G1 SEMINAR: ULTRAPRODUCTS

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Abstract. This is a set of notes accompanying a talk given at the G1 seminar in Princeton in September 2025.

I will explain how to put together fields of positive characteristic to obtain the field of complex numbers, and show how this newfound presentation of $\mathbf{C}$ can be used to give a straightforward proof of a curious theorem in complex algebraic geometry.

0.1. **Preamble.** The overarching goal of this talk is to make sense of the statement

$$\mathbf{C} = \lim_{p\to\infty} \overline{\mathbf{F}}_p$$

where $\mathbf{C}$ is the field of complex numbers and $\overline{\mathbf{F}}_p$ is the algebraic closure of the finite field $\mathbf{F}_p$ of cardinality p (a prime). The first order of business will be to quickly review the basic theory of ultrafilters; this will naturally lead to the definition of an ultraproduct, and the above 'equality' will turn out to be true when the 'limit' is replaced with the ultraproduct of $\overline{\mathbf{F}}_p$, relative to a fixed non-principal ultrafilter on the set of all prime numbers $p > 0$. We shall conclude by deducing an interesting result from complex algebraic geometry (namely, the Ax-Grothendieck theorem).

Everything covered in this talk, and much more, can (essentially) be found in [Gol22]. Another excellent reference is [Sch10], though it does exclusively cater for the commutative algebraist. For applications of ultraproducts to number theory, see [Pan22] and the references contained therein.

0.2. **Ultrafilters.** Let us warm up by doing some set theory. Fix a (non-empty) set I. We say a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(I)$ of subsets of I is a *filter* on I if:

(1) for any inclusion $I' \subseteq I''$ of subsets of I, if $I' \in \mathcal{F}$ then $I'' \in \mathcal{F}$;
(2) for any $I', I'' \in \mathcal{F}$, we have $I' \cap I'' \in \mathcal{F}$.

An *ultrafilter* is a filter $\mathcal{F} \neq \mathcal{P}(I)$ satisfying the following additional axiom:

(3) for any partition $I = I' \sqcup I''$ of I, precisely one of $I' \in \mathcal{F}$ or $I'' \in \mathcal{F}$ holds.

The more general notion of a filter will be of little consequence to us, and it is ultrafilters that will play first fiddle. In fact, we will only need filters to prove the existence of 'interesting' ultrafilters.

Remark 0.2.1. As soon as I has more than one element, any filter $\mathcal{F}$ satisfying (1)-(3) is an ultrafilter. (If $i \in I$ then $\mathcal{P}(I)$ fails (3) with respect to $I = \{i\} \sqcup (I \setminus \{i\})$.)

One ought to think of ultrafilters as sieves deciding if a given subset of I is 'large' or 'small', according to whether or not it belongs to $\mathcal{F}$. We may thus rephrase the above axioms as follows: (1) no small set can contain a large set; (2) the property of being large is stable under finite intersections; and (3) a set is large if and only if

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its complement is small. In keeping with this philosophy, we say a property (*blah*) of elements of I holds for $\mathcal{F}$-*many* $i \in I$ if the set

$$\{\text{elements } i \in I \text{ for which } (\textit{blah}) \text{ holds}\}$$

belongs to $\mathcal{F}$.

Example 0.2.2. Given $i \in I$, we define $\mathcal{F}_i = \{I' \subseteq I \mid i \in I'\}$. This is manifestly an ultrafilter on I, called the *principal ultrafilter* generated by i.

The 'size heuristic' outlined above breaks down for principal ultrafilters, since we would seldom wish for the class of large subsets of I to include a singleton set. It is thus natural to focus our attention on *non-principal ultrafilters*, that is, those not of type $\mathcal{F}_i$. Fact of life: non-principal ultrafilters are hard (if not impossible) to write down. They do still exist, however.

