$ and $i = 1, 2$ one can define the boundary map for $\widehat{CF}(\mathcal{H}_1; \mathcal{H}_2; T)$ as

$$\partial^T(\mathbf{x}_1, \mathbf{x}_2) = \sum_{(\mathbf{y}_1, \mathbf{y}_2)} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \operatorname{ind}(B)=1}} \#\big(\mathcal{MM}^B(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2; T)\big) \cdot (\mathbf{y}_1, \mathbf{y}_2)$$

where we have finiteness for the sum if both $\mathcal{H}_1$ and $\mathcal{H}_2$ are provincially admissible, and at least one is admissible. Here is the key lemma of this construction.

Lemma 3.1. *For each $T > 0$, ∂_T is a differential. Furthermore for any $T > 0$ the chain complex $\widehat{CF}(\mathcal{H}_1; \mathcal{H}_2; T)$ is homotopy equivalent to $\widehat{CF}(\mathcal{H}_1 \cup \mathcal{H}_2)$.*

We discuss the proof of such a result.

The first part is proved as usual by considering the ends of a 1-dimensional moduli spaces $\mathcal{MM}^B(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2; T)$. The compactification one constructs is the space of *T-matched combs*, i.e. the space of pairs of holomorphic combs with matching evaluations up to a T-scaling.

The second part is similar in spirit to the change of the almost complex structure, and is proved by constructing a chain map $F_T : \widehat{CF}(\mathcal{H}_1; \mathcal{H}_2; T) \to \widehat{CF}(\mathcal{H}_1; \mathcal{H}_2)$ defined by counting pairs of holomorphic curves with asymptotics matching up to a nonlinear rescaling. In particular one considers a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi' > 0$ and

$$\psi(t) = \begin{cases} T \cdot t & \text{if } t \leq -1 \\ t & \text{if } t \geq 1. \end{cases}$$

and the fibered products of spaces of curves

$$\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1; S_1; \mathbf{x}_2, \mathbf{y}_2; S_2; \psi) = \widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1) \times_{\psi \circ \mathrm{ev}_1 = \mathrm{ev}_2} \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2).$$

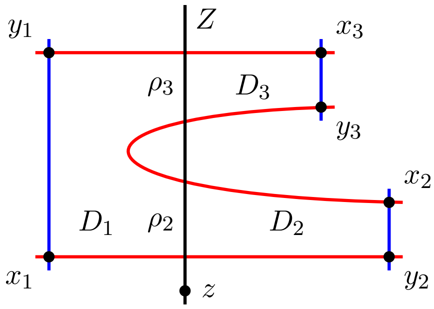
As in the proof on the invariance with respect to the change of complex structure one defines F_T by counting rigid embedded holomorphic curves, and proves that is a chain map considering the end of 1-dimensional moduli spaces. The homotopy inverse of F_T is constructed in a totally analogue way.

As the homotopy type of $\widehat{CF}(\mathcal{H}_1; \mathcal{H}_2; T)$ is independent of T, one may look to its behaviour for $T \to \infty$. Intuitively, on the left side (the one involved in the type A modules) some Reeb chords may collide while on the right side (the one involved in the type D modules) some Reeb chords may become infinitely apart. This is made rigorous with the introduction of the so called *ideal matchings*, which are limits of T-matched holomorphic curves. The main result is the following.

Proposition 3.2. *For sufficiently big $T > 0$, $\widehat{CF}(\mathcal{H}_1; \mathcal{H}_2; T)$ is a chain complex isomorphic to $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$.*

In particular, in light of the algebraic result of proposition 2.12 this implies theorem 0.2. We will not enter the complicated details and definitions needed for the proof of this result (see [**LOT11b**]). Instead, we illustrate a quite long local example which shows the phenomena which occur and how they are described by the algebra we have defined.

Example 3.3. Consider the hexagonal domain cut as in the next figure



which topologically connects $\mathbf{x} = \{x_1, x_2, x_3\}$ to $\mathbf{y} = \{y_1, y_2, y_3\}$ in a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ corresponding to the domain $D_1 + D_2 + D_3$.

From a global point of view, a result due to Rasmussen ([**Ras03**], lemma 9.11) assures that $\#\mathcal{M}^B(\mathbf{x}, \mathbf{y}) = 1$ so this hexagon always contributes to the differential of $\mathbf{x}$. Let us see what happens when one slices the diagram along Z.

