TALK 5: PERFECTOID RINGS

FRANKIE KNYSZEWSKI

Abstract. These notes keep track of the things I will (most likely not have time to) say about perfectoid rings at the ReMp-AHT Seminar in Princeton on October 6th, 2025. Use at your own risk!

0.1. **Preliminaries.** According to MAT 517, a perfectoid ring is a quotient of a perfect prism by its distinguished ideal. My aim for today is to discuss an alternative (one might even say intrinsic) characterisation of perfectoid rings. Time permitting, the tilting equivalence and Tate perfectoid rings should also make an appearance. The main references are [BMS18] and [BS22].

Fix a prime number $p > 0$ forever. Given a ring R, we let

$$R^\flat = \lim_{x \mapsto x^p} R \cong \mathrm{Hom}(\mathbf{Z}[X^{1/p^\infty}], R)$$

denote its *tilt*. A priori, this is only a multiplicative monoid, but if $p = 0$ in R then $R^\flat$ is the universal perfect $\mathbf{F}_p$-algebra mapping to R.

0.2. **Reminders on Witt vectors.** To fix notation, let us quickly review the relevant aspects of the theory of Witt vectors from Hari's talk before moving onto perfectoid rings. Fix a perfect $\mathbf{F}_p$-algebra S. Then there is a ring $W(S)$, called the *ring of (p-typical) Witt vectors over* S, such that:

- $W(S)$ is p-adically complete and p-torsion-free;
- there is an isomorphism $W(S)/pW(S) \cong S$.

We refer to Section II.6 of [Ser79] for an explicit construction of $W(S)$. Let

$$[-] : S \to W(S)$$

denote the *Teichmüller character*, i.e., the unique set-theoretic section of $W(S) \to S$ satisfying $[x^p] = [x]^p$ for all $x \in S$. Note that $[-]$ is a map of multiplicative monoids. (We deduce the existence and uniqueness of $[-]$ from Proposition II.4.8 in op. cit.) An easy inductive argument then shows that any $\xi \in W(S)$ can be written as

$$\xi = [\xi_0] + [\xi_1]p + [\xi_2]p^2 + \cdots$$

for unique 'coefficients' $\xi_0, \xi_1, \xi_2, ... \in S$. This is the *Teichmüller expansion* of ξ. On the other hand, the *n-th Witt coordinate* of ξ is defined as $\xi_n^{p^n}$; we write

$$\xi = (\xi_0, \xi_1^p, \xi_2^{p^2}, ...)$$

if we wish to emphasise that we are working with Witt coordinates, rather than the Teichmüller expansion. The benefit of representing elements of $W(S)$ in this way

Date: October 6, 2025.

is that we can write down the first few Witt coordinates of a sum/product of two Witt vectors explicitly. For example,

$$(x_0, x_1, ...) \cdot (y_0, y_1, ...) = (x_0y_0, x_0^p y_1 + x_1 y_0^p, ...)$$

for any $(x_0, x_1, ...), (y_0, y_1, ...) \in W(S)$.

The reason we care about $W(S)$ in the context of perfectoid rings is that it controls the infinitesimal deformation theory of S.

Proposition 0.2.1 (Universal property of Witt vectors)**.** *Let R be ring which is complete with respect to an ideal $I \subseteq R$ containing p. Then the reduction map*

$$\mathrm{Hom}(W(S), R) \to \mathrm{Hom}(S, R/I)$$

is well-defined and bijective.

Proof. When R/I is perfect, this is a special case of [Ser79, Proposition II.5.10]. More generally, we consider $R' = R \times_{R/I} (R/I)^\flat$. Then R' is I'-adically complete, where I' is the kernel of $R' \to (R/I)^\flat$, and $R'/I' \cong (R/I)^\flat$ is a perfect $\mathbf{F}_p$-algebra. We may thus identify

$$\begin{aligned}\mathrm{Hom}(S, R/I) &\cong \mathrm{Hom}(S, (R/I)^\flat) \\ &\cong \mathrm{Hom}(W(S), R') \\ &\cong \mathrm{Hom}(W(S), R)\end{aligned}$$

by the universal property of tilts of $\mathbf{F}_p$-algebras, the special case of Proposition 0.2.1 considered at the start, and the universal property of fibre products, respectively. This is enough. □

Remark 0.2.2. Let $\alpha : S \to R/I$ be a map of rings. Then, chasing through the identifications from Proposition 0.2.1, we see that the associated homomorphism $\widetilde{\alpha} : W(S) \to R$ is given by

$$\widetilde{\alpha}([\xi_0] + [\xi_1]p + [\xi_2]p^2 + \cdots) = \alpha^\sharp(\xi_0) + \alpha^\sharp(\xi_1)p + \alpha^\sharp(\xi_2)p^2 + \cdots$$

in terms of Teichmüller expansions. Here, $\alpha^\sharp : S \to R$ takes $x \in S$ to $\lim_{n\to\infty} r_n^{p^n}$, where $r_n \in R$ is an arbitrary lift of $\alpha(x^{1/p^n}) \in R/I$. In particular, $\widetilde{\alpha}$ depends only on the adic topology we put on R, and not on the particular choice of I.