Lemma 0.2.3. *If I is infinite then it supports at least one non-principal ultrafilter.*

Proof. This is an application of Zorn's lemma. Consider the *Fréchet filter*

$$\mathrm{Fr}(I) = \{\text{subsets } I' \subseteq I \text{ whose complement is finite}\}.$$

This is indeed a filter on I; crucially, we have $\mathrm{Fr}(I) \neq \mathcal{P}(I)$ thanks to the assumption that I is infinite. It thus makes sense to contemplate

$$S = \{\text{filters } \mathcal{F} \text{ on } I \text{ such that } \mathrm{Fr}(I) \subseteq \mathcal{F} \subsetneq \mathcal{P}(I)\},$$

which is a non-empty poset under inclusion. Given a non-empty totally-ordered subset $T \subseteq S$, it is a straightforward exercise to verify that $\mathcal{F}_T = \bigcup T$ is an element of S. (Verifying (2) is the only place where we need T totally-ordered. Note that, once we have verified (1), the condition $\mathcal{F}_T \neq \mathcal{P}(I)$ amounts to the fact that $\emptyset \notin \mathcal{F}_T$.) In view of this, Zorn's lemma implies that S contains at least one maximal element, call it $\mathcal{F}$.

We claim $\mathcal{F}$ is an ultrafilter on I. Pick a partition $I = I' \sqcup I''$. If both $I', I'' \in \mathcal{F}$ then $\emptyset \in \mathcal{F}$ by (2), contradicting $\mathcal{F} \neq \mathcal{P}(I)$. So we are left with showing that at least one of I', I'' lies in $\mathcal{F}$. Suppose $I' \notin \mathcal{F}$. Let $\mathcal{G}$ be the filter 'generated by $\mathcal{F} \cup \{I'\}$'. Concretely, we first consider the closure

$$\mathcal{G}_0 = \mathcal{F} \cup \{A \cap I' \mid A \in \mathcal{F}\} \cup \{I'\}$$

of $\mathcal{F} \cup \{I'\}$ under finite intersections, and then define

$$\mathcal{G} = \{A \subseteq I \mid \text{there is some } B \subseteq A \text{ such that } B \in \mathcal{G}_0\}.$$

By construction, $\mathcal{G}$ is a filter on I properly containing $\mathcal{F}$. Our choice of $\mathcal{F}$ then forces us to conclude that $\mathcal{G} = \mathcal{P}(I)$. In particular, $\emptyset \in \mathcal{G}_0$. Since $I' \neq \emptyset$ and $\mathcal{F} \neq \mathcal{P}(I)$, this implies that $A \cap I' = \emptyset$ for some $A \in \mathcal{F}$, or equivalently, $A \subseteq I''$. Hence $I'' \in \mathcal{F}$ by (1), as required. Therefore, $\mathcal{F}$ is an ultrafilter.

The last order of business is to check that the ultrafilter $\mathcal{F}$ above is non-principal. But this follows from the fact that $\mathrm{Fr}(I)$ is not contained in any principal ultrafilter: indeed, if $i \in I$ then $I \setminus \{i\} \in \mathrm{Fr}(I)$ so $\mathrm{Fr}(I) \not\subseteq \mathcal{F}_i$ since $\{i\} \in \mathcal{F}_i$. □

Remark 0.2.4. One could show that, conversely, if $\mathcal{F}$ is an ultrafilter on I then the only filter properly containing $\mathcal{F}$ is $\mathcal{P}(I)$. Note also that any non-principal ultrafilter on I must necessarily contain $\mathrm{Fr}(I)$.

0.3. **Ultraproducts.** From now on we fix an infinite set I and a non-principal ultrafilter $\mathcal{F}$ on I. The particular choice of $\mathcal{F}$ will be immaterial to us: we ultimately only care about the fact that one *can* partition $\mathcal{P}(I)$ into 'large' and 'small' subsets, not about how one might go about doing so.

Let $(X_i)_{i\in I}$ be an I-indexed collection of sets/groups/rings/... (the construction below works in any category with products and filtered colimits). Then we define the *ultraproduct* of $(X_i)_{i\in I}$ as

$$\operatorname*{ulim}_{i\in I} X_i = \Big(\prod_{i\in I} X_i\Big)/\sim$$

where $\sim$ is the equivalence relation on $\prod_{i\in I} X_i$ which identifies $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ if and only if $x_i = y_i$ for $\mathcal{F}$-many $i \in I$. This is again a set/group/ring/... We write $[x_i]_{i\in I}$ for the image of $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ in $\operatorname{ulim}_{i\in I} X_i$.