First of all we consider the naïve decoupling, i.e. the chain complex $\widehat{CF}(\mathcal{H}_1; \mathcal{H}_2)$ of subsection 1.1. Here, $\mathcal{M}^{D_1}(\{x_1\}, \{y_1\})$ consists of a single curve and the difference of the evaluations is a fixed real number $t_0 = ev_{\rho_3} - ev_{\rho_2} > 0$ depending on the complex structure, while $\mathcal{M}^{D_2+D_3}(\{x_2, x_3\}, \{y_2, y_3\})$ is a one dimensional space parametrized by $ev_{\rho_2} - ev_{\rho_2}$ (which is a positive number because of strong boundary monotonicity). It is then clear that there is only one pair of curves with matching asymptotics, so $\#\mathcal{MM}^B(\{x_1\}, \{y_1\}; \{x_2, x_3\}, \{y_2, y_3\}) = 1$.

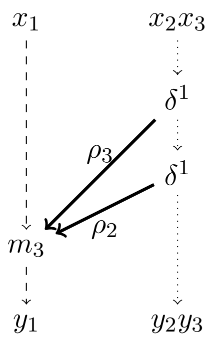
Then we consider the effects of time dilation. The type A module $\widehat{CFA}(\mathcal{H}_1)$ has two generators x_1 and x_2 and only one non trivial operation

$$m_3(x_1, \rho_2, \rho_3) = y_1$$

while the type D module $\widehat{CFD}(\mathcal{H}_2)$ has four generators x_2x_3, x_2y_3, y_2x_3 and y_2y_3 and differentials

$$\begin{aligned}
\partial x_2x_3 &= \rho_2 \cdot y_2x_3 + \rho_3 \cdot x_2y_3 \\
\partial x_2y_3 &= \rho_2 \cdot y_2y_3 \\
\partial y_2x_3 &= \rho_3 \cdot y_2y_3.
\end{aligned}$$

The module $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$ has only two generators $x_1 \otimes x_2x_3$ and $y_1 \otimes y_2y_3$, and the only non trivial differential is graphically given by



so we recover exactly the global chain complex. The analytical interpretation of this result is that the moduli space $\mathcal{MM}^B(\{x_1\},\{y_1\};\{x_2,x_3\},\{y_2,y_3\};T)$ contains a single pair given by the disk in $\mathcal{H}_1$ and the pair of disks in $\mathcal{H}_2$ with height difference $T\cdot t_0$. For $T\to\infty$ this difference goes to infinity and the curve breaks in a two-story holomorphic building, which are exactly the two δ^1s.

Finally we see the different behaviour when we switch the roles of $\mathcal{H}_1$ and $\mathcal{H}_2$. The type A module $\widehat{CFA}(\mathcal{H}_2)$ has four generators x_2x_3, x_2y_3, y_2x_3 and y_2y_3 and non trivial operations

$$\begin{aligned} m_2(x_2x_3,\rho_2) &= y_2x_3\\ m_2(x_2y_3,\rho_2) &= y_2y_3\\ m_2(x_2x_3,\rho_3) &= x_2y_3\\ m_2(y_2x_3,\rho_3) &= y_2y_3\\ m_2(x_2x_3,\{\rho_2,\rho_3\}) &= y_2y_3 \end{aligned}$$

while the type D module $\widehat{CFD}(\mathcal{H}_1)$ has two generators x_1 and y_1 and differential

$$\partial x_1 = \rho_{23}\cdot y_1.$$

Then $\widehat{CFA}(\mathcal{H}_2)\boxtimes\widehat{CFD}(\mathcal{H}_1)$ has two generators $x_2x_3\otimes x_1$ and $y_2y_3\otimes x_1$, and there is only one non trivial differential represented graphically as

$$\begin{tikzcd}
x_2x_3 \arrow[dd, dashed] & x_1 \arrow[d, dotted] \\
 & \delta^1 \arrow[dd, dotted] \arrow[dl, "\rho_{23}"'] \\
m_2 \arrow[d, dashed] & \\
y_2y_3 & y_1
\end{tikzcd}$$

Geometrically, $\mathcal{MM}^B(\{x_2,x_3\},\{y_2,y_3\};\{x_1\},\{y_1\};T)$ consists of a single pair with a disk in $\mathcal{H}_2$ and a pair of disks in $\mathcal{H}_1$ with height difference t_0/T. For $T\to\infty$, this separation goes to 0, so the Reeb chords collide returning us the only non trivial differential in the tensor product.

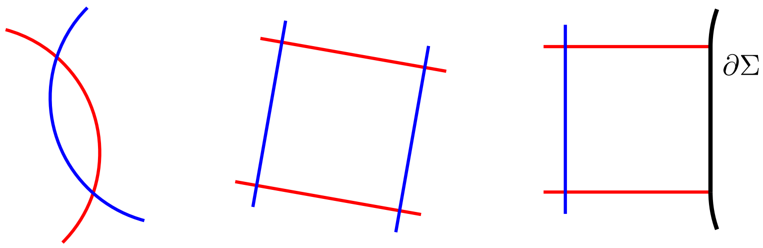
4. Nice Heegaard diagrams

We now discuss another proof, via the notion of nice Heegaard diagrams. These were introduced for closed 3-manifolds by Sarkar and Wang (in [**SW10**]), and their importance is that in such diagrams the differential of the Floer chain complex can be determined in a completely combinatorial way. This was a big breakthrough, as Heegaard Floer homology turned out to be the first completely computable Floer theory. In this section introduce the basic facts about nice Heegaard diagrams and show how to use them to prove the pairing result.

