Integral Parts of Reciprocals of the Hurwitz Zeta Function and the Polynomial Families That Arise in the Study Thereof

Franciszek Knyszewski

d'Overbroeck's College, Oxford, England

Abstract

The main purpose of this paper is to investigate the existence of a general evaluatory formula for integral parts of reciprocals of tails of the series defining the Riemann ζ-function for a fixed value of s: a non-unitary positive integer. To that end, we introduce two families of polynomials - one given by a convolution recurrence involving Bernoulli numbers, and the other defined in terms of the previous family. Through an asymptotic argument, we provide the desired formula in terms of said polynomials. Subsequently, using the methodology established, a direct derivation of some particular formulae (corresponding to the cases $s = 7$ and $s = 9$) is made. We then proceed to study the properties of the polynomials mentioned, in particular obtaining a rather surprising symmetry property about the 'critical line'.

Keywords: Computational Formula, Floor Function, Hurwitz Zeta Function, Tails of Series, Polynomials, Polynomial Symmetry, Quasipolynomials, Quasiperiods

1. Introduction

Let $a > 0$ and $s \in \mathbb{C}$ with $\sigma := \mathrm{Re}(s) > 1$. We define the *Hurwitz ζ-function* by

$$\zeta(s,a) := \sum_{n\geq 0} \frac{1}{(n+a)^s}.$$

Note that, when $a = 1$, we recover the *Riemann ζ-function* evaluated at s: $\zeta(s)$. In order to better our understanding of the nature of $\zeta(s)$ for integers $s \geq 2$, it is argued that it would be beneficial to approximate the 'tails' $\zeta(s,n)$ for $n \in \mathbb{Z}^+$. Since $1/\zeta(s,n)$ is unbounded as $n \to \infty$, a natural question to ask would be to find an interval of the form $[k, k+1)$, such that it contains $1/\zeta(s,n)$, where k is an integer dependent on s and n. In this paper, motivated by the problem stated and some of the recent publications (see §2), we consider the question of the existence of a general computational formula for $[\![1/\zeta(m+1,n)]\!]$, where $[\![\cdot]\!]$ denotes the floor function and both m and n are positive integers. In other words, we seek to investigate whether, for a fixed positive integer m, $[\![1/\zeta(m+1,n)]\!]$ can be expressed in terms of a 'simple' function of n for all sufficiently large integers n. As the reader will come to realise, this question naturally gives rise to a certain family of polynomials, which will be referred to as the *ξ-polynomials*. We define $\xi_n(x)$ by the recurrence relation

$$\xi_n(x) = -\sum_{k=1}^{n} \binom{-x}{k} B_k \xi_{n-k}(x), \qquad n \geq 1$$

with $\xi_0(x) = 1$, where (B_k) are the Bernoulli numbers (see §3). The first few ξ-polynomials are

$$\begin{aligned}
\xi_0(x) &= 1 \\
\xi_1(x) &= 0 - \tfrac{1}{2}x \\
\xi_2(x) &= 0 - \tfrac{1}{12}x + \tfrac{1}{6}x^2 \\
\xi_3(x) &= 0 + 0x + \tfrac{1}{12}x^2 - \tfrac{1}{24}x^3 \\
\xi_4(x) &= 0 + \tfrac{1}{120}x + \tfrac{1}{45}x^2 - \tfrac{29}{720}x^3 + \tfrac{1}{120}x^4 \\
\xi_5(x) &= 0 + 0x - \tfrac{1}{120}x^2 - \tfrac{37}{1440}x^3 + \tfrac{1}{80}x^4 - \tfrac{1}{720}x^5 \\
\xi_6(x) &= 0 - \tfrac{1}{252}x - \tfrac{79}{7560}x^2 - \tfrac{23}{4032}x^3 + \tfrac{3}{224}x^4 - \tfrac{17}{6048}x^5 + \tfrac{1}{5040}x^6 \\
\xi_7(x) &= 0 + 0x + \tfrac{1}{252}x^2 + \tfrac{337}{30240}x^3 + \tfrac{125}{12096}x^4 - \tfrac{25}{6048}x^5 + \tfrac{1}{2016}x^6 - \tfrac{1}{40320}x^7
\end{aligned}$$

Email address: franciszek.knyszewski@gmail.com (Franciszek Knyszewski)

Our main result is the following.

Theorem 1. *Fix $m \in \mathbb{Z}^+$. There exist natural numbers N_m and τ_m, such that, for all integers $n \geq N_m$, we have*

$$\left[\!\left[\frac{1}{\zeta(m+1,n)}\right]\!\right] = \left[\!\left[\sum_{k=0}^{m} m\xi_k(m)n^{m-k}\right]\!\right] - c_m(n), \tag{1.1}$$

where $c_m : \mathbb{Z} \longrightarrow \{0,1\}$ is a τ_m-periodic correction term.

Given the structure of Theorem 1, it is reasonable to introduce a family of auxiliary polynomials (the Ξ-*polynomials*), defined by

$$\Xi_m(x) := \sum_{k=0}^{m} m\xi_k(m)x^{m-k}, \qquad m \geq 1.$$

The first few such polynomials are given below.

$$\begin{aligned}
\Xi_1(x) &= -\tfrac{1}{2} + x\\
\Xi_2(x) &= 1 - 2x + 2x^2\\
\Xi_3(x) &= -\tfrac{9}{8} + \tfrac{15}{4}x - \tfrac{9}{2}x^2 + 3x^3\\
\Xi_4(x) &= -\tfrac{2}{9} - \tfrac{16}{3}x + \tfrac{28}{3}x^2 - 8x^3 + 4x^4\\
\Xi_5(x) &= \tfrac{25}{96} + \tfrac{185}{48}x - \tfrac{125}{8}x^2 + \tfrac{75}{4}x^3 - \tfrac{25}{2}x^4 + 5x^5\\
\Xi_6(x) &= \tfrac{75}{4} - \tfrac{27}{10}x + \tfrac{177}{10}x^2 - 36x^3 + 33x^4 - 18x^5 + 6x^6\\
\Xi_7(x) &= -\tfrac{65611}{3456} + \tfrac{69685}{1728}x - \tfrac{1715}{96}x^2 + \tfrac{7399}{144}x^3 - \tfrac{1715}{24}x^4 + \tfrac{637}{12}x^5 - \tfrac{49}{2}x^6 + 7x^7\\
\Xi_8(x) &= -\tfrac{3324}{7} - \tfrac{512}{7}x + \tfrac{624}{7}x^2 - 64x^3 + 120x^4 - 128x^5 + 80x^6 - 32x^7 + 8x^8
\end{aligned}$$

For the sake of completeness, one may also define $\Xi_0(x) := 0$. If we further put $K_m := \min\{k > m \mid \xi_k(m) \neq 0\}$, then the correction term in Theorem 1 admits the explicit representation

$$c_m(n) = \begin{cases} 0, & \xi_{K_m}(m) > 0 \vee \Xi_m(n) \notin \mathbb{Z} \\ 1, & \xi_{K_m}(m) < 0 \wedge \Xi_m(n) \in \mathbb{Z} \end{cases}, \qquad m \geq 1.$$

Note that, whilst the appearance of c_m in Theorem 1 is necessary due to technical reasons, the author is not aware of a positive integer m for which c_m is not a constant function. In either case, as $\Xi_m(x) \in \mathbb{Q}[x]$ for all $m \in \mathbb{Z}^+$ (refer to §3), the correction term is indeed τ_m-periodic, where τ_m is the period of the fractional part of $\Xi_m(x)$ and is given by

$$\tau_m = \min\{d > 0 \mid d\Xi_m(x) - d\Xi_m(0) \in \mathbb{Z}[x]\}$$

(see §5). The table below provides values of τ_m, $1 \leq m \leq 16$, along with their prime decompositions.

m	τ_m	prime decomposition of τ_m	m	τ_m	prime decomposition of τ_m
1	1	-	9	256	2^8
2	1	-	10	648	$2^3 \times 3^4$
3	4	2^2	11	230400	$2^{10} \times 3^2 \times 5^2$
4	3	$2^0 \times 3^1$	12	55	$2^0 \times 3^0 \times 5^1 \times 7^0 \times 11^1$
5	48	$2^4 \times 3^1$	13	14929920	$2^{12} \times 3^6 \times 5^2$
6	10	$2^1 \times 3^0 \times 5^1$	14	11232	$2^5 \times 3^3 \times 5^0 \times 7^0 \times 11^0 \times 13^1$
7	1728	$2^6 \times 3^3$	15	802816	$2^{14} \times 3^0 \times 5^0 \times 7^2$
8	7	$2^0 \times 3^0 \times 5^0 \times 7^1$	16	382725	$2^0 \times 3^7 \times 5^2 \times 7^1$

Having the above terminology at our disposal, we remark on an interesting interpretation of Theorem 1. First, recall that a function $q : \mathbb{Z} \longrightarrow \mathbb{C}$ is called a *quasipolynomial* if it can be represented by polynomials on residue classes with respect to some modulus τ (termed its *quasiperiod*), i.e., if for each $r \in \{0, ..., \tau - 1\}$, there exists a polynomial $q_r(x)$, such that $q(x) = q_r(x)$ whenever $x \equiv r \pmod{\tau}$. Now, fix a positive integer m and note that Theorem 1 is equivalent to the quasipolynomial $Q_m(n) := [\![\Xi_m(n)]\!] - c_m(n)$ (of quasiperiod τ_m) coinciding with $[\![1/\zeta(m+1,n)]\!]$ for all sufficiently large integers n. Whilst it certainly does suffice, τ_m may not necessarily be the minimal quasiperiod of $Q_m(n)$, which in either case is a factor of τ_m (trivial to verify).

Upon deriving Theorem 1, we continue to obtain a result which provides a practical way of finding the lower bound N_m in particular cases. Due to its heavy reliance on auxiliary terminology which we have yet to introduce, we postpone the presentation of said theorem until §6. However, the key conclusion to be drawn from said result is that the lowest integer for which formula (1.1) holds is different across different residue classes modulo τ_m and depends on the fractional part of $\Xi_m(r)$, where $r \in \{0, ..., \tau_m - 1\}$ is an element of the residue class considered. Another (rather informal) consequence regarding the size of N_m is that, if, for some residue class $r + \tau_m\mathbb{Z}$, the value of $\Xi_m(r)$ is not an integer but close to one, then N_m should be expected to be 'quite big'.

In order to provide the reader with a more concrete understanding of the applicability of Theorem 1, it might be useful to consider the following particular example (note the evident appearance of Ξ_6 and τ_6 in the statement).

Corollary 1. *Suppose that $n = 10k + r$ with $k \in \mathbb{Z}$ and $r \in \{0, ..., 9\}$. The computational formula*

$$\left[\!\left[\frac{1}{\zeta(7,n)}\right]\!\right] = 6n^6 - 18n^5 + 33n^4 - 36n^3 + 17n^2 - 3n + 18 + \left[\!\left[\frac{7}{10}n^2 + \frac{3}{10}n + \frac{3}{4}\right]\!\right] \tag{1.2}$$

holds for

(1) $k \geq 1$ *if* $r \in \{3, 5, 6, 8\}$;
(2) $k \geq 2$ *if* $r \in \{0, 1, 7, 9\}$;
(3) $k \geq 3$ *if* $r \in \{2, 4\}$.

In particular, it holds for all $n \geq 25$.