Examples 0.2.3. The above proof goes through verbatim if we replace $W(S)$ with any p-adically complete p-torsion-free ring W such that $W/pW \cong S$. Therefore, these properties characterise $W = W(S)$ up to a unique isomorphism, allowing us to spot what $W(S)$ should be for our favourite perfect $\mathbf{F}_p$-algebras S.

(a) We have $W(\mathbf{F}_p) = \mathbf{Z}_p$;
(b) More ambitiously, consider $S = \mathbf{F}_p[X^{1/p^\infty}]$. Then $W(S) = \mathbf{Z}_p[X^{1/p^\infty}]^\wedge$, where the completion is p-adic.

0.3. **Integral perfectoid rings.** The universal property of Witt vectors can be used to equip the tilt of a p-complete ring with the structure of a perfect $\mathbf{F}_p$-algebra.

Lemma 0.3.1. *Let R be a ring which is complete with respect to an ideal $I \subseteq R$ containing p. Then the reduction map*

$$R^\flat \to (R/I)^\flat$$

is a monoid isomorphism.

Proof. The diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathbf{Z}_p[X^{1/p^\infty}]^\wedge, R) & \longrightarrow & \mathrm{Hom}(\mathbf{F}_p[X^{1/p^\infty}], R/I) \\ \big\downarrow\cong & & \big\downarrow\cong \\ R^\flat & \longrightarrow & (R/I)^\flat \end{array}$$

commutes. (It is universally agreed upon that one should never write down in explicit terms what all the arrows involved in a commutative diagram are, or indeed check whether the diagram itself is actually commutative.) The top horizontal map is bijective by Proposition 0.2.1. We conclude by noting that everything in sight is compatible with the attendant multiplicative structures. □

In particular, in the situation of Lemma 0.3.1, the operation of addition on $(R/I)^\flat$ can be transported to $R^\flat$. This is independent of the choice of I (see Remark 0.2.2). We write $(-)^\sharp : R^\flat \to R$ for the multiplicative map $x \mapsto x_0$.

Moving on, let us contemplate Fontaine's ring

$$\mathbf{A}_{\mathrm{inf}}(R) = W(R^\flat)$$

of *infinitesimal periods* of R. This is objectively the single best object in all of mathematics, if not in all of life (cf. [Col19]). The way that $\mathbf{A}_{\mathrm{inf}}(R)$ remembers R (and not just $R^\flat$) is through the projection

$$\theta_R : \mathbf{A}_{\mathrm{inf}}(R) \to R, \quad ([x_0] + [x_1]p + [x_2]p^2 + \cdots) \mapsto (x_0^\sharp + x_1^\sharp p + x_2^\sharp p^2 + \cdots).$$

Note that $\theta_R : \mathbf{A}_{\mathrm{inf}}(R) \to R$ is the unique lift of $R^\flat \to R/I$, by Proposition 0.2.1.

Example 0.3.2. If the $\mathbf{F}_p$-algebra R/pR is semi-perfect then θ_R is onto. Indeed, R is a p-complete ring since $pR \subseteq I$ [Sta24, Tag 090T], so we need only check surjectivity after reduction mod p. But $\theta_R \bmod p = (-)^\sharp \bmod p$ can be factored as

$$R^\flat \to (R/pR)^\flat \to R/pR,$$

with the first map being an isomorphism (by Lemma 0.3.1) and the second map being surjective (by assumption).

One way to think about perfectoid rings is that they are the correct generalisation of perfect $\mathbf{F}_p$-algebras outside characteristic p. For this, we would certainly wish to remain within the tranquil realm of p-complete rings. So that we do not stray too far from characteristic p, we should also like to have good control over the infinitesimal deformation theory of perfectoid rings: this is where Fontaine's $\mathbf{A}_{\mathrm{inf}}(R)$ comes in. We have already seen that R/pR being semi-perfect suffices for R to be a quotient of $\mathbf{A}_{\mathrm{inf}}(R)$, and it is in the structure of $\mathrm{Ker}(\theta_R)$ that we find the last obstacles to obtaining a good theory of perfectoid rings.

Definition 0.3.3. A ring R is *(integral) perfectoid* if there is an element $\varpi \in R$, referred to in the body of the text as a *pseudo-uniformiser*, such that:

(1) R is ϖ-adically complete;
(2) $R/\varpi^p R$ is a semi-perfect $\mathbf{F}_p$-algebra (in particular, $\varpi^p \mid p$);
(3) $\mathrm{Ker}(\theta_R)$ is principal.