4.1. Nice Heegaard diagrams. We will treat the case of nice bordered Heegaard diagrams. The case of closed Heegaard diagrams is completely analogue (and simpler).

Definition 4.1. We say that a bordered Heegaard diagram is *nice* if each region in which $\overline{\Sigma}$ is cut out by $\overline{\boldsymbol{\alpha}}$ and $\boldsymbol{\beta}$ not adjacent to z is topologically a disk with at most four corners, so in particular:

- each region in the interior of Σ is a bigon or a quadrilateral
- each region at the boundary of Σ except the one adjacent to z is a quadrilateral with boundary two α-arcs, a β-curve and $\partial\overline{\Sigma}$.



The following is the key result regarding nice Heegaard diagrams.

Proposition 4.2. *Let $\mathcal{H}$ be a nice bordered Heegaard diagram, and let* $\operatorname{ind}(B, \vec{\boldsymbol{\rho}}) = 1$*. Then any holomorphic curve $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})$ has one of the following forms:*

(1) the source of u consists of g bigons, $g-1$ of which are mapped trivially and the other is mapped by π_Σ to the interior of Σ;
(2) the source of u consists of $g-1$ bigons and a quadrilateral, with the bigons mapped trivially and the quadrilateral mapped by π_Σ to the interior of Σ;
(3) the source of g consists of g bigons, which are mapped trivially or with a single puncture at $e\infty$. Furthermore all those punctures are at the same height, i.e. $|\vec{\boldsymbol{\rho}}| = 1$.

Conversely, if there are topological maps satisfying the previous conditions, there is a unique holomorphic map in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})$.

This implies that in a nice Heegaard diagram the non trivial differentials are obtained by counting special bigons and quadrilaterals. In particular, the differentials between $\mathbf{x} = \{x_1, \dots, x_g\}$ and $\mathbf{y} = \{y_1, \dots, y_g\}$ are obtained by counting empty bigons and quadrilaterals connecting them, i.e. domains $B \in \pi_2(\mathbf{x}, \mathbf{y})$ which are topologically embedded bigons and quadrilaterals and do not contain any of the x_i's or y_i's in their interior.

Notation 4.3. In particular the differentials in a nice Heegaard diagram depend only on the diagram and not on the almost complex structure, so we will always drop the latter from our notations.

Proof. By the index formula of proposition 3.2 of chapter 2 we know that

$$1 = g - \chi(S) + 2e(D(u)) + |\vec{\boldsymbol{\rho}}|.$$

Since the diagram is nice, each region (except the one adjacent to z, which our curves cannot intersect) has Euler measure 0 or $1/2$, and so $e(D(u)) \geq 0$, and as S has at most g components, none of which closed, $\chi(S) \leq g$. So there are the three possibilities:

(1) $\chi(S) = g$, $e(D(u)) = 1/2$ and $|P| = 0$;
(2) $\chi(S) = g - 1$, $e(D(u)) = 0$ and $|P| = 0$;

(3) $\chi(S) = g$, $e(D(u)) = 0$ and $|P| = 1$;

which clearly correspond to the cases in the statement. Such domains admit holomorphic representatives by the Riemann mapping theorem, and it is clear that this is unique in the moduli space. The only non straightforward case are the rectangles. For these one can use Rasmussen's lemma ([**Ras03**], lemma 9.11) or the classical fact in conformal geometry that a rectangle admits a double branched covering over the bigon, unique up to translations of the image. □

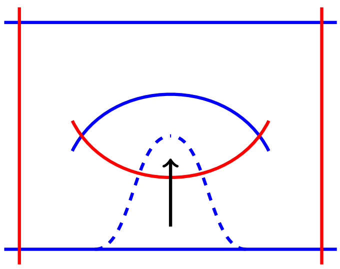
The natural question that arises is whether or not any bordered 3-manifold admits a nice Heegaard diagram. This is answered in the following algorithmic way.