In §7.2, we acquire yet another explicit computational formula (corresponding to the case $s = 9$). Thereafter, we commence the study of the properties of the polynomial families introduced in this paper. Using arguments based on the theory of formal power series, we acquire a curious identity involving the ξ-polynomials and the binomial coefficients.

Theorem 2. *We have*

$$\sum_{k=0}^{n} \binom{x-k}{n-k} \xi_k(x) \equiv (-1)^n \xi_n(x) \tag{1.3}$$

for all integers $n \geq 0$.

Perhaps one of our most surprising discoveries is the following symmetry property about the *critical line* $\{x \mid \mathrm{Re}(x) = 1/2\}$.

Theorem 3. *The Ξ-polynomials satisfy the reflection formula*

$$\Xi_m(x) \equiv (-1)^m \Xi_m(1-x)$$

for all $m \in \mathbb{Z}^+$. Equivalently, the parity of $\Xi_m(x + 1/2)$ as a function agrees with the parity of m as an integer.

Moreover, in §10, different representations of $\xi_n(x)$ are provided. Furthermore, we derive a general recurrence relation for the coefficients of the ξ-polynomials and use it to deduce some explicit formulae for said coefficients (see §11).

Finally, we would like to direct the reader's attention towards §12 where we give our concluding conjectures regarding the concepts considered in this paper. There, we identify a series of p-adic patterns involving the τ-quasiperiods. Further, we hypothesise an explicit formula for K_m and a criterion for determining the sign of $\xi_{K_m}(m)$. Lastly, we state a conjecture about the complex zeroes of the Ξ-polynomials bearing a similarity to the Riemann Hypothesis.

2. Results in Literature

Some progress regarding the evaluation of the integral parts of $1/\zeta(s, n)$ for particular cases of s has been made in a number of publications of late. In particular, in their article published in the *Journal of Inequalities and Applications* titled 'Some Identities Related to Riemann Zeta-Function' [1], Xin proves that, in the notation of this article,

$$\left[\!\left[\frac{1}{\zeta(2,n)}\right]\!\right] = n - 1 \qquad \& \qquad \left[\!\left[\frac{1}{\zeta(3,n)}\right]\!\right] = 2n^2 - 2n$$

for all positive integers n. Moreover, they raise a question of the existence of a generalised result for any $s \in \mathbb{Z}^+/\{1\}$. Not long after, Xu [2] provides formulae corresponding to $s = 4$

$$\left[\!\!\left[\frac{1}{\zeta(4,n)}\right]\!\!\right] = 3n^3 - 5n^2 + 4n - 1 + \left[\!\!\left[\frac{(2n+1)(n-1)}{4}\right]\!\!\right], \qquad n \geq 2,$$

and $s = 5$

$$\left[\!\!\left[\frac{1}{\zeta(5,n)}\right]\!\!\right] = 4n^4 - 8n^3 + 9n^2 - 5n + \left[\!\!\left[\frac{(n+1)(n-2)}{3}\right]\!\!\right], \qquad n \geq 4.$$

Notably, their formulae are the first to feature quasipolynomials of nontrivial quasiperiods. It should be mentioned that Xin and Xiaoxue [3] give a different formula for $[\![1/\zeta(4,n)]\!]$, where they split into cases according to the parity of n. We remark that, although both representations are equally valid, Xu's formula is considerably more compact, whilst also conforming with Theorem 1. Finally, Hwang and Song [4] give an 'unusual' result concerning $[\![1/\zeta(6,n)]\!]$. Unusual in that it is directly dependent on the residue of n modulo 48, labelled in their paper as n_{48}. The formula they obtain is

$$\left[\!\!\left[\frac{1}{\zeta(6,n)}\right]\!\!\right] = 5n^5 - \frac{25}{2}n^4 + \frac{75}{4}n^3 - \frac{125}{8}n^2 + \frac{185}{48}n - \frac{5n_{48}}{48} - \begin{cases} [\![\frac{35-5n_{48}}{48}]\!], & n \equiv 0 \pmod 2 \\ \frac{3}{8} + [\![\frac{17-5n_{48}}{48}]\!], & n \equiv 1 \pmod 2 \end{cases},$$

where $n \geq 829$. Further, they state a conjecture that $[\![1/\zeta(s,n)]\!]$ with $s \geq 6$; an integer, is directly dependent on the residue of n modulo a multiple of $s-2$. We refer the reader to Appendix A, where we provide a refinement of the above formula - one that neither depends directly on n_{48}, nor splits into different cases according to the parity of n. At the time of writing this article, no further developments for non-unitary integral values of s were known to the author.

3. Definitions

We begin by recalling some standard concepts. Define

(1) the set $\langle m\rangle := \{0, ..., m\}$ for a given $m \in \mathbb{Z}_0^+$.
(2) the *Pochhammer symbol* by $(z)_n := \Gamma(z+n)/\Gamma(z)$, where $\Gamma(\cdot)$ is the Γ-*function*, $z \in \mathbb{C}$ and $n \in \mathbb{Z}_0^+$.
(3) a sequence $\left(\left[{n \atop k}\right] \mid n \in \mathbb{Z}_0^+, k \in \langle n\rangle\right)$ of *unsigned Stirling numbers of the 1st kind* by the identity $(z)_n \equiv \sum_{k=0}^n \left[{n \atop k}\right] z^k$.
(4) the *binomial coefficient* by $\binom{z}{n} = (-1)^n(-z)_n/n!$, where $z \in \mathbb{C}$ and $n \in \mathbb{Z}_0^+$.
(5) a sequence $(B_n \mid n \in \mathbb{Z}_0^+)$ of *Bernoulli numbers* as the coefficients of $z(\exp(z)-1)^{-1} \sim \sum_{n\geq 0} B_n z^n/n!$ as $z \to 0$.
(6) the *unit impulse function* $\delta : \mathbb{R} \longrightarrow \{0,1\}$ by $\delta(x) = 1$ when $x = 0$ and $\delta(x) = 0$ otherwise.
(7) the *difference operator* by $\Delta : f(x) \mapsto f(x+1) - f(x)$. Additionally, let Δ^n be the n^{th} iteration of Δ.

We will now give formal definitions of all concepts introduced in this paper.

Definition 1. Let $(\xi_n(x))_{n\geq 0}$ be a sequence of polynomials given by the convolution recurrence

$$\sum_{m=0}^{n} \binom{-x}{m} B_m \xi_{n-m}(x) = \delta(n). \tag{3.1}$$

Further, put $\xi_{n,k}$ for the coefficient of x^k in $\xi_n(x)$, so that $\xi_n(x) \equiv \sum_{k=0}^n \xi_{n,k}x^k$. Trivially, (3.1) and the definition given in §1 are equivalent. Also, note that, as all Bernoulli numbers are rational [5, §4.1], we have $\xi_n(x) \in \mathbb{Q}[x]$, $n \geq 0$.

For the remainder of this section, we fix m to be a positive integer.

Definition 2. Let $\Xi_m(x)$ be an auxiliary polynomial in x given by

$$\Xi_m(x) := \sum_{k=0}^{m} m\xi_k(m)x^{m-k}.$$

Note that Ξ_m is exactly the polynomial inside the floor function in formula (1.1).

Definition 3. Let $\tau_m := \min\{d \in \mathbb{Z}^+ \mid \forall k \in \langle m-1\rangle : dm\xi_k(m) \in \mathbb{Z}\}$. Again, it is easy to verify that this and the definition given in §1 are equivalent. As $\xi_n(x) \in \mathbb{Q}[x]$, it follows that $m\xi_k(m) \in \mathbb{Q}$ for all $k \geq 0$, which implies that τ_m is well-defined. Also, let $K_m := \min\{k > m \mid \xi_k(m) \neq 0\}$ and $\mathcal{R}_m := \{r \in \langle \tau_m - 1\rangle \mid \Xi_m(r) \notin \mathbb{Z}\}$.

Definition 4. Let ϵ be a positive real number. Further, put

$$\vartheta_m^-(\epsilon) := \begin{cases} m\xi_{K_m}(m) - \epsilon, & \xi_{K_m}(m) < 0 \\ 0, & \xi_{K_m}(m) > 0 \end{cases} \qquad \& \qquad \vartheta_m^+(\epsilon) := \begin{cases} 0, & \xi_{K_m}(m) < 0 \\ m\xi_{K_m}(m) + \epsilon, & \xi_{K_m}(m) > 0 \end{cases}. \tag{3.2}$$

We define the auxiliary polynomials $\Pi_m^{\pm} : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$\begin{aligned}\Pi_m^{\pm}(a,\epsilon) := &\left[a^{K_m-m}\Xi_m(a) + \vartheta_m^{\pm}(\epsilon)\right]\left[(a+1)^{K_m-m}\Xi_m(a+1) + \vartheta_m^{\pm}(\epsilon)\right] \\ &- a^{K_m+1}(a+1)^{K_m-m}\left[\Xi_m(a+1) - \Xi_m(a)\right] + \vartheta_m^{\pm}(\epsilon)\left[a^{m+1}(a+1)^{K_m-m} - a^{K_m+1}\right].\end{aligned}$$

Definition 5. Let $\Xi_m^* : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be an auxiliary polynomial given by $\Xi_m^*(a,\epsilon) = a^{K_m-m}\Xi_m(a) + \vartheta_m^-(\epsilon)$.

4. Supplementary Theorems

Following are some of the lemmas we shall use in the subsequent sections.

Lemma 1. *The Hurwitz ζ-function admits the following asymptotic expansion*

$$\zeta(s,a) \sim \frac{1}{(s-1)a^{s-1}} \sum_{n\geq 0} \binom{1-s}{n} \frac{B_n}{a^n}$$

as $a \to \infty$.