Remark 0.3.1. The ultraproduct of $(X_i)_{i\in I}$ with respect to a *principal* ultrafilter, say $\mathcal{F}_j$, is just X_j. To get interesting examples of ultraproducts, we are thus forced to work with non-principal ultrafilters.

The utility of ultrafilters in commutative algebra comes from the fact that proofs relying on the pigeonhole principle can often be rephrased in terms of the language of ultraproducts. This has the stylistic advantage that one does not rely on choices, once a suitable $\mathcal{F}$ has been fixed. The following lemma is crucial to executing this strategy; it should also provide some intuition for working with ultraproducts.

Lemma 0.3.2 (Pigeonhole principle)**.** *Let $(X_i)_{i\in I}$ be sequence of sets/groups/rings. Assume in addition that each X_i has finite cardinality, with $\sup_{i\in I} |X_i|$ finite. Then*

$$\operatorname*{ulim}_{i\in I} X_i \cong X_j$$

for $\mathcal{F}$-many $j \in I$.

Proof. By assumption, there are only finitely many isomorphism classes among the X_i. Therefore, there is an object X (belonging to the same category as each X_i) such that $X \cong X_i$ for $\mathcal{F}$-many $i \in I$. We thus have

$$\operatorname*{ulim}_{i\in I} X_i \cong \operatorname*{ulim}_{i\in I} X,$$

and so we are reduced to the constant case. But this is settled by proving that the diagonal map

$$X \to \operatorname*{ulim}_{i\in I} X$$

is an isomorphism: injectivity is immediate, whereas surjectivity follows from the fact that X has finite cardinality. See [Man18, Proposition A.1.3] for details. □

We are finally within reach of the main goal of this talk:

Proposition 0.3.3. *Choose a non-principal ultrafilter $\mathcal{F}$ on* {primes $p \in \mathbf{N}$}*. Then, the ring $K = \operatorname{ulim}_p \overline{\mathbf{F}}_p$ is isomorphic to $\mathbf{C}$.*

Proof. The proof is by cheating: we show that K is an algebraically-closed field of characteristic 0 such that $|K| = |\mathbf{C}|$. (This will be enough by the existence of transcendence bases and the uniqueness of algebraic closures: see, for instance, [Sch10, Theorem 2.4.7].) Let $[x_p]_p \in K$ be nonzero (that is, $x_p \neq 0$ for $\mathcal{F}$-many p). Define $(y_p)_p \in \prod_p \overline{\mathbf{F}}_p$ by $y_p = x_p^{-1}$ if $x_p \neq 0$ and $y_p = 0$ otherwise. Then $[y_p]_p \in K$ is an inverse for $[x_p]_p$, since $x_p y_p = 1$ for $\mathcal{F}$-many p. So K is a field.

To see why $\mathrm{char}(K) = 0$, simply notice that if $n \in \mathbf{Z}$ dies in K then we must have $n = 0$ in $\overline{\mathbf{F}}_p$ for $\mathcal{F}$-many p. In particular, $p \mid n$ for infinitely many primes p, implying that $n = 0$ in $\mathbf{Z}$.

That K is algebraically closed follows from the fact that it admits a surjection from $\prod_p \overline{\mathbf{F}}_p$; being a product of algebraically closed fields, the latter ring has the property that every monic polynomial splits.

Finally, we attend to the cardinality of K. We know that

$$|K| \leq \Big| \prod_p \overline{\mathbf{F}}_p \Big| = |\mathbf{N}|^{|\mathbf{N}|} = |\mathbf{C}|$$

since there are $|\mathbf{N}|$ primes in $\mathbf{N}$, and each $|\overline{\mathbf{F}}_p| = |\mathbf{N}|$. (The last equality is a pleasant exercise in cardinal arithmetic.) To conclude, it suffices to show that $\mathrm{ulim}_p \mathbf{F}_p$ has cardinality at least $|\mathbf{C}|$. We follow [Gol22, Theorem 6.8.3]. Let