PROPOSITION 4.4. *Any bordered Heegaard diagram $\mathcal{H} = (\overline{\Sigma}, \overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}, z)$ can be turned into a nice one by a suitable finite sequence of*

- *isotopies of the β-curves;*
- *handleslides among the β-curves.*

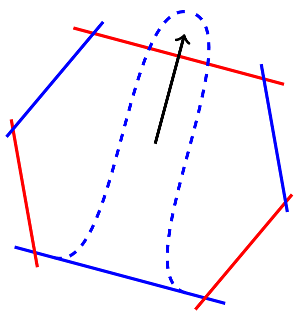
PROOF. We first describe how the proof goes in the closed case, referring the reader to the original paper [**SW10**] for the details. This is done in two steps.

In step 1, one makes all the regions topological disks. To do this, after performing some isotopies one can assume that no curve β-curve is disjoint from $\boldsymbol{\alpha}$ and viceversa no α-curve is disjoint from $\boldsymbol{\beta}$. As each region in which the surface is cut by the curves is planar (by homological assumptions), one can perform finger moves as in the following figure to obtain the result.

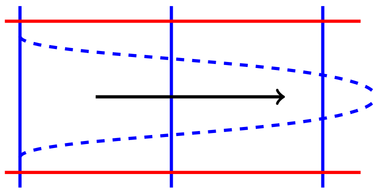


In step 2 we obtain a nice Heegaard diagram. First one defines an appropriate *complexity* function on Heegaard diagrams with values in a well ordered set which attains its minimum if and only if the diagram is nice, and a related notion of *distance* for a region.

Then, starting from a non disk or rectangle region with maximal distance, one performs a finger move like



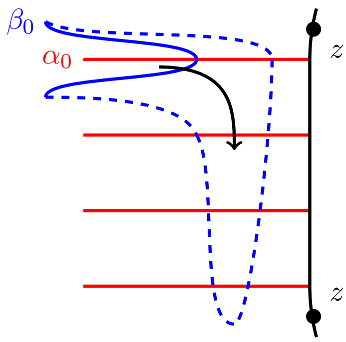
splitting the region in two regions with less edges, and pushes it through all square regions as far as possible.



From the definition of complexity then if the region one reaches at the end is different from the starting one the complexity of the new Heegaard diagram is smaller. If the starting region is reached, one has to perform a suitable handleslide of the β-curves is order to decrease the complexity.

In any case, if $\mathcal{H}$ is not nice then one can perform some moves as in the statement to decrease its complexity. Because of the well-orderedness of the values of the complexity, this algorithm will terminate, yielding a nice Heegaard diagram.

In the closed setting, one just starts by doing a finger move as

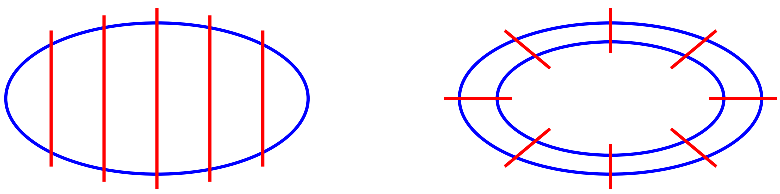


where α_0 is the α-arc intersecting $\partial\overline{\Sigma}$ in the 'highest' point, and β_0 is the arc which intersects α_0 closest to the boundary, and then one just proceeds as in the closed case. $\square$

We conclude the paragraph with the following simple result.

Lemma 4.5. *A nice bordered Heegaard diagram is admissible.*

Proof. First one notices that in a nice diagram the region adjacent to z has to occur to both left and right of each β-curve. Otherwise on one side of the β-curve there would be only bigons and rectangles (possibly intersecting the boundary), and by analyzing the cases one shows that the only possibilities are



which are impossible as in each of them the β curves turn out to be not homologically independent.

Now, suppose we have a periodic domain D with $n_z(D) = 0$ and all positive coefficients. Chosen a β-curve on its boundary (without loss of generality in the left), as D is periodic all the domains on the left of this curve have positive coefficient, which clashes with the assumption $n_z(D) = 0$. □

REMARK 4.6. Even if this algorithm has a great theoretical importance, in the practice this is far from being satisfactory, as:

- the computations are pretty complicated. For example we remarked that one can not tell the differentials in the Heegaard diagram for the Poincaré homology sphere, while the Sarkar and Wang algorithm yields a nice Heegaard diagram with computable chain complex, but with 335 generators and 505 non trivial differentials;
- the algorithm works really ad hoc for each Heegaard diagram. In particular this method does not fit well when one wants to compute Heegaard Floer homology for infinite families of 3-manifolds with special properties (for example, all the Dehn fillings of the complement of a knot).

Both these reasons are indeed motivations behind the construction of bordered Heegaard Floer homology.

4.2. The pairing theorem via nice diagrams. We now show how one can use nice Heegaard diagrams in order to give a neat proof of the of the pairing theorem (via proposition 2.12).