Proof. Fix s with $\sigma > 1$ and let $f(t) := t^{s-1}/(1-\exp(-t))$. Recall the integral representation [6, p. 23]

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty f(t)\exp(-at)\,\mathrm{d}t$$

Note that f is locally absolutely integrable on $(0,\infty)$ as $|f(t)| = t^{\sigma-1}/(1-\exp(-t))$ is continuous there. Further, we have that $f(t) = O(\exp(t))$ as $t \to \infty$, which follows from the trivial fact that $f(t) = o(\exp(t))$ as $t \to \infty$. Accordingly, as $\sigma > 1$, we may use Watson's Lemma, as given in [7, §4.1], along with the expansion $f(t) \sim \sum_{n\geq 0}(-1)^n B_n t^{s+n-2}/n!$ as $t \to 0$, to obtain

$$\zeta(s,a) \sim \frac{1}{\Gamma(s)} \sum_{n\geq 0} \frac{(-1)^n B_n \Gamma(s+n-1)}{a^{s+n-1} n!} = \frac{1}{(s-1)a^{s-1}} \sum_{n\geq 0} \left(\frac{(-1)^n \Gamma(s+n-1)}{\Gamma(s-1)n!}\right) \frac{B_n}{a^n} = \frac{1}{(s-1)a^{s-1}} \sum_{n\geq 0} \binom{1-s}{n} \frac{B_n}{a^n},$$

concluding the proof. ☐

Lemma 2. *We have the following asymptotic expansion as $a \to \infty$*

$$\frac{1}{\zeta(s,a)} \sim (s-1)a^{s-1} \sum_{n\geq 0} \frac{\xi_n(s-1)}{a^n}.$$

Proof. Define an auxiliary function ζ^* by $\zeta^*(s,a) = sa^s\zeta(s+1,a)$. Note that, by Lemma 1, $\zeta^*(s,a) \sim \sum_{n\geq 0}\binom{-s}{n}B_n/a^n$ as $a \to \infty$. Accordingly, suppose that $1/\zeta^*(s,a) \sim \sum_{n\geq 0} \xi_n^*(s)/a^n$ as $a \to \infty$ for some unique constants $\xi_0^*(s), \xi_1^*(s), \ldots$. We need to show these satisfy equation (3.1). Fix an arbitrary integer $N \geq 0$. By definition of an asymptotic expansion,

$$\frac{1}{\zeta^*(s,a)} = \sum_{n=0}^{N} \frac{\xi_n^*(s)}{a^n} + o\left(\frac{1}{a^N}\right) \iff 1 - \zeta^*(s,a)\sum_{n=0}^{N} \frac{\xi_n^*(s)}{a^n} = o\left(\frac{\zeta^*(s,a)}{a^N}\right) \qquad (a \to \infty). \tag{4.1}$$

But then, since $\zeta^*(s,a) = 1 + o(1)$ as $a \to \infty$ by Lemma 1, (4.1) can be reduced to $1 - \zeta^*(s,a)\sum_{n=0}^{N}\xi_n^*(s)/a^n = o\big(1/a^N\big)$ as $a \to \infty$. Now, utilising Lemma 1 yet again, we obtain

$$\begin{aligned} 0 &= \lim_{a\to\infty}\left[a^N\left(1 - \left(\sum_{n=0}^{N}\binom{-s}{n}\frac{B_n}{a^n} + O_s\left(\frac{1}{a^{N+1}}\right)\right)\sum_{n=0}^{N}\frac{\xi_n^*(s)}{a^n}\right)\right] \\ &= \lim_{a\to\infty}\left[a^N - \sum_{n=0}^{N} a^{N-n}\sum_{m=0}^{n}\binom{-s}{m}B_m\xi_{n-m}^*(s) + O_s\left(\frac{1}{a}\right)\right]. \end{aligned} \tag{4.2}$$

In order for the expression under the limit in (4.2) to converge to 0, we need $1 - \xi_0^*(s) = -\sum_{m=0}^{n}\binom{-s}{m}B_m\xi_{n-m}^*(s) = 0$ for all integers n with $1 \leq n \leq N$. Since the choice of N was arbitrary, this is equivalent to (3.1), as required. ☐

Accordingly, the ξ-polynomials evaluated at $s-1$ hold the key to the large-a asymptotic expansion of the reciprocal of the Hurwitz ζ-function evaluated at s. This somewhat arbitrary shift was introduced as it makes it easier to study the properties of $\xi_n(s)$, $n \geq 0$. In particular, it gives us a recursive formula for their coefficients (see Theorem 7) that is relatively easier to work with than with the one obtained if the shift was not present in the original definition.

Lemma 3. *For all $n \in \mathbb{Z}_0^+$, we have the representation*

$$\Delta^n[f](x) \equiv \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+n-k)$$

Proof. The proof is a trivial application of induction; refer to [8, §6] for a symbolic argument. $\square$

Lemma 4. *Let $n \in \mathbb{Z}_0^+$ and f, g be nonzero functions differentiable at least n times with $g(x) \neq 0$. We have*

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{f(x)}{g(x)}\right] \equiv \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{\mathrm{D}_x^n\left[f(x)g(x)^k\right]}{g(x)^{k+1}} \tag{4.3}$$

Proof. With the use of Lemma 3, statement (4.3) is equivalent to

$$\begin{aligned} 0 \equiv \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{f(x)}{g(x)}\right] - \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{\mathrm{D}_x^n\left[f(x)g(x)^k\right]}{g(x)^{k+1}} &= \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{\mathrm{D}_x^n\left[f(x)g(x)^{k-1}\right]}{g(x)^k} \\ &= (-1)^{n+1}\, \Delta^{n+1}\left[\frac{\partial_x^n\left[f(x)g(x)^{y-1}\right]}{g(x)^y}\right](y)\bigg|_{y=0}. \end{aligned}$$

But then, if $P(y)$ is a polynomial in y, then $\deg(\Delta[P](y)) = \deg(P(y)) - 1$, which extends to $\deg(\Delta^n[P](y)) = \deg(P(y)) - n$ for all $n \in \mathbb{Z}_0^+$. Since a polynomial of negative degree is identically 0, in order to prove Lemma 4, it would be sufficient to prove that $g(x)^{-y}\partial_x^n\left[f(x)g(x)^{y-1}\right]$ is a polynomial in y of degree at most n. As such, let $\mathrm{S}(n)$ be that statement for a given nonnegative integer n. We induct on n.

(1) *Base case.* For $n = 0$, we have $g(x)^{-y}\partial_x^0\left[f(x)g(x)^{y-1}\right] = f(x)/g(x)$, which is a polynomial in y of negative degree given that $f(x) = 0$ and of degree 0 otherwise. Consequently, $\mathrm{S}(0)$ holds.
(2) *Inductive step.* Suppose that $\mathrm{S}(m)$ is true for some $m \in \mathbb{Z}_0^+$. Let $Q(x,y) := g(x)^{-y}\partial_x^m\left[f(x)g(x)^{y-1}\right]$. We acquire

$$g(x)^{-y}\frac{\partial^{m+1}}{\partial x^{m+1}}\left[f(x)g(x)^{y-1}\right] = g(x)^{-y}\frac{\partial}{\partial x}\left[Q(x,y)g(x)^y\right] = Q_x(x,y) + g'(x)g(x)^{-1}yQ(x,y) =: Q^*(x,y).$$

But then, by the induction hypothesis, $Q(x,y)$ is a polynomial in y of degree at most m. This implies that $Q_x(x,y)$ is a polynomial in y of degree at most m and that $g'(x)g(x)^{-1}yQ(x,y)$ is a polynomial in y of degree at most $m+1$. This implies that $Q^*(x,y)$ is a polynomial in y of degree at most $m+1$. Hence, $\forall m \in \mathbb{Z}_0^+ : [\mathrm{S}(m) \implies \mathrm{S}(m+1)]$, completing the inductive step.

By the principle of induction, $\mathrm{S}(n)$ holds for all nonnegative integers n, in consequence proving Lemma 4. $\square$

Lemma 5. *Let $n \in \mathbb{Z}_0^+$ and $k \in \mathbb{Z}^+$, and let $(f_1, ..., f_k)$ be a sequence of functions differentiable at least n times. Then,*

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\prod_{m=1}^{k} f_m(x)\right] = n! \sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \prod_{m=1}^{k} \frac{1}{j_m!}\frac{\mathrm{d}^{j_m} f_m}{\mathrm{d}x^{j_m}}(x)$$

Proof. See [9]. Note that one may easily prove the statement by induction. The case of $k = 1$ is trivial, whilst the case of $k = 2$ is the general Leibniz rule. The inductive step follows by applying the general Leibniz rule to $(f_{k+1}(x)u(x))^{(n)}$, where $u(x) := \prod_{m=1}^{k} f_m(x)$, subsequently utilising the inductive hypothesis, and then rearranging. $\square$

Lemma 6. *Let $n \in \mathbb{Z}_0^+$. We have the formulae $\left[{n \atop 0}\right] = \delta(n)$, $\left[{n+1 \atop 1}\right] = n!$, $\left[{n+2 \atop 2}\right] = (n+1)!H_{n+1}$, $\left[{n+1 \atop n}\right] = \binom{n+1}{2}$ and $\left[{n \atop n}\right] = 1$, where (H_k) denotes the sequence of harmonic numbers given by $H_k = H_{k-1} + 1/k$ for $k \geq 2$ with $H_1 = 1$.*

Proof. Refer to [10, §6.1] or [11, §5.5]. $\square$

5. Proof of Theorem 1

First, we will prove that $\llbracket 1/\zeta(s+1,n)\rrbracket = \llbracket \Xi_m(n) + m\xi_{K_m}(m)n^{m-K_m}\rrbracket$. To achieve that, it would be sufficient to show that there exist distinct real numbers θ^- and θ^+ with $\theta^- \le 0 \le \theta^+$, such that, for all sufficiently large reals a,

$$\Xi_m(a) + (m\xi_{K_m}(m) + \theta^-)a^{m-K_m} < 1/\zeta(m+1,a) < \Xi_m(a) + (m\xi_{K_m}(m) + \theta^+)a^{m-K_m} \tag{5.1}$$

and $\llbracket \Xi_m(n) + (m\xi_{K_m}(m) + \theta^-)n^{m-K_m}\rrbracket = \llbracket \Xi_m(n) + (m\xi_{K_m}(m) + \theta^+)n^{m-K_m}\rrbracket$ for all sufficiently large integers n. But then, by definition of K_m and by Lemma 2, we know that $1/\zeta(m+1,a) = \Xi_m(a) + \xi_{K_m}(m)a^{m-K_m} + o(a^{m-K_m})$ as $a \to \infty$, which is equivalent to $a^{K_m-m}/\zeta(m+1,a) - a^{K_m-m}\Xi_m(a) - \xi_{K_m}(m) \to 0$ as $a \to \infty$. From first principles, this means that

$$\forall \varepsilon > 0\, \exists A > 0\, \forall a \ge A : \left| \frac{a^{K_m-m}}{\zeta(m+1,a)} - a^{K_m-m}\Xi_m(a) - \xi_{K_m}(m) \right| < \varepsilon,$$

implying that $\Xi_m(a) + m\xi_{K_m}(m)a^{m-K_m} - \varepsilon a^{m-K_m} < 1/\zeta(m+1,a) < \Xi_m(a) + m\xi_{K_m}(m)a^{m-K_m} + \varepsilon a^{m-K_m}$ for all $a \ge A$. Thus, setting $\theta^\pm = \pm\varepsilon$ ensures the first property required. Since the choice of $\varepsilon > 0$ was arbitrary, we only need to verify that

$$\exists \varepsilon > 0\, \exists N > 0\, \forall n \in \mathbb{Z} : \left[n \ge N \implies \llbracket \Xi_m(n) + (m\xi_{K_m}(m) - \varepsilon)n^{m-K_m}\rrbracket = \llbracket \Xi_m(n) + (m\xi_{K_m}(m) + \varepsilon)n^{m-K_m}\rrbracket \right]. \tag{5.2}$$

Now, note that, as $m\xi_k(m)\tau_m \in \mathbb{Z}$ provided $k < m$ (by definition), Ξ_m is τ_m-periodic in the additive quotient group $\mathbb{R}/\mathbb{Z}$. Indeed, the fact that $(n+\tau_m)^k \equiv n^k \pmod{\tau_m}$ gives us $m\xi_k(m)(n+\tau_m)^k \equiv m\xi_k(m)n^k$ modulo $\mathbb{Z}$, which yields

$$\Xi_m(n+\tau_m) = m\xi_m(m) + \sum_{k=0}^{m-1} m\xi_k(m)(n+\tau_m)^{m-k} \equiv m\xi_m(m) + \sum_{k=0}^{m-1} m\xi_k(m)n^{m-k} = \Xi_m(n) \qquad (\text{in } \mathbb{R}/\mathbb{Z}).$$

As such, put $\{x\} := x - \llbracket x\rrbracket$ for the fractional part of a real number x and note that, by subtracting $\llbracket \Xi_m(n)\rrbracket$ from both sides of the requirement in (5.2), it is equivalent to

$$\llbracket \{\Xi_m(r)\} + (m\xi_{K_m}(m) - \varepsilon)n^{m-K_m}\rrbracket = \llbracket \{\Xi_m(r)\} + (m\xi_{K_m}(m) + \varepsilon)n^{m-K_m}\rrbracket, \tag{5.3}$$

where r is the remainder of the division of n by τ_m. Accordingly, let us consider the required property for different residue classes modulo τ_m separately. We have two essentially different cases to examine.