$$s_p : \mathbf{F}_p \to \{0, 1/p, ..., (p-1)/p\} \subseteq [0,1]$$

be the normalisation of the 'standard section' of $\mathbf{Z} \to \mathbf{F}_p$. Consider also the map

$$l : \prod_p [0,1] \to [0,1]$$

sending a sequence $(\alpha_p)_p$ of elements of $[0,1]$ to the unique $\alpha \in [0,1]$ such that whenever $U \subseteq [0,1]$ is an open neighbourhood of α, then $\alpha_p \in U$ for $\mathcal{F}$-many p. This makes sense since $[0,1]$ is a compact T2 space (see [Gol22, Theorem 3.1.10]). One checks that l factors through $\mathrm{ulim}_p[0,1]$; it thus makes sense to define

$$f = l \circ s : \operatorname*{ulim}_p \mathbf{F}_p \to \operatorname*{ulim}_p [0,1] \to [0,1].$$

We show that $[0,1)$ lies in the image of f. Pick $\alpha \in [0,1)$. For each p, we find a unique element $x_p \in \mathbf{F}_p$ such that $s_p(x_p) \leq \alpha < s_p(x_p) + 1/p$. It is easy to see that $x = [x_p]_p$ maps to α under f. Indeed, if $\epsilon > 0$ then we have $p \geq 1/\epsilon$ for $\mathcal{F}$-many p (in fact, this happens co-finitely often), in which case we have

$$|s_p(x_p) - \alpha| < 1/p \leq \epsilon.$$

Therefore $f(x) = \alpha$, whence the claim: $|K| \geq |\mathbf{C}|$.

Putting everything together, we conclude that $K \cong \mathbf{C}$. □

Remark 0.3.4. The proof did not use in any essential way the fact that we are working with the particular fields $\overline{\mathbf{F}}_p$; in fact, whenever $(K_i)_{i \in I}$ is a countable collection of algebraically-closed fields such that

- for each prime $p > 0$, $\mathrm{char}(K_i) \neq p$ for $\mathcal{F}$-many $i \in I$;
- $|K_i| \leq |\mathbf{C}|$ for $\mathcal{F}$-many $i \in I$,

then $K = \mathrm{ulim}_{i \in I} K_i$ is isomorphic to $\mathbf{C}$. In particular, $\mathrm{ulim}_{n \in \mathbf{N}} \overline{\mathbf{Q}} \cong \mathbf{C}$. (However, the algebraic-geometric application of Proposition 0.3.3, discussed overleaf, *does* require us to work with the fields $\overline{\mathbf{F}}_p$.)

This fits into a much more general framework of transferring properties of the factors of an ultraproduct to the ultraproduct itself. The buzzword is Łoś' theorem, which states that any first-order formula holds in $\mathrm{ulim}_{i \in I} X_i$ if and only if it holds in X_i for $\mathcal{F}$-many $i \in I$ [Gol22, Theorem 6.4.1]. This for example implies the curious fact that $K = \mathrm{ulim}_p \mathbf{F}_p$ is an uncountably-infinite field of characteristic 0 such that for every $n \geq 1$, there is a unique extension L/K of degree n, and the latter is moreover cyclic. In particular, $\mathrm{Gal}(\overline{K}/K) \cong \widehat{\mathbf{Z}}$.

0.4. **An application.** We conclude with a cute application of the theory developed so far to complex algebraic geometry. Recall that an *affine variety* over $\mathbf{C}$ is a subset

$$V = \{(z_1, ..., z_n) \in \mathbf{C}^n \mid f_1(z_1, ..., z_n) = \cdots = f_m(z_1, ..., z_n) = 0\}$$

of $\mathbf{C}^n$ (for some integer $n \geq 0$) given by the simultaneous vanishing of a finite list $f_1, ..., f_m$ of n-variate polynomials over $\mathbf{C}$. A function $\phi : V \to V$ is called *regular* if there are polynomials $\phi_1, ..., \phi_n \in \mathbf{C}[x_1, ..., x_n]$ such that

$$\phi(z_1, ..., z_n) = (\phi_1(z_1, ..., z_n), ..., \phi_n(z_1, ..., z_n))$$

for all $(z_1, ..., z_n) \in V$.