LEMMA 4.7. *If $\mathcal{H}$ is a nice bordered Heegaard diagram, $\widehat{CFA}(\mathcal{H})$ is a differential module.*

PROOF. The higher operations m_i with $i > 2$ count curves in the index 1 moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})$ with $|\vec{\boldsymbol{\rho}}| > 1$, which in a nice Heegaard diagram are always empty in light of proposition 4.2. □

PROOF OF THEOREM 0.2. The union of nice bordered Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ is obviously a nice Heegaard diagram $\mathcal{H}$, and they are all admissible by lemma 4.5. The differential of $\widehat{CF}(\mathcal{H})$ counts index 1 empty bigons and rectangles, and we want to show that this is exactly what the chain complex $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$ does. This will let us conclude thanks to proposition 2.12.

First of all, notice that because of the definition of the actions of the subring of the idempotents $\mathcal{I}(\mathcal{Z})$ there is an obvious identification between pairs of generators $(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1 \in \mathfrak{S}(\mathcal{H}_1)$ and $\mathbf{x}_2 \in \mathfrak{S}(\mathcal{H}_2)$ with $\mathbf{x}_1 \otimes \mathbf{x}_2$ non zero and generators $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, and so $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$ and $\widehat{CF}(\mathcal{H})$ are naturally identified as vector spaces over $\mathbb{F}_2$.

Because of proposition 4.2, one has that $\partial \mathbf{x}_2 = \sum_{\mathbf{y}_2} a_{\mathbf{x}_2,\mathbf{y}_2} \otimes \mathbf{y}_2$ where

$$a_{\mathbf{x}_2,\mathbf{y}_2} = \begin{cases} 1 & \text{if it comes from a provincial bigon or rectangle;} \\ a(-\rho) & \text{if it comes from a bigon with asymptotic at east infinity } \rho. \end{cases}$$

By definition the differential of $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$ has the form

$$\partial^{\boxtimes}(\mathbf{x}_1 \otimes \mathbf{x}_2) = (m_1(\mathbf{x}_1)) \otimes \mathbf{x}_2 + \sum_{\mathbf{y}_2} m_2(\mathbf{x}_2, a_{\mathbf{x}_2,\mathbf{y}_2}) \otimes \mathbf{y}_2$$

which corresponds exactly to the differential in $\widehat{CF}(\mathcal{H})$. In fact, the first terms corresponds to empty rectangles and bigons entirely contained in Σ_1, the second terms where $a_{\mathbf{x}_2,\mathbf{y}_2} = 1$ corresponds to empty rectangles and bigons entirely contained in Σ_2, while the ones where $a_{\mathbf{x}_2,\mathbf{y}_2} = a(-\rho)$ are the correspond to the empty rectangles intersecting the splitting circle at the Reeb chord ρ. □

5. An application

In this section we show a quick proof of the surgery exact sequence for Heegaard Floer homology using the tools of bordered Heegaard Floer homology. The existence of such an exact sequence was already proved in [**OS04b**], relying on the count of some special holomorphic curves similar to those of subsection 6.1 of chapter 1. In this case, with all the machinery we have constructed, this turns out to be a completely algebraic argument.

Recall that given a framed knot in a 3-manifold $(K, \lambda) \subset Y$ and a pair of coprime integers p, q, the *p/q-surgery along* K is the 3 manifold $Y_{p/q}$ obtained gluing to the manifold with torus boundary $Y \setminus \mathrm{nbd}(K)$ the solid torus $S^1 \times D^2$ with a diffeomorphism $\varphi : \partial(S^1 \times D^2) \to \partial(\mathrm{nbd}(K))$ which sends the meridian of $S^1 \times D^2$ to a curve in the homology class $p[\lambda] + q[\mu]$, where $[\mu]$ is the meridian of $\mathrm{nbd}(K)$. Here we allow also the pairs $(0, 1)$ (the 0-surgery) and $(1, 0)$ (the ∞-surgery). The 3-manifold $Y_{p/q}$ is well defined because of the same argument of example 1.4 of chapter 1.

Example 5.1. Here are some easy examples:

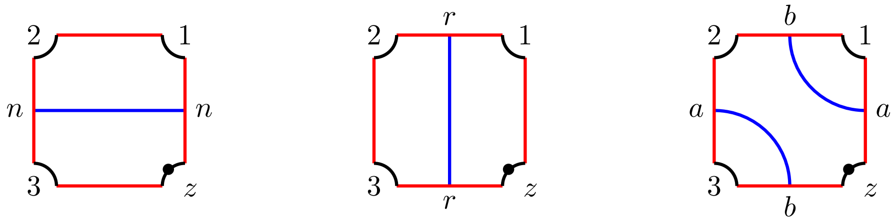
- we always have $Y_0 = Y$;
- if $K \subset S^3$ is the unknot with the obvious framing, then $Y_{p/q}$ is the lens space $L(p, q)$ of example 1.4 of chapter 1 (by definition);
- this example is more complicated. If K is the trefoil knot with the 0-framing (i.e. the framing curve has linking number 0 with K), then Y_{+1} is the Poincaré homology sphere presented in example 1.5 of chapter 1. A nice graphical proof of this can be found in [**Rol76**].