(1) *Residue classes of the form $r + \tau_m\mathbb{Z}$, where $r \notin \mathcal{R}_m$.*
Fix an arbitrary $r \notin \mathcal{R}_m$ and consider the property for $n \in r + \tau_m\mathbb{Z}$. As $\{\Xi_m(r)\} = 0$, it would suffice to show that we can choose some $\varepsilon > 0$ and $N > 0$ so that $\llbracket (m\xi_{K_m}(m) - \varepsilon)n^{m-K_m}\rrbracket = \llbracket (m\xi_{K_m}(m) + \varepsilon)n^{m-K_m}\rrbracket$ for all $n \ge N$. In order for that to be possible, $m\xi_{K_m}(m) - \varepsilon$ and $m\xi_{K_m}(m) + \varepsilon$ have to have the same sign, implying that $\varepsilon < m|\xi_{K_m}(m)|$. Accordingly, taking $N > \sqrt[K_m-m]{m|\xi_{K_m}(m)| + \varepsilon}$, gives us

$$\left| \frac{m\xi_{K_m}(m) \pm \varepsilon}{n^{K_m-m}} \right| \le \left| \frac{m\xi_{K_m}(m) \pm \varepsilon}{N^{K_m-m}} \right| \overset{(*)}{\le} \frac{m|\xi_{K_m}(m)| + |\pm\varepsilon|}{N^{K_m-m}} < \frac{m|\xi_{K_m}(m)| + \varepsilon}{\left(\sqrt[K_m-m]{m|\xi_{K_m}(m)| + \varepsilon}\right)^{K_m-m}} = \frac{m|\xi_{K_m}(m)| + \varepsilon}{m|\xi_{K_m}(m)| + \varepsilon} < 1,$$

where $(*)$ follows from the triangle inequality. Thus, we have $\llbracket (m\xi_{K_m}(m) - \varepsilon)n^{m-K_m}\rrbracket = \llbracket (m\xi_{K_m}(m) + \varepsilon)n^{m-K_m}\rrbracket$ for all $n \ge N$, further implying that $\llbracket 1/\zeta(m+1,n)\rrbracket = \Xi_m(n) + \llbracket m\xi_{K_m}(m)n^{m-K_m}\rrbracket$ for all sufficiently large $n \in r + \tau_m\mathbb{Z}$. But then, for all $n \ge N$, $\llbracket m\xi_{K_m}(m)n^{m-K_m}\rrbracket = 0$ if $\xi_{K_m}(m) > 0$ and $\llbracket m\xi_{K_m}(m)n^{m-K_m}\rrbracket = -1$ otherwise. Thus, for all sufficiently large elements of $r + \tau_m\mathbb{Z}$,

$$\left\llbracket \frac{1}{\zeta(m+1,n)} \right\rrbracket = \Xi_m(n) - \begin{cases} 0, & \xi_{K_m}(m) > 0 \\ 1, & \xi_{K_m}(m) < 0 \end{cases}.$$

As the choice of $r \notin \mathcal{R}_m$ was arbitrary, this agrees with the statement of Theorem 1, as required.

(2) *Residue classes of the form $r + \tau_m\mathbb{Z}$, where $r \in \mathcal{R}_m$.*
Fix an arbitrary $r \in \mathcal{R}_m$ and consider the property for $n \in r + \tau_m\mathbb{Z}$. Note that $0 < \{\Xi_m(r)\} < 1$. We have to show that we can choose $\varepsilon > 0$ and $N > 0$ so that (5.3) holds for all $n \ge N$. Let us again set $\varepsilon < m|\xi_{K_m}(m)|$ and consider the following two possibilities separately.

(i) $\xi_{K_m}(m) > 0$.
Take note that $\{\Xi_m(r)\}+(m\xi_{K_m}(m)\pm\varepsilon)n^{m-K_m}$ will approach $\{\Xi_m(r)\}$ from above as n increases. Accordingly, taking $N > \sqrt[K_m-m]{(m\xi_{K_m}(m)+\varepsilon)/(1-\{\Xi_m(r)\})}$ gives us

$$\{\Xi_m(r)\} + \frac{m\xi_{K_m}(m)\pm\varepsilon}{n^{K_m-m}} \leq \{\Xi_m(r)\} + \frac{m\xi_{K_m}(m)+\varepsilon}{N^{K_m-m}} < \{\Xi_m(r)\} + \frac{m\xi_{K_m}(m)+\varepsilon}{\left(\sqrt[K_m-m]{\frac{m\xi_{K_m}(m)+\varepsilon}{1-\{\Xi_m(r)\}}}\right)^{K_m-m}}$$
$$= \{\Xi_m(r)\} + \frac{(1-\{\Xi_m(r)\})(m\xi_{K_m}(m)+\varepsilon)}{m\xi_{K_m}(m)+\varepsilon} = \{\Xi_m(r)\} + 1 - \{\Xi_m(r)\} = 1.$$

Thus, we have $\left[\!\left[\{\Xi_m(r)\} + (m\xi_{K_m}(m)\pm\varepsilon)n^{m-K_m}\right]\!\right] = 0$ for $n \geq N$, further giving us $[\![1/\zeta(m+1,n)]\!] = [\![\Xi_m(n)]\!]$ for all sufficiently large elements of $r+\tau_m\mathbb{Z}$. As the choice of $r \in \mathcal{R}_m$ was arbitrary, we arrive at the desired.

(ii) $\xi_{K_m}(m) < 0$.
Take note that $\{\Xi_m(r)\}+(m\xi_{K_m}(m)\pm\varepsilon)n^{m-K_m}$ will approach $\{\Xi_m(r)\}$ from below as n increases. Accordingly, taking $N > \sqrt[K_m-m]{(-m\xi_{K_m}(m)+\varepsilon)/\{\Xi_m(r)\}}$ gives us

$$\{\Xi_m(r)\} + \frac{m\xi_{K_m}(m)\pm\varepsilon}{n^{K_m-m}} \geq \{\Xi_m(r)\} + \frac{m\xi_{K_m}(m)-\varepsilon}{N^{K_m-m}} > \{\Xi_m(r)\} + \frac{m\xi_{K_m}(m)-\varepsilon}{\left(\sqrt[K_m-m]{\frac{-m\xi_{K_m}(m)+\varepsilon}{\{\Xi_m(r)\}}}\right)^{K_m-m}}$$
$$= \{\Xi_m(r)\} + \frac{\{\Xi_m(r)\}(m\xi_{K_m}(m)-\varepsilon)}{-m\xi_{K_m}(m)+\varepsilon} = \{\Xi_m(r)\} - \{\Xi_m(r)\} = 0.$$

Thus, we have $\left[\!\left[\{\Xi_m(r)\} + (m\xi_{K_m}(m)\pm\varepsilon)n^{m-K_m}\right]\!\right] = 0$ for $n \geq N$, further giving us $[\![1/\zeta(m+1,n)]\!] = [\![\Xi_m(n)]\!]$ for all sufficiently large elements of $r+\tau_m\mathbb{Z}$. As the choice of $r \in \mathcal{R}_m$ was arbitrary, we arrive at the desired.

The fact that all possible cases agree with Theorem 1 completes the proof. □

6. The lower bound N_m

Theorem 4. *Let ϵ be a positive real number, and m and n positive integers. If A is a constant, such that, for all $a \geq A$, we have $\Xi^*_m(a,\epsilon) > 0$, $\Pi^-_m(a,\epsilon) < 0$ and $\Pi^+_m(a,\epsilon) > 0$, then the computational formula* (1.1) *holds for all n satisfying*

$$n > \max\left(A, \sqrt[K_m-m]{(m|\xi_{K_m}(m)|+\epsilon) \div \begin{Bmatrix} 1-\{\Xi_m(r)\}, & \xi_{K_m}(m) > 0 \wedge \exists r \in \mathcal{R}_m : n \equiv r \pmod{\tau_m} \\ \{\Xi_m(r)\}, & \xi_{K_m}(m) < 0 \wedge \exists r \in \mathcal{R}_m : n \equiv r \pmod{\tau_m} \\ 1, & \text{otherwise} \end{Bmatrix}}\right). \tag{6.1}$$

Proof. By similar logic as in the proof of Theorem 1, for θ^+ approaching $\vartheta^+_m(\epsilon)$ from below and θ^- approaching $\vartheta^-_m(\epsilon)$ from above, we have that $\left[\!\left[\Xi_m(n)+\theta^- n^{m-K_m}\right]\!\right] = \left[\!\left[\Xi_m(n)+\theta^+ n^{m-K_m}\right]\!\right]$ for all n with (6.1) (where A is arbitrary for now). As such, in order for (1.1) to hold, we only need (5.1) to hold for sufficiently large a. As $\Xi^*_m(a,\epsilon) = ma^{K_m} + O_{m,\epsilon}(a^{K_m-1})$ as $a \to \infty$, we know that Ξ^*_m will 'eventually stay positive'. Accordingly, let A^* be a real number, such that, for all real numbers $a > A^*$, $\Xi^*_m(a,\epsilon) > 0$. Note that, as $\vartheta^-_m(\epsilon) < \vartheta^+_m(\epsilon)$, we also have $\Xi_m(a) + \vartheta^+_m(\epsilon)a^{m-K_m} > 0$ for $a > A^*$, implying that requirement (5.1) is equivalent to

$$\frac{a^{K_m-m}}{a^{K_m-m}\Xi_m(a)+\vartheta^+_m(\epsilon)} < \zeta(m+1,a) < \frac{a^{K_m-m}}{a^{K_m-m}\Xi_m(a)+\vartheta^-_m(\epsilon)}. \tag{6.2}$$

But then, $\zeta(m+1,a) = \sum_{k\geq 0}(n+a)^{-m-1}$ and $f(a) \equiv \sum_{k\geq 0} -\Delta[f](k+a)$ for any function f with $f(a) = o(1)$ as $a \to \infty$. Consequently, condition (6.2) is implied by

$$\frac{1}{a^{m+1}} < \frac{a^{K_m-m}}{a^{K_m-m}\Xi_m(a)+\vartheta^-_m(\epsilon)} - \frac{(a+1)^{K_m-m}}{(a+1)^{K_m-m}\Xi_m(a+1)+\vartheta^-_m(\epsilon)} \tag{6.3}$$

and

$$\frac{a^{K_m-m}}{a^{K_m-m}\Xi_m(a)+\vartheta^+_m(\epsilon)} - \frac{(a+1)^{K_m-m}}{(a+1)^{K_m-m}\Xi_m(a+1)+\vartheta^+_m(\epsilon)} < \frac{1}{a^{m+1}}. \tag{6.4}$$

Multiplying both sides of (6.3) by $a^{m+1}\left[a^{K_m-m}\Xi_m(a)+\vartheta_m^-(\epsilon)\right]\left[(a+1)^{K_m-m}\Xi_m(a+1)+\vartheta_m^-(\epsilon)\right]$; a positive real number (as all its factors are positive due to the requirement $a > A^*$), we acquire

$$\begin{aligned}&\left[a^{K_m-m}\Xi_m(a)+\vartheta_m^-(\epsilon)\right]\left[(a+1)^{K_m-m}\Xi_m(a+1)+\vartheta_m^-(\epsilon)\right]\\&< a^{K_m+1}\left[(a+1)^{K_m-m}\Xi_m(a+1)+\vartheta_m^-(\epsilon)\right]-a^{m+1}(a+1)^{K_m-m}\left[a^{K_m-m}\Xi_m(a)+\vartheta_m^-(\epsilon)\right].\end{aligned}$$

After some straightforward algebraic manipulation, this reduces to $\Pi_m^-(a,\epsilon)<0$. We similarly obtain $\Pi_m^+(a,\epsilon)>0$ from (6.4). As such, let A^- and A^+ be real numbers such that $\Pi_m^-(a,\epsilon)<0$ for $a>A^-$ and $\Pi_m^+(a,\epsilon)>0$ for $a>A^+$ respectively (note that the existence of $A^\pm$ is implicitly implied by Theorem 1). Taking $A\geq \max(A^-,A^*,A^+)$ gives us the desired result, concluding the proof. □

In practise, we will aim to have both ϵ and A minimised in order to achieve the biggest range of values for which the computational formula holds.