Theorem 0.4.1 (Ax-Grothendieck)**.** *Let $\phi : V \to V$ be a regular endomorphism of an affine variety $V \subseteq \mathbf{C}^n$. If ϕ is injective then it is surjective.*

Proof. Since an injective map between finite sets of the same cardinality is bijective, the theorem holds with $\mathbf{C}$ replaced by any finite field. Next, since $\overline{\mathbf{F}}_p$ is a union of all of its finite subfields (containing the coefficients of the polynomials defining a fixed affine variety over $\overline{\mathbf{F}}_p$), the theorem holds for $\overline{\mathbf{F}}_p$ as well. Theorem 0.4.1 follows at once from Łoś's theorem.

Explicitly, we can find affine varieties $V_p \subseteq \overline{\mathbf{F}}_p^n$ and regular maps $\phi_p : V_p \to V_p$ such that V comes from $(V_p)_p$ and ϕ from $(\phi_p)_p$ under the identification

$$\mathbf{C} \cong \operatorname*{ulim}_p \overline{\mathbf{F}}_p$$

of Proposition 0.3.3. For this, first enlarge the set $\{f_1, ..., f_m\}$ so that it generates

$$\{f \in \mathbf{C}[x_1, ..., x_n] \mid f(z_1, ..., z_n) = 0 \text{ for all } (z_1, ..., z_n) \in V\}.$$

Then, there is an $m \times m$ matrix M over $\mathbf{C}[x_1, ..., x_n]$ such that

$$(f_1(\phi_1, ..., \phi_n), ..., f_m(\phi_1, ..., \phi_n)) = (f_1, ..., f_m)M.$$

Choosing polynomials $f_{i,p}$, $\phi_{j,p}$ and $M_{\lambda\mu,p}$ in $\overline{\mathbf{F}}_p[x_1, ..., x_n]$ such that $(f_{i,p})_p$ maps to f_i, $(\phi_{j,p})_p$ to ϕ_j and $(M_p)_p = ((M_{\lambda\mu,p})_{\lambda,\mu})_p$ to $M = (M_{\lambda\mu})_{\lambda,\mu}$, we find that

$$(f_{1,p}(\phi_{1,p}, ..., \phi_{n,p}), ..., f_{m,p}(\phi_{1,p}, ..., \phi_{n,p})) = (f_{1,p}, ..., f_{m,p})M_p$$

holds for $\mathcal{F}$-many p. Letting V_p be the zero set of $f_{1,p}, ..., f_{m,p}$ in $\overline{\mathbf{F}}_p^n$, we thus see that $\phi_p = (\phi_{1,p}, ..., \phi_{n,p})$ maps V_p to itself for $\mathcal{F}$-many p. Adjusting the definition of ϕ_p on an $\mathcal{F}$-small subset of all the primes p (say, by declaring it to be a constant map taking values in V_p), we may assume that $\phi_p(V_p) \subseteq V_p$ for all p, as required.

We claim that ϕ_p is injective for $\mathcal{F}$-many p. If not, then there are elements $x_p, y_p \in V_p$ such that $\phi_p(x_p) = \phi_p(y_p)$ for all p but such that $x_p \neq y_p$ for $\mathcal{F}$-many p. Then, $[x_p]_p$ and $[y_p]_p$ are two distinct points of V whose images under ϕ coincide, a contradiction. By the discussion above, we see that ϕ_p is bijective for $\mathcal{F}$-many p, and hence that ϕ is bijective. So we win! □

Remark 0.4.2. Theorem 0.4.1 is a special case of a much more general theorem of algebraic geometry: if $X \to S$ is a finite-type morphism of schemes then any radical S-endomorphism of X is bijective [Gro66, (10.4.11)]. We were able to prove this for affine varieties over $\mathbf{C}$ using the language of ultrafilters, thereby sidestepping EGA's discussion of Jacobson schemes. (Note that Grothendieck also proves his (10.4.11) by reduction to the finite-field case, but requires much more preparation.)

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