Theorem 5.2. *Given a framed knot $K \subset Y$, there is an exact sequence*

$$\begin{tikzcd} \widehat{HF}(Y_{-1}) \arrow[rr] & & \widehat{HF}(Y_0) \arrow[dl] \\ & \widehat{HF}(Y_\infty) \arrow[ul] & \end{tikzcd}$$

between the Heegaard Floer homologies of the $0, \infty$ *and* -1 *surgery on* K.

From the bordered point of view, surgery can be seen as the operation of gluing to a given 3-manifold with torus boundary a solid torus with different framing of the boundary, and so a different bordered Heegaard diagram. In particular, one may use the following provincially admissible Heegaard diagrams $\mathcal{H}_0, \mathcal{H}_\infty$ and $\mathcal{H}_{-1}$



respectively for the $0, \infty$ and -1 surgery. Here the vertical and horizontal edges have to be identified in order to get a genus 1 diagram.

REMARK 5.3. Because these diagrams are only provincially admissible, we will always suppose that the one for the bordered 3-manifold $Y \setminus \mathrm{nbd}(K)$ is admissible.

Here comes the algebra. As we will deal with type D structures (N, δ^1) over a differential algebra $\mathcal{A}$, we will always have that $\mathcal{N} \cong \mathcal{A} \boxtimes N$ is a differential module.

DEFINITION 5.4. Given a differential algebra $\mathcal{A}$, and three type D structures (N_1, δ_1),(N_2, δ_2) and (N_3, δ_3), we say that two type D morphisms $\varphi^1 : N_1 \to A \otimes N_2$ and $\psi^1 : N_2 \to A \otimes N_3$ form a *short exact sequence*, and write

$$0 \to N_1 \xrightarrow{\varphi} N_2 \xrightarrow{\psi} N_3 \to 0,$$

if the induced maps $\varphi : \mathcal{N}_1 \to \mathcal{N}_2$ and $\psi : \mathcal{N}_2 \to \mathcal{N}_3$ between the associated differential modules form a short exact sequence.

LEMMA 5.5. *Given a short exact sequence of type D structures $0 \to N_1 \xrightarrow{\varphi} N_2 \xrightarrow{\psi} N_3 \to 0$ over a differential algebra $\mathcal{A}$ with $(N_3, \delta^1_{N_3})$ equivalent to a bounded type D structure and a bounded $\mathcal{A}_\infty$ module $\mathcal{M}$ over $\mathcal{A}$, there is the following exact sequence in homology.*

$$\begin{tikzcd} H_*(\mathcal{M} \boxtimes N_1) \arrow[rr] & & H_*(\mathcal{M} \boxtimes N_2) \arrow[dl] \\ & H_*(\mathcal{M} \boxtimes N_3) \arrow[ul] & \end{tikzcd}$$

PROOF. From the short exact sequence of type D structures we can form a 3-step filtered type D structure N with underlying $\mathbf{k}$-module $N_1 \oplus N_2 \oplus N_3$ and structure map

$$N_1 \xrightarrow{\delta^1_{N_1} \oplus \varphi^1} (A \otimes N_1) \oplus (A \otimes N_2)$$
$$N_2 \xrightarrow{\delta^1_{N_2} \oplus \psi^1} (A \otimes N_2) \oplus (A \otimes N_3)$$
$$N_3 \xrightarrow{\delta^1_{N_3}} A \otimes N_3.$$

In particular $\mathcal{A} \boxtimes N$ is a chain complex with a 3-step filtration

$$0 \subset \mathcal{A} \boxtimes N_3 \subset \mathcal{A} \boxtimes (N_2 \oplus N_3) \subset \mathcal{A} \boxtimes (N_1 \oplus N_2 \oplus N_3) = \mathcal{A} \boxtimes N.$$

We have that $\mathcal{A} \boxtimes N_3 = \mathcal{N}_3$ is a projective differential $\mathcal{A}$-module, as it is homotopy equivalent to its bar resolution which is always projective ([**BL94**]). So the short exact sequence

$$0 \to \mathcal{N}_1 \to \mathcal{N}_2 \to \mathcal{N}_3 \to 0$$

is split, and this splitting implies that $\mathcal{N}$ is nullhomotopic.