7. Particular Examples

We will now show how one could practically use the methodology derived in the above section in direct proofs of some particular computational formulae. First, will prove Corollary 1, and then give a formula for $[\![1/\zeta(9,n)]\!]$.

7.1. Proof of Corollary 1

Note that we have $\Xi_6(n)=6n^6-18n^5+33n^4-36n^3+177n^2/10-27n/10+75/4$. Accordingly, we get $\tau_6=10$. But then, $\forall r\in\langle 9\rangle:\Xi_6(r)\notin\mathbb{Z}$, implying that $\mathcal{R}_6=\emptyset$. As such, by Theorem 1, we know that (1.2) holds for all sufficiently large n. Note that $K_6=8$ as $\xi_7(6)=0$ and $\xi_8(6)=-6459/400<0$. Now, take $\epsilon=19/5$ and observe that

$$\Xi_6^*\left(a,\frac{19}{5}\right)=6a^7(a-3)+33a^5\left(a-\frac{12}{11}\right)+\frac{177}{10}a^3\left(a-\frac{9}{59}\right)+\frac{75}{4}\left(a^2-\frac{20137}{3750}\right)>0 \tag{7.1}$$

for $a>3$ as every expression in brackets in (7.1) is positive for $a>3$. Additionally, note that

$$\Pi_6^-\left(a,\frac{19}{5}\right)=-\frac{266}{5}a^5\left(a^3-\frac{56155}{2128}a^2+\frac{886113}{5320}a+\frac{51033}{380}\right)-\frac{26256303}{2000}a^4 \tag{7.2}$$

$$-\frac{810723}{125}a^3-\frac{11499897}{2000}a^2-\frac{4047537}{1000}\left(a-\frac{16387}{8040}\right)<0 \tag{7.3}$$

for $a>3$. Indeed, the cubic in brackets in (7.2) is positive for $a>0$ (one may compute its discriminant to deduce that it only has one real root, and then use the intermediate value theorem [12, p. 26] to verify that it lies in $(-1,0)$), whilst the linear expression in brackets in (7.3) is positive for $a>3$. Further, we have

$$\Pi_6^+\left(a,\frac{19}{5}\right)=\frac{135639}{100}a^8+\frac{135639}{50}a^7+\frac{40257}{20}a^6+\frac{32823}{25}a^5+\frac{403209}{400}a^4+\frac{5625}{8}a^3+\frac{5625}{16}a^2>0$$

for $a>0$. Accordingly, by Theorem 4, we know that (1.2) holds for

$$n>\max\left(3,\sqrt{\frac{20137}{200\{\Xi_6(r)\}}}\right), \tag{7.4}$$

where r is the remainder of the division of n by 10. Therefore, in order to complete the proof of Corollary 1, we only need to verify that, in the 10 possible cases, (7.4) gives rise to the intervals stated. We leave this straightforward task as a simple exercise for the reader, thus concluding the proof. □

7.2. A Computational Formula Corresponding to $s=9$

Corollary 2. *Suppose that $n=7k+r$ with $k\in\mathbb{Z}$ and $r\in\langle 6\rangle$. The computational formula*

$$\left[\!\!\left[\frac{1}{\zeta(9,n)}\right]\!\!\right]=8n^8-32n^7+80n^6-128n^5+120n^4-64n^3+89n^2-74n-475+\left[\!\!\left[\frac{n^2+6n+1}{7}\right]\!\!\right] \tag{7.5}$$

holds for

(1) *$k\geq 9$ if $r=5$;*

(2) $k \geq 10$ *if* $r = 3$;
(3) $k \geq 11$ *if* $r \in \{0, 1\}$;
(4) $k \geq 12$ *if* $r = 6$;
(5) $k \geq 13$ *if* $r = 2$;
(6) $k \geq 25$ *if* $r = 4$.

In particular, it holds for all $n \geq 173$.

Proof. First, note that $\Xi_8(n) = 8n^8 - 32n^7 + 80n^6 - 128n^5 + 120n^4 - 64n^3 + 624n^2/7 - 512a/7 - 3324/7$ and $K_8 = 10$ as $\xi_9(8) = 0$ and $\xi_{10}(8) = 11288/21 > 0$. As such, by Theorem 1, formula (7.5) holds for all sufficiently large n. Now, take $\epsilon = 60$ and observe that

$$\Xi_8^* (a, 60) = 8a^9 (a - 4) + 80a^7 \left(a - \frac{8}{5}\right) + 120a^5 \left(a - \frac{8}{15}\right) + \frac{624}{7}a^2 \left(a^2 - \frac{32}{39}a - \frac{277}{52}\right) > 0 \tag{7.6}$$

for $a > 4$ as it is easy to check that every expression in brackets in (7.6) is positive for $a > 4$. Additionally, note that

$$\Pi_8^- (a, 60) = -a^2 (a + 1)^2 \left(\frac{541824}{7}a^6 + \frac{5653696}{49}a^4 + \frac{4410496}{49}a^2 - \frac{11048976}{49}\right) < 0 \tag{7.7}$$

for $a > 1$. Indeed, by Decartes' rule of signs [13, p. 319], the quartic in the brackets in (7.7) has exactly one positive root which lies in the interval $(0, 1)$ by the intermediate value theorem. Further, it is trivial to check that

$$\begin{aligned}\Pi_8^+ (a, 60) =& 1080a^8 \left(a^2 - \frac{423589}{5670}a + \frac{15045068}{19845}\right) + \frac{29713280}{49}a^7 + \frac{112997824}{49}a^6 + \frac{209405888}{147}a^5 \\ &+ \frac{41877488}{21}a^4 + \frac{31655200}{21}\left(a^3 - \frac{26542067}{13849150}a^2 - \frac{17557397}{6924575}a + \frac{33352187}{2967675}\right) > 0\end{aligned}$$

for $a > 63$. Accordingly, taking $\epsilon = 60$ and $A = 63$ suffices for the antecedent of Theorem 4 to hold. Now, note that we have $\tau_8 = 7$. It is also straightforward to verify that $\mathcal{R}_8 = \{3, 5\}$. As such, formula (7.5) holds for

$$n > \max\left(63, \sqrt{\frac{91564}{21} \div \begin{cases} 1, & r \in \{3, 5\} \\ 1 - \{\Xi_m(r)\}, & \text{otherwise} \end{cases}}\right) \tag{7.8}$$

where r is the remainder of the division of n by 7. Thus, in order to complete the proof of Corollary 2, we only need to verify that, in the 7 possible cases, (7.8) gives rise to the intervals stated. We leave this straightforward task as a simple exercise for the reader, thus concluding the proof. □

8. Proof of Theorem 2

Let us first introduce two formal power series, given by $G(x; q) := \sum_{n \geq 0} \xi_n(x) q^n$ and $H(x; q) := \sum_{n \geq 0} \binom{-x}{n} B_n q^n$ with q a formal variable. By the Cauchy Product [14, §2.1],

$$G(x; q)H(x; q) = \left(\sum_{n \geq 0} \xi_n(x) q^n\right) \left(\sum_{n \geq 0} \binom{-x}{n} B_n q^n\right) = \sum_{n \geq 0} \left[\sum_{m=0}^{n} \binom{-x}{m} B_m \xi_{n-m}(x)\right] q^n = \sum_{n \geq 0} \delta(n) q^n = 1.$$

Thus, $G(x; q) \equiv H(x; q)^{-1}$. Now, the desired proposition holds iff the Ordinary Generating Functions of the LHS and the RHS of (1.3) coincide. The OGF of the RHS is trivially $G(x; -q)$, whilst the OGF of the LHS is

$$\begin{aligned}\sum_{n \geq 0} \sum_{k=0}^{n} \binom{x-k}{n-k} \xi_k(x) q^n &= \sum_{k \geq 0} \sum_{n \geq k} \binom{x-k}{n-k} \xi_k(x) q^n = \sum_{k \geq 0} \xi_k(x) \left[\sum_{n \geq k} \binom{x-k}{n-k} q^n\right] = \sum_{k \geq 0} \xi_k(x) q^k \left[\sum_{n \geq 0} \binom{x-k}{n} q^n\right] \\ &= \sum_{k \geq 0} \xi_k(x) q^k \times (1+q)^{x-k} = (1+q)^x \sum_{k \geq 0} \xi_k(x) \left(\frac{q}{1+q}\right)^k = (1+q)^x G\left(x; \frac{q}{1+q}\right).\end{aligned}$$

Accordingly, it suffices to show that $G(x;-q) \equiv (1+q)^x G(x;q/(1+q))$. But then, $G(x;q) \equiv H(x;q)^{-1}$, implying that we only need to prove that $H(x;-q) = H(x;q/(1+q))/(1+q)^x$. First, note that $H(x;-q) \equiv \sum_{n\geq 0}(-1)^n\binom{-x}{n}B_n q^n$. Also, we have

$$\begin{aligned}(1+q)^{-x}H\left(x;\frac{q}{1+q}\right) &= (1+q)^{-x}\sum_{k\geq 0}\binom{-x}{k}B_k\left(\frac{q}{1+q}\right)^k = \sum_{k\geq 0}\binom{-x}{k}B_k q^k(1+q)^{-x-k}\\ &= \sum_{k\geq 0}\binom{-x}{k}B_k q^k\left[\sum_{n\geq 0}\binom{-x-k}{n}q^n\right] = \sum_{k\geq 0}\binom{-x}{k}B_k\left[\sum_{n\geq k}\binom{-x-k}{n-k}q^n\right]\\ &= \sum_{k\geq 0}\sum_{n\geq k}\binom{-x}{k}\binom{-x-k}{n-k}B_k q^n = \sum_{n\geq 0}\sum_{k=0}^{n}\binom{-x}{k}\binom{-x-k}{n-k}B_k q^n.\end{aligned}$$

Therefore, it would be sufficient to show that

$$\sum_{k=0}^{n}\binom{-x}{k}\binom{-x-k}{n-k}B_k = (-1)^n\binom{-x}{n}B_n \tag{8.1}$$

for all $n \geq 0$. Now, take note that, by definition, we have

$$\binom{-x}{k}\binom{-x-k}{n-k} = \frac{(-1)^k(x)_k}{k!}\times\frac{(-1)^{n-k}(x+k)_{n-k}}{(n-k)!} = \frac{(-1)^n\Gamma(x+k)\Gamma(x+n)}{k!(n-k)!\Gamma(x)\Gamma(x+k)} = \frac{(-1)^n\Gamma(x+n)}{k!(n-k)!\Gamma(x)} = \frac{(-1)^n(x)_n}{k!(n-k)!},$$

implying that we can rewrite (8.1) as

$$\sum_{k=0}^{n}\frac{B_k}{k!}\times\frac{1}{(n-k)!} = \frac{(-1)^n B_n}{n!}.$$

But then, by the Cauchy product, this is equivalent to

$$\frac{q}{\exp(q)-1}\times\exp(q) = \frac{q}{1-\exp(-q)},$$

which is identically true, completing the proof. □

Corollary 3. *There exists a sequence of functions $(\hat{\xi}_n)_{n\geq 0}$ such that*

$$\xi_n(x) = (-1)^n\sum_{k=0}^{[\![n/2]\!]}\binom{x-2k}{n-2k}\frac{\hat{\xi}_k(x)}{2^{n-2k}}$$

for all $n \geq 0$.

Proof. Fix $n \in \mathbb{Z}_0^+$. First, note that, by definition,

$$\binom{x-k}{n-k} = (-1)^{n-k}\frac{(k-x)_{n-k}}{(n-k)!} = \frac{(-1)^{n-k}\Gamma(n-x)}{(n-k)!\Gamma(k-x)}. \tag{8.2}$$

Accordingly, we can rewrite (1.3) as

$$\sum_{k=0}^{n}\frac{(-1)^k\xi_k(x)}{\Gamma(k-x)}\times\frac{1}{(n-k)!} = \frac{\xi_n(x)}{\Gamma(n-x)}. \tag{8.3}$$

Now, put $K(x;q) := \sum_{m\geq 0}\xi_m(x)q^m/\Gamma(m-x)$, where q is a formal variable. As such, by the Cauchy product, (8.3) is equivalent to $K(x;-q)\exp(q) = K(x;q)$, which implies that $\mathcal{K}(x;q) := K(x;q)\exp(-q/2)$ satisfies $\mathcal{K}(x;q) = \mathcal{K}(x;-q)$. Indeed, we readily acquire

$$\mathcal{K}(x;q) = K(x;q)\exp\left(-\frac{q}{2}\right) = K(x;-q)\exp(q)\exp\left(-\frac{q}{2}\right) = K(x;-q)\exp\left(\frac{q}{2}\right) = \mathcal{K}(x;-q).$$

Consequently, there exists a sequence $(\kappa_n(x))_{n\geq 0}$ such that $\mathcal{K}(x;q) \equiv \sum_{m\geq 0}\kappa_m(x)q^{2m}$. This further gives us that

$$K(x;q) = \mathcal{K}(x;q)\exp\left(\frac{q}{2}\right) = \left(\sum_{m\geq 0}\kappa_m(x)q^{2m}\right)\left(\sum_{m\geq 0}\frac{q^m}{m!2^m}\right) = \sum_{m\geq 0}\left[\sum_{k=0}^{m}\frac{1}{(m-k)!2^{m-k}}\times\begin{Bmatrix}\kappa_{k/2}(x), & 2\mid k\\ 0, & 2\nmid k\end{Bmatrix}\right]q^m. \tag{8.4}$$

Recovering the coefficient of q^n from the LHS and the RHS of (8.4) we acquire

$$\xi_n(x) = \Gamma(n-x)\sum_{\substack{k\in\langle n\rangle\\ 2\mid k}}\frac{\kappa_{k/2}(x)}{(n-k)!2^{n-k}} = \Gamma(n-x)\sum_{k=0}^{[\![n/2]\!]}\frac{\kappa_k(x)}{(n-2k)!2^{n-2k}}. \tag{8.5}$$

Now, define $\hat{\xi}_n(x) := \Gamma(2n-x)\kappa_n(x)$. Representation (8.5) can therefore be rewritten as

$$\xi_n(x) = \sum_{k=0}^{[\![n/2]\!]}\frac{\Gamma(n-x)}{(n-2k)!\Gamma(2k-x)}\times\frac{\hat{\xi}_k(x)}{2^{n-2k}} \overset{(8.2)}{=} \sum_{k=0}^{[\![n/2]\!]}\binom{x-2k}{n-2k}\frac{(-1)^{n-2k}\hat{\xi}_k(x)}{2^{n-2k}} = (-1)^n\sum_{k=0}^{[\![n/2]\!]}\binom{x-2k}{n-2k}\frac{\hat{\xi}_k(x)}{2^{n-2k}},$$

as required. □

Remark. By recovering the even coefficients of the LHS and the RHS of $\mathcal{K}(x;q) = K(x;q)\exp(-q/2)$, we acquire

$$\hat{\xi}_n(x) = \sum_{k=0}^{2n}\binom{x-k}{2n-k}\frac{\xi_k(x)}{2^{2n-k}}.$$

As $\xi_k(x)$ and $\binom{x-k}{2n-k}$ are always polynomials in x, it inductively follows that $\hat{\xi}_n(x)$ is always a polynomial in x. The first such polynomial is $\hat{\xi}_0(x) = 1$, followed by

$$\hat{\xi}_1(x) = \frac{x(x+1)}{24}, \hat{\xi}_2(x) = \frac{x(x+1)(3x^2-25x-42)}{5760}, \hat{\xi}_3(x) = \frac{x(x+1)(3x^4-14x^3+1193x^2+4186x+3720)}{967680}, \cdots$$

By recovering the odd coefficients of the LHS and the RHS of $\mathcal{K}(x;q) = K(x;q)\exp(-q/2)$, we acquire the following identity.

Corollary 4. *We have*

$$\sum_{k=0}^{2n+1}\binom{x-k}{2n+1-k}2^k\xi_k(x) \equiv 0$$

for all integers $n \geq 0$.

9. Proof of Theorem 3

First, note that

$$\Xi_m(1-x) = \sum_{k=0}^{m}m\xi_k(m)(1-x)^{m-k} = \sum_{k=0}^{m}m\xi_k(m)\left[\sum_{r=0}^{m-k}(-1)^r\binom{m-k}{r}x^r\right] = \sum_{k=0}^{m}\sum_{r=0}^{m-k}(-1)^r\binom{m-k}{r}m\xi_k(m)x^r.$$

In order for $(-1)^m\Xi_m(1-x) \equiv \Xi_m(x)$ to hold, all coefficients of powers of x on the LHS and the RHS must coincide. Accordingly, recovering the coefficient of x^{m-n}, $n\in\langle m\rangle$, we need

$$(-1)^m\sum_{\substack{k=0\\ m-k\geq m-n}}^{m}(-1)^{m-n}\binom{m-k}{m-n}m\xi_k(m) = m\xi_n(m) \iff \sum_{k=0}^{n}\binom{m-k}{n-k}\xi_k(m) = (-1)^n\xi_n(m),$$

which follows by setting $x = m$ in Theorem 2, as required. □

Note that Theorem 3 immediately implies that all odd Ξ-polynomials have a zero at $x = 1/2$.

10. Representations of the $\boldsymbol{\xi}$-Polynomials

Theorem 5. *For $n \geq 1$, we have the following semi-explicit formula*

$$\xi_n(x) = (-1)^n \det \begin{bmatrix} -xB_1 & 1 & 0 & \cdots & 0 \\ \binom{-x}{2}B_2 & -xB_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{-x}{n-1}B_{n-1} & \binom{-x}{n-2}B_{n-2} & \binom{-x}{n-3}B_{n-3} & \cdots & 1 \\ \binom{-x}{n}B_n & \binom{-x}{n-1}B_{n-1} & \binom{-x}{n-2}B_{n-2} & \cdots & -xB_1 \end{bmatrix}$$

Proof. Note that we can rewrite (3.1) as $\mathbf{M}(x)\boldsymbol{\xi}(x) = \boldsymbol{\delta}$, where

$$\mathbf{M}(x) := \begin{bmatrix} 1 & & & & \mathbf{0} \\ -xB_1 & 1 & & & \\ \binom{-x}{2}B_2 & -xB_1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{-x}{n}B_n & \binom{-x}{n-1}B_{n-1} & \binom{-x}{n-2}B_{n-2} & \cdots & 1 \end{bmatrix}, \qquad \boldsymbol{\xi}(x) := \begin{bmatrix} \xi_0(x) \\ \xi_1(x) \\ \xi_2(x) \\ \vdots \\ \xi_n(x) \end{bmatrix} \qquad \& \qquad \boldsymbol{\delta} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By Cramer's rule, we have $\xi_n(x) \det \mathbf{M}(x) = \det \mathbf{M}^\star(x)$, where $\mathbf{M}^\star(s)$ is the matrix obtained by replacing the n^{th} column of $\mathbf{M}(x)$ by the column vector $\boldsymbol{\delta}$. But then, since $\mathbf{M}(x)$ is a lower triangular matrix, we have $\det \mathbf{M}(x) = \prod_{i=1}^{n+1} 1 = 1$. As such, $\xi_n(x) = \det \mathbf{M}^\star(x)$. Using Laplace's expansion along the last column of $\mathbf{M}^\star(x)$, we arrive at the desired. ☐

Theorem 6. *We have the following formula*

$$\xi_n(x) \equiv \delta(n) + \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \prod_{m=1}^{k} \binom{-x}{j_m} B_{j_m}$$

for all $n \geq 0$.

Proof. Note that $\partial_q^n[G(x;q)]/n! \to \xi_n(x)$ as $q \to 0$. Thus, using the fact that $G(x;q) \equiv H(x;q)^{-1}$ and Lemmas 4 and 5,

$$\begin{aligned} \xi_n(x) &= \lim_{q\to 0} \left[\frac{1}{n!} \frac{\partial^n}{\partial q^n} \left[\frac{1}{\sum_{i=0}^{\infty} \binom{-x}{i} B_i q^i} \right] \right] = \lim_{q\to 0} \left[\frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{\partial_q^n \left[\left(\sum_{i=0}^{\infty} \binom{-x}{i} B_i q^i \right)^k \right]}{\left(\sum_{i=0}^{\infty} \binom{-x}{i} B_i q^i \right)^{k+1}} \right] \\ &= \frac{n+1}{n!} \frac{\partial^n 1}{\partial q^n} + \frac{1}{n!} \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \frac{\lim_{q\to 0} \left[\partial_q^n \left[\left(\sum_{i=0}^{\infty} \binom{-x}{i} B_i q^i \right)^k \right] \right]}{\left(\lim_{q\to 0} \left[\sum_{i=0}^{\infty} \binom{-x}{i} B_i q^i \right] \right)^{k+1}} \\ &= \delta(n) + \frac{1}{n!} \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \lim_{q\to 0} \left[n! \sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \prod_{m=1}^{k} \frac{\partial^{j_m}}{\partial q^{j_m}} \left[\sum_{i=0}^{\infty} \binom{-x}{i} \frac{B_i q^i}{j_m!} \right] \right] \\ &= \delta(n) + \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \prod_{m=1}^{k} \lim_{q\to 0} \left[\sum_{i=0}^{\infty} \binom{-x}{i} \frac{(-1)^{j_m} (-i)_{j_m} B_i q^{i-j_m}}{j_m!} \right] \\ &= \delta(n) + \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \prod_{m=1}^{k} \binom{-x}{j_m} B_{j_m}, \end{aligned}$$

as required. ☐

11. Coefficients of the ξ-Polynomials

Theorem 7. *Coefficients of the ξ-polynomials satisfy the following recurrence*

$$\sum_{r=0}^{k}\sum_{m=r}^{r+n-k}\frac{(-1)^{n-m}B_{n-m}}{(n-m)!}\begin{bmatrix}n-m\\k-r\end{bmatrix}\xi_{m,r}=\delta(n)\delta(k), \tag{11.1}$$

where $n\in\mathbb{Z}_0^+$ and $k\in\langle n\rangle$.