We then turn our attention to the 3-step filtered complex $\mathcal{M} \boxtimes N$, which has as associate graded complex $\bigoplus_{i=1}^{3} \mathcal{M} \boxtimes N_i$. The nullhomotopy of $\mathcal{N}$ induces a nullhomotopy of $\mathcal{M} \boxtimes N$, so it is acyclic. This implies that there is an exact sequence between the homologies of the terms of its associated graded complex. This is a small generalization of the fact that an short exact sequence of chain complexes induces a long exact sequence in homology, and comes from the Leray spectral sequence. □

Remark 5.6. Because of the definitions the differential of elements $\mathcal{M} \boxtimes N_1$ may also have part in $\mathcal{M} \boxtimes N_3$.

In order to prove theorem 5.2 it suffices then to study the type D modules associated to the Heegaard diagrams introduced before, whose generators are indicated in the figures. The differentials are

$$\begin{aligned} \partial n &= \rho_{12} n \\ \partial r &= \rho_{23} r \\ \partial a &= (\rho_1 + \rho_3) b \\ \partial b &= 0 \end{aligned}$$

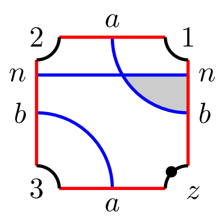
and there is a short exact sequence

$$0 \longrightarrow \widehat{CFD}(\mathcal{H}_\infty) \xrightarrow{\varphi} \widehat{CFD}(\mathcal{H}_{-1}) \xrightarrow{\psi} \widehat{CFD}(\mathcal{H}_0) \longrightarrow 0$$

where the maps are defined as

$$\begin{aligned} \varphi(r) &= b + \rho_2 a \\ \psi(a) &= n \\ \psi(b) &= \rho_2 n. \end{aligned}$$

Actually one can guess those maps by counting holomorphic triangles (which is indeed the original way of constructing the surgery exact sequence). For example, $\psi(a)$ is obtained by counting the following triangle.



Further developments

The theory we have described in the present work is just the beginning the whole big theory of bordered Heegaard Floer homology. In fact, despite its nice properties, the construction of the invariants $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ and their pairing do not solve the two main problems we pointed out in the introduction, i.e. the creation of efficient computational tools and the categorification of the restricted TQFT. Indeed, among the weak points of this construction, the objects we constructed have the same computational complexity of traditional Heegaard Floer homology, and their complete antisymmetry does not fit at all in the quantum invariants big picture.

Anyway, there have been many improvements of the theory in subsequent papers by Lipshitz, Ozsváth and Thurston, and we describe the most interesting ones among them.

Bimodules and 3-manifolds with two boundary components. In the present work we have considered only 3-manifolds with one boundary component, and it is an interesting problem to extend the construction to manifolds with disconnected boundary. In this direction, in the paper [**LOT11a**] the authors associate bimodules to *strongly bordered* 3*-manifolds* Y *with two boundary components* $\partial_L Y$ and $\partial_R Y$, which require the additional data of parameterizations $\phi_L : F(\mathcal{Z}_L) \to \partial_L Y$ and $\phi_R : F(\mathcal{Z}_R) \to \partial_R Y$ and a framed arc γ from $\partial_L Y$ to $\partial_R Y$ compatible with the parameterizations in a suitable sense. To such an object the authors associate:

- a differential bimodule $\widehat{CFDD}(Y)$ with commuting left actions of $\mathcal{A}(-\mathcal{Z}_L)$ and $\mathcal{A}(-\mathcal{Z}_R)$;
- and $\mathcal{A}_\infty$ bimodule $\widehat{CFDA}(Y)$ with a left action of $\mathcal{A}(-\mathcal{Z}_L)$ and a right $\mathcal{A}_\infty$ action of $\mathcal{A}(\mathcal{Z}_R)$;
- an $\mathcal{A}_\infty$ bimodule $\widehat{CFAA}(Y)$ with commuting right $\mathcal{A}_\infty$ actions of $\mathcal{A}(\mathcal{Z}_L)$ and $\mathcal{A}(\mathcal{Z}_R)$,

each defined up to quasi-isomorphism.

Those invariants behave in an expected way with respect to gluing. For example, given 3-manifolds Y_1 and Y_2 with $\partial Y_1 = -F_1 \coprod F_2$ and $\partial Y_2 = -F_2 \coprod F_3$ one has the pairings

$$\begin{aligned}\widehat{CFDD}(Y_1 \cup_{F_2} Y_2) &\cong \widehat{CFDA}(Y_1)\widetilde{\otimes}_{\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(Y_2)\\ &\cong \widehat{CFDA}(Y_2)\widetilde{\otimes}_{\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(Y_1)\end{aligned}$$

and in general all the compatible matchings of A and Ds return a correct pairing theorem.

Remark 5.7. Here some of the boundary components may be empty, returning the well known invariants $\widehat{CF}, \widehat{CFD}, \widehat{CFA}$.