Proof. Fix $n\in\mathbb{Z}_0^+$. Note that we can rewrite the left-hand side of (3.1) as

$$\sum_{m=0}^{n}\binom{-x}{m}B_m\xi_{n-m}(x)=\sum_{m=0}^{n}\binom{-x}{n-m}B_{n-m}\xi_m(x)=\sum_{m=0}^{n}\frac{(-1)^{n-m}(x)_{n-m}}{(n-m)!}B_{n-m}\xi_m(x).$$

Now, by definition of the Stirling numbers of the 1st kind, $(x)_{n-m}\equiv\sum_t\begin{bmatrix}n-m\\t\end{bmatrix}x^t$, where we conveniently take the series over $\mathbb{Z}$, but $\begin{bmatrix}n-m\\t\end{bmatrix}=0$ whenever $t\notin\langle n-m\rangle$. Further, we have $\xi_m(x)\equiv\sum_{r=0}^{m}\xi_{m,r}x^r$ and $\delta(n)=\sum_{l=0}^{n}\delta(n)\delta(l)x^l$. Thus,

$$\sum_{m=0}^{n}\sum_{r=0}^{m}\sum_{t}\frac{(-1)^{n-m}B_{n-m}}{(n-m)!}\begin{bmatrix}n-m\\t\end{bmatrix}\xi_{m,r}x^{t+r}=\sum_{l=0}^{n}\delta(n)\delta(l)x^l.$$

Extracting the coefficient of x^k, $k\in\langle n\rangle$, from both sides we get

$$\sum_{m=0}^{n}\sum_{\substack{t+r=k\\r\in\langle m\rangle}}\frac{(-1)^{n-m}B_{n-m}}{(n-m)!}\begin{bmatrix}n-m\\t\end{bmatrix}\xi_{m,r}=\delta(n)\delta(k)\implies\sum_{m=0}^{n}\sum_{r=0}^{m}\frac{(-1)^{n-m}B_{n-m}}{(n-m)!}\begin{bmatrix}n-m\\k-r\end{bmatrix}\xi_{m,r}=\delta(n)\delta(k)$$

since t runs over the entirety of $\mathbb{Z}$. Interchanging the order of summation, we arrive at

$$\sum_{r=0}^{n}\sum_{m=r}^{n}\frac{(-1)^{n-m}B_{n-m}}{(n-m)!}\begin{bmatrix}n-m\\k-r\end{bmatrix}\xi_{m,r}=\delta(n)\delta(k).$$

But then, note that $\begin{bmatrix}n-m\\k-r\end{bmatrix}=0$ for $k-r\notin\langle n-m\rangle$, implying that we need $r\le k$ and $m\le r+n-k$. As $k\in\langle n\rangle$, we know that $[0,k]\subseteq[0,n]$, implying that we may restrict r to run over $\langle k\rangle$. This further gives us $[r,r+n-k]\subseteq[r,n]$, implying that we we may restrict m to run over $r+\langle n-k\rangle$. Thus, we finally obtain

$$\sum_{r=0}^{k}\sum_{m=r}^{r+n-k}\frac{(-1)^{n-m}B_{n-m}}{(n-m)!}\begin{bmatrix}n-m\\k-r\end{bmatrix}\xi_{m,r}=\delta(n)\delta(k),$$

as required. □

Corollary 5. *Let $n\ge 0$. We have $\xi_{n,0}=\delta(n)$.*

Proof. Fix $n\ge 0$ and let $k=0$. Then, recurrence (11.1) reduces to

$$\sum_{m=0}^{n}\frac{(-1)^{n-m}B_{n-m}}{(n-m)!}\begin{bmatrix}n-m\\0\end{bmatrix}\xi_{m,0}=\delta(n). \tag{11.2}$$

But then, by Lemma 6, $\begin{bmatrix}n-m\\0\end{bmatrix}=\delta(n-m)$, implying that the only entry of the partial sum in (11.2) for which $\begin{bmatrix}n-m\\0\end{bmatrix}\neq 0$ is the one corresponding to $m=n$. As such, we have $(-1)^0B_0\xi_{n,0}/0!=\delta(n)$, giving us $\xi_{n,0}=\delta(n)$. □

Corollary 6. *We have*

$$\xi_{n,1}=\frac{(-1)^{n-1}B_n}{n}$$

for all $n\ge 1$.

Proof. Fix $n \geq 1$ and let $k = 1$. Then, recurrence (11.1) reduces to

$$\sum_{m=0}^{n-1} \frac{(-1)^{n-m} B_{n-m}}{(n-m)!} \begin{bmatrix} n-m \\ 1 \end{bmatrix} \xi_{m,0} + \sum_{m=1}^{n} \frac{(-1)^{n-m} B_{n-m}}{(n-m)!} \begin{bmatrix} n-m \\ 0 \end{bmatrix} \xi_{m,1} = 0. \tag{11.3}$$

By Lemma 6 and Corollary 5, $\left[{n-m \atop 0}\right] = \delta(n-m)$, $\left[{n-m \atop 1}\right] = (n-m-1)!$ and $\xi_{m,0} = \delta(m)$. As such, (11.3) reduces to

$$\frac{(-1)^n B_n}{n!} \times (n-1)! + \frac{(-1)^0 B_0}{0!} \times \xi_{n,1} = 0 \implies \xi_{n,1} = \frac{(-1)^{n-1} B_n}{n},$$

as required. □

Corollary 7. *We have*

$$\xi_{n,2} = \frac{(-1)^{n-1} B_n H_{n-1}}{n} + (-1)^n \sum_{m=1}^{n-1} \frac{B_m B_{n-m}}{m(n-m)},$$

for all $n \geq 2$.

Proof. Fix $n \geq 2$ and let $k = 2$. Then, recurrence (11.1) reduces to

$$\sum_{m=0}^{n-2} \frac{(-1)^{n-m} B_{n-m}}{(n-m)!} \begin{bmatrix} n-m \\ 2 \end{bmatrix} \xi_{m,0} + \sum_{m=1}^{n-1} \frac{(-1)^{n-m} B_{n-m}}{(n-m)!} \begin{bmatrix} n-m \\ 1 \end{bmatrix} \xi_{m,1} + \sum_{m=2}^{n} \frac{(-1)^{n-m} B_{n-m}}{(n-m)!} \begin{bmatrix} n-m \\ 0 \end{bmatrix} \xi_{m,2} = 0. \tag{11.4}$$

By Lemma 6, $\left[{n-m \atop 0}\right] = \delta(n-m)$, $\left[{n-m \atop 1}\right] = (n-m-1)!$, $\left[{n-m \atop 2}\right] = (n-m-1)! H_{n-m-1}$. By Corollaries 5 and 6, we have $\xi_{m,0} = \delta(m)$ and $\xi_{m,1} = (-1)^{m-1} B_m / m$. Accordingly, (11.4) reduces to

$$\frac{(-1)^n B_n}{n!} \times (n-1)! H_{n-1} + \sum_{m=1}^{n-1} \frac{(-1)^{n-m} B_{n-m}}{(n-m)!} \times (n-m-1)! \times \frac{(-1)^{m-1} B_m}{m} + \frac{(-1)^0 B_0}{0!} \times \xi_{n,2} = 0,$$

which, after simplifying, finally gives us

$$\xi_{n,2} = \frac{(-1)^{n-1} B_n H_{n-1}}{n} + (-1)^n \sum_{m=1}^{n-1} \frac{B_m B_{n-m}}{n(n-m)},$$

as required. □

Before stating the final two corollaries of Theorem 7, let us first define an auxiliary formal power series, given by $F_h(q) := \sum_{m \geq 0} \xi_{m+h,m} q^m$, where h is a given positive integer. Moving on,

Corollary 8. *We have*

$$\xi_{n,n} = \frac{(-1)^n}{(n+1)!}$$

for all $n \geq 0$.

Proof. Fix $n \in \mathbb{Z}_0^+$ and let $k = n$. Then, recurrence (11.1) reduces to

$$\sum_{r=0}^{n} \frac{(-1)^{n-r} B_{n-r}}{(n-r)!} \begin{bmatrix} n-r \\ n-r \end{bmatrix} \xi_{r,r} = \delta(n)^2 \overset{\text{Lemma 6}}{\Longrightarrow} \sum_{r=0}^{n} \frac{(-1)^{n-r} B_{n-r}}{(n-r)!} \times \xi_{r,r} = \delta(n).$$

By the Cauchy product, this is equivalent to

$$\left(\sum_{m \geq 0} \frac{(-1)^m B_m}{m!} q^m \right) \left(\sum_{m \geq 0} \xi_{m,m} q^m \right) \equiv 1.$$

But then, by definition, $\sum_{m \geq 0} (-1)^m B_m q^m / m! \equiv q / (1 - \exp(-q))$. Also, recall that $\exp(q) \equiv \sum_{m \geq 0} q^m / m!$ [15, §4.2]. As such, we acquire

$$F_0(q) = \frac{1 - \exp(-q)}{q} = \frac{1 - \sum_{m \geq 0} (-q)^m / m!}{q} = \frac{1}{q} \sum_{m \geq 1} \frac{(-1)^{m-1}}{m!} q^m = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m!} q^{m-1} = \sum_{m \geq 0} \frac{(-1)^m}{(m+1)!} q^m. \tag{11.5}$$

Recovering the coefficient of q^n from the LHS and the RHS of (11.5) completes the proof. □

Corollary 9. *We have*

$$\xi_{n+1,n} = \frac{(-1)^n(n-1) - 2B_{n+1}}{2(n+1)!}$$

for all $n \geq 0$.

Proof. Fix $k \in \mathbb{Z}_0^+$ and let $n = k+1$. Then, recurrence (11.1) reduces to

$$\sum_{r=0}^{k} \left[\frac{(-1)^{k+1-r} B_{k+1-r}}{(k+1-r)!} \begin{bmatrix} k+1-r \\ k-r \end{bmatrix} \xi_{r,r} + \frac{(-1)^{k-r} B_{k-r}}{(k-r)!} \begin{bmatrix} k-r \\ k-r \end{bmatrix} \xi_{r+1,r} \right] = 0,$$

which, after some simplifications using Lemma 6 and Corollary 8, further gives us

$$\sum_{r=0}^{k} \frac{(-1)^{k-r} B_{k-r}}{(k-r)!} \times \xi_{r+1,r} = \frac{1}{2} \sum_{r=0}^{k} \frac{(-1)^r B_{r+1}}{(r-1)!} \times \frac{(-1)^{k-r}}{(k-r+1)!}.$$

Utilising the Cauchy product again, this is further equivalent to

$$\left(\sum_{m \geq 0} \frac{(-1)^m B_m}{m!} q^m \right) \left(\sum_{m \geq 0} \xi_{m+1,m} q^m \right) \equiv \frac{1}{2} \left(\sum_{m \geq 0} \frac{(-1)^m B_{m+1}}{(m-1)!} q^m \right) \left(\sum_{m \geq 0} \frac{(-1)^m}{(m+1)!} q^m \right). \tag{11.6}$$

Now, note that

$$\sum_{m \geq 0} \frac{(-1)^m B_{m+1}}{(m-1)!} q^m = -q \times \frac{\mathrm{d}^2}{\mathrm{d}q^2} \left[\frac{q}{1 - \exp(-q)} \right] = -\frac{q \exp(q)(q-2)}{(\exp(q)-1)^2} - \frac{2q^2 \exp(q)}{(\exp(q)-1)^3}.$$

Accordingly, (11.6) implies that

$$\begin{aligned} F_1(q) &= \frac{1-\exp(-q)}{q} \times \frac{1}{2} \times \left(-\frac{q \exp(q)(q-2)}{(\exp(q)-1)^2} - \frac{2q^2 \exp(q)}{(\exp(q)-1)^3} \right) \times \frac{1-\exp(-q)}{q} \qquad (11.7) \\ &= \frac{-1}{2q \exp(q)} \left(q - 2 + \frac{2q}{\exp(q)-1} \right) = \frac{1}{2q \exp(q)} \left(q + 2 - \frac{2q \exp(q)}{\exp(q)-1} \right) \\ &= \frac{1}{2} \exp(-q) + \frac{1}{q} \exp(-q) - \frac{1}{\exp(q)-1} = \sum_{m \geq 0} \frac{(-1)^m q^m}{2m!} + \sum_{m=-1}^{\infty} \frac{(-1)^{m+1} q^m}{(m+1)!} - \sum_{m=-1}^{\infty} \frac{B_{m+1} q^m}{(m+1)!} \\ &= \sum_{m \geq 0} \left[\frac{(-1)^m}{2m!} + \frac{(-1)^{m+1}}{(m+1)!} - \frac{B_{m+1}}{(m+1)!} \right] q^m = \sum_{m \geq 0} \frac{(-1)^m (m-1) - 2B_{m+1}}{2(m+1)!} q^m. \qquad (11.8) \end{aligned}$$

Recovering the coefficient of q^k from the LHS of (11.7) and the RHS of (11.8) yields the desired result. ☐

12. Conjectures

We now give some final observations regarding the concepts considered during the course of this paper. First, let us focus on the τ-quasiperiods. We refine Hwang's and Song's conjecture

Conjecture 1. *For all* $m \geq 2$, $(m-1) \mid \tau_m$ *and* $m \nmid \tau_m$.

Further, we observe a series of interesting p-adic patterns

Conjecture 2. *For an arbitrary prime* p *and natural number* n*, let* $\nu_p(n)$ *be the* p*-adic valuation of* n*. Then,*

(1) $\forall p$ prime $\forall m \geq 0 : \nu_p(\tau_{p(p-1)m+1}) = pm$
(2) $\forall p > 2$ prime $\forall m \geq 0 : \nu_p(\tau_{p(p-1)m+p+1}) = pm + 1$
(3) $\forall m \geq 1 : \nu_2(\tau_{4m+2}) = 2m - 1$
(4) $\forall m \geq 0 : \nu_5(\tau_{20m+11}) = \nu_5(\tau_{20m+16}) = 5m + 2$.
(5) *This 'linear exponent pattern' can be further generalised, i.e.,* $\log(\tau_m) = O(m)$ *as* $m \to \infty$.

The next two conjectures give insights into the practical structure of the computational formulae.

Conjecture 3. *For all* $m \geq 1$*, we have* $K_m = 2 \llbracket m/2 \rrbracket + 2$.

Conjecture 4. *For all* $m \geq 1$*, we have* $\xi_{K_m}(m) > 0$ *iff* $m \equiv 0 \pmod 4$ *or* $m \equiv 1 \pmod 4$.

We conclude by stating a rather curious observation, reminiscent of the Riemann Hypothesis.

Conjecture 5. *Let* m *be an arbitrary positive integer. All complex solutions to the equation* $\Xi_m(x) = 0$ *contained in the critical strip* $\{x \mid 0 \leq \mathrm{Re}(x) \leq 1\}$ *lie on the critical line* $\{x \mid \mathrm{Re}(x) = 1/2\}$.

Acknowledgements

The author would like to sincerely thank Professor Minhyong Kim for making the creation of this paper possible. The author expresses his gratitude towards the anonymous referee for constructive and helpful comments that greatly improved the presentation of this article. In particular, we thank the referee for the quasipolynomial perspective on the author's results.

Appendix A. A Refinement of Hwang's and Song's Formula

Corollary 10. *Suppose that $n = 48k + r$ with $k \in \mathbb{Z}$ and $r \in \langle 47 \rangle$. The computational formula*

$$\left[\!\left[\frac{1}{\zeta(6,n)}\right]\!\right] = 5n^5 - 13n^4 + 18n^3 - 16n^2 + 3n + \left[\!\left[\frac{1}{2}n^4 + \frac{3}{4}n^3 + \frac{3}{8}n^2 + \frac{41}{48}n + \frac{25}{96}\right]\!\right] \tag{A.1}$$

holds for

(1) $k \geq 0$ *if* $r \in \{15, 17, 18, 20, 22, 23, 25, 27, 28, 29, 30, 32, 33, 35, 36, 37, 38, 39, 40, 42, 43, 45, 46, 47\}$*;*
(2) $k \geq 1$ *if* $r \in \{0, 1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 14, 19, 21, 24, 31, 34, 44\}$*;*
(3) $k \geq 2$ *if* $r \in \{6, 41\}$*;*
(4) $k \geq 3$ *if* $r = 16$*;*
(5) $k \geq 4$ *if* $r = 3$*;*
(6) $k \geq 6$ *if* $r = 26$*;*
(7) $k \geq 18$ *if* $r = 13$.

In particular, it holds for all $n \geq 830$.

Proof. First, note that $\Xi_5(n) = 5n^5 - 25n^4/2 + 75n^3/4 - 125n^2/8 + 185n/48 + 25/96$ and $K_5 = 6$ as $\xi_6(5) = 325/192 > 0$ As such, by Theorem 1, formula (A.1) holds for all sufficiently large n. Now, take $\epsilon = 1/5$ and observe that

$$\Xi_5^*\left(a, \frac{1}{5}\right) = 5a^5\left(a - \frac{5}{2}\right) + \frac{75}{4}a^3\left(a - \frac{5}{6}\right) + \frac{185}{48}a^2 + \frac{25}{96}a > 0 \tag{A.2}$$

for $a > 5/2$ as all expressions in the brackets in (A.2) are positive in that interval. Additionally, we have

$$\begin{aligned}\Pi_5^-\left(a, \frac{1}{5}\right) &= -\frac{17875}{192}a^6 - \frac{17875}{192}a^5 + \frac{52975}{2304}a^4 + \frac{52975}{2304}a^3 - \frac{625}{9216}a^2 - \frac{625}{9216}a \\ &= -\frac{17875}{192}a\,(a+1)\left(a^4 - \frac{163}{660}a^2 + \frac{5}{6864}\right) < 0 \end{aligned} \tag{A.3}$$

for $a > 1/2$ by applying the quadratic formula to the biquadratic in the brackets of (A.3). Further, note that

$$\begin{aligned}\Pi_5^+\left(a, \frac{1}{5}\right) &= \frac{11}{5}a^6 - \frac{1593}{32}a^5 + \frac{1051015}{2304}a^4 + \frac{53405}{288}a^3 + \frac{1862383}{9216}a^2 + \frac{3199}{96}a + \frac{22364413}{307200} \\ &= \frac{11}{5}a^4\left(a^2 - \frac{7965}{352}a + \frac{5255075}{25344}\right) + \frac{53405}{288}a^3 + \frac{1862383}{9216}a^2 + \frac{3199}{96}a + \frac{22364413}{307200} > 0\end{aligned} \tag{A.4}$$

for $a > 0$ by applying the quadratic formula to the expression in the brackets of (A.4). Accordingly, taking $\epsilon = 1/5$ and $A = 5/2$ suffices for the antecedent of Theorem 4 to hold. Now, note that $\tau_5 = 48$. It is also straightforward to verify that $\mathcal{R}_5 = \emptyset$. As such, formula (A.1) holds for

$$n > \max\left(\frac{5}{2}, \frac{8317}{960(1 - \{\Xi_5(r)\})}\right), \tag{A.5}$$

where r is the remainder of the division of n by 48. Thus, in order to complete the proof of Theorem 10, we only need to verify that, in the 48 possible cases, (A.5) gives rise to the intervals stated. We leave this straightforward task as a simple exercise for the reader, thus concluding the proof. ☐

References

[1] L. Xin, Some identities related to riemann zeta-function, Journal of Inequalities and Applications 2016 (2016) 32. doi:DOI: 10.1186/s13660-016-0980-9.

[2] H. Xu, Some computational formulas related the riemann zeta-function tails, Journal Of Inequalities and Applications 2016 (2016) 132. doi:DOI: 10.1186/s13660-016-1068-2.

[3] L. Xin, L. Xiaoxue, A reciprocal sum related to the riemann ζ-function, J. Math. Inequal 11 (2017) 209–215. doi:DOI: 10.7153/jmi-11-20.

[4] W. Hwang, K. Song, A reciprocal sum related to the riemann zeta function at $s = 6$, arXiv preprint arXiv:1709.07994 (2017).

[5] N. Larson, B. Balof, et al., The Bernoulli numbers: a brief primer, Master's thesis, Whitman College, 2019.

[6] W. Magnus, F. Oberhettinger, R. P. Soni, Formulas and theorems for the special functions of mathematical physics, volume 52, Springer Science & Business Media, 2013.

[7] N. Bleistein, R. A. Handelsman, Asymptotic expansions of integrals, Courier Corporation, 1986.

[8] C. Jordan, Calculus of finite differences, volume 33, American Mathematical Soc., 1965.

[9] D. Mazkewitsch, The n-th derivative of a product, The American Mathematical Monthly 70 (1963) 739–742. doi:DOI: 10.2307/2312259.

[10] R. L. Graham, D. E. Knuth, O. Patashnik, Concrete Mathematics: A Foundation for Computer Science, 2nd ed., Addison-Wesley Longman Publishing Co., Inc., USA, 1994.

[11] L. Comtet, Advanced Combinatorics: The art of finite and infinite expansions, Springer Science & Business Media, 2012.

[12] H. L. Royden, P. Fitzpatrick, Real analysis, volume 32, Macmillan New York, 1988.

[13] Q. I. Rahman, G. Schmeisser, et al., Analytic theory of polynomials, 26, Oxford University Press, 2002.

[14] H. S. Wilf, Chapter 2 - series, in: Generatingfunctionology, Academic Press, 1990, pp. 27–63. doi:DOI: 10.1016/B978-0-12-751955-5.50005-8.

[15] M. Abramowitz, I. A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, volume 55, US Government printing office, 1948.