Duality properties and morphism spaces. In the paper [**LOT11d**] the authors prove a number of duality properties up to homotopy equivalence relating the bordered modules and bimodules. First of all $\mathcal{A}(-\mathcal{Z})$ is the opposite algebra of $\mathcal{A}(\mathcal{Z})$, so one can exchange left actions of $\mathcal{A}(-\mathcal{Z})$ with right actions of $\mathcal{A}(\mathcal{Z})$. With this in mind, the module $\widehat{CFD}(Y)$ is dual over $A(\mathcal{Z})$ to $\widehat{CFA}(-Y)$ in the sense that

$$\widehat{CFD}(Y) \cong \mathrm{Hom}_{\mathcal{A}(-\mathcal{Z})}\big(\widehat{CFA}(-Y), \mathcal{A}(-\mathcal{Z})\big)$$
$$\widehat{CFA}(Y) \cong \mathrm{Hom}_{\mathcal{A}(\mathcal{Z})}\big(\widehat{CFD}(-Y), \mathcal{A}(\mathcal{Z})\big)$$

where the morphism spaces have the structure of chain complexes given by

$$(\partial f)(\mathbf{x}) = \partial(f(\mathbf{x})) + f(\partial \mathbf{x}).$$

Similarly $\widehat{CFDD}(-Y)$ is the one-sided dual of $\widehat{CFAA}(Y)$, i.e.

$$\widehat{CFAA}(Y) \cong \mathrm{Hom}_{\mathcal{A}(-\mathcal{Z}_2)}\big(\widehat{CFDD}(-Y), \mathcal{A}(-\mathcal{Z}_2)\big),$$

and also the symmetric relation holds.

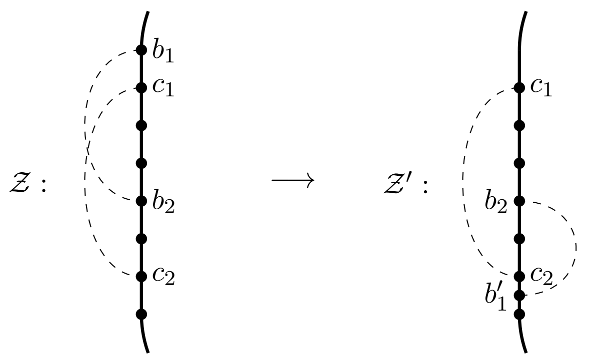
In particular the first duality properties permit to reformulate the pairing theorem as

$$\begin{aligned}\widehat{CF}(-Y_1 \cup Y_2)) &\cong \mathrm{Hom}_{\mathcal{A}(-\mathcal{Z})}\big(\widehat{CFD}(Y_1), \widehat{CFD}(Y_2))\big)\\ &\cong \mathrm{Hom}_{\mathcal{A}(\mathcal{Z})}\big(\widehat{CFA}(Y_1), \widehat{CFA}(Y_2))\big)\end{aligned}$$

which is more symmetric and involves only type D modules or type A modules.

The action of mapping classes. Note that every 3-manifold can be obtained by gluing two handlebodies along their boundaries so, in order to compute the closed Heegaard Floer invariants one just needs to understand how the reparameterization of the boundary acts on the type D invariants. In particular, one can address the action of a reparameterization by a diffeomorphism $\phi : F \to F$ by studying the DD-bimodule associated to the bordered 3-manifold $F \xrightarrow{\phi} F \times [0,1] \xleftarrow{\mathbb{I}_F} F$ with 2 boundary components.

In the paper [**LOT11c**] the authors describe explicitly these bimodules for certain special reparameterizations of the boundary, namely the *arc slides*. An arc slide on a pointed matched circle $\mathcal{Z}$ takes two matched pairs $\{b_1, b_2\}$ and $\{c_1, c_2\}$ where b_1 and c_1 are adjacent and replaces it with a new pointed matched circle $\mathcal{Z}'$ which coincides with $\mathcal{Z}$ everywhere except for the matched pair $\{b_1, b_2\}$ which is replaced by a new matched pair $\{b_1', b_2\}$, with b_1' a new point adjacent to c_2 in the direction opposite to b_2.



These moves on pointed matched circles come with naturally associated diffeomorphisms $F(\mathcal{Z}) \to F(\mathcal{Z}')$ between the associated pointed matched circles. As one can prove that the arc slides generate the strongly based mapping class grupoid (i.e. the grupoid of diffeomorphisms $\varphi : F(\mathcal{Z}) \to F(\mathcal{Z}')$ which map the 0-handle to the 0-handle considered up to isotopies which are constant on the 0-handle) this solves the problem of computing Heegaard Floer homology in a more efficient way, leading for example to the first computer program able to compute it.

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