(It seems there is no standard definition of a pseudo-uniformiser in the literature; we work with the one above for convenience.) We often permit ourselves to be lazy and drop the epithet 'integral'; this will become illegal in Section 0.5.

Remark 0.3.4. Let R be a ring admitting an element $\varpi \in R$ satisfying (1)–(2). Then, a unit multiple of ϖ lies in the image of $(-)^\sharp$, i.e., we can always ensure that pseudo-uniformisers come equipped with a compatible systems of p-power roots. The argument is somewhat similar to that of Example 0.3.2. The reduction map

$$R^\flat \to (R/\varpi^p R)^\flat$$

is an isomorphism (Lemma 0.3.1). Noting that the Frobenius on $R/\varpi^p R$ is onto, we can find an element $\varpi' \in R$ admitting all p-power roots such that

$$\varpi' \equiv \varpi \pmod{\varpi^p}.$$

But this means that $\varpi' = \varpi u$ for some $u \in 1 + \varpi^{p-1}R \subseteq R^\times$, whence the claim.

Before giving examples of perfectoid rings, let us talk in a bit more detail about the properties of the Frobenius in the context of Definition 0.3.3.

Lemma 0.3.5. *Let R be a ϖ-adically complete ring, where $\varpi \in R$ has $\varpi^p \mid p$. Then, the following are equivalent:*

(i) *every element of $R/\varpi pR$ is a p-th power;*
(ii) *every element of R/pR is a p-th power;*
(iii) *every element of $R/\varpi^p R$ is a p-th power;*

Proof. This is part of [BMS18, Lemma 3.9]. Since $\varpi pR \subseteq pR \subseteq \varpi^p R$, the only nontrivial thing to check is that if (iii) holds then so does (i). But then, by induction, any $x \in R$ can be written as

$$x = x_0^p + x_1^p \varpi^p + x_2^p \varpi^{2p} + \cdots \equiv (x_0 + x_1\varpi + x_2\varpi^2 + \cdots)^p \pmod{\varpi p}$$

for some $x_n \in R$. □

Lemma 0.3.6 ([BMS18, Lemma 3.9])**.** *Let R be a ϖ-adically complete ring, where $\varpi \in R$ has $\varpi^p \mid p$. Assume moreover that the Frobenius $R/\varpi R \to R/\varpi^p R$ is onto.*

(a) *If $\mathrm{Ker}(\theta_R) \subseteq \mathbf{A}_{\mathrm{inf}}(R)$ is principal then $R/\varpi R \to R/\varpi^p R$ is an isomorphism and any generator of $\mathrm{Ker}(\theta_R)$ is a non-zero-divisor.*
(b) *If $R/\varpi R \to R/\varpi^p R$ is an isomorphism and and ϖ is a non-zero-divisor then $\mathrm{Ker}(\theta_R) \subseteq \mathbf{A}_{\mathrm{inf}}(R)$ is a principal ideal.*

Proof. We require some preparation before getting on with the proof proper. First, replacing ϖ with a unit-multiple, we may assume that ϖ comes from some $\varpi^\flat \in R^\flat$ (Remark 0.3.4). Picking $x \in \mathbf{A}_{\mathrm{inf}}(R)$ such that $p = \varpi^p \theta_R(-x)$ using Example 0.3.2, we see that $\xi = p + [\varpi^\flat]^p x$ is an element of $\mathrm{Ker}(\theta_R)$ such that if $g = 1, p$ then

$$\begin{array}{ccc} \mathbf{A}_{\mathrm{inf}}(R)/\xi\mathbf{A}_{\mathrm{inf}}(R) & \xrightarrow{\theta_R} & R \\ \downarrow & & \downarrow \\ \mathbf{A}_{\mathrm{inf}}(R)/(\xi, [\varpi^\flat]^g) \cong R^\flat/\varpi^{\flat g}R^\flat & \longrightarrow & R/\varpi^g R \end{array} \tag{0.3.7}$$

commutes. We claim that if $\mathrm{Ker}(\theta_R)$ is principal then it is generated by ξ. Let ξ' be a generator, and write $\xi = a\xi'$ for $a \in \mathbf{A}_{\mathrm{inf}}(R)$. Comparing the Witt coordinates, we find that

$$1 + \varpi^{\flat p^2} x_1 = a_0^p \xi_1' + a_1 \xi_0'^p.$$

This implies that $a_0^p \xi_1'$ is a unit in $R^\flat = \lim_{x \mapsto x^p}(R/\varpi R)$ since both $\varpi^\flat$ and ξ_0' have trivial image in $R/\varpi R$. In particular, $a \in \mathbf{A}_{\mathrm{inf}}(R)^\times$ so that ξ generates $\mathrm{Ker}(\theta_R)$, as claimed. In fact, the same argument shows that if $\mathrm{Ker}(\theta_R)$ is principal then

$\xi' \in \mathrm{Ker}(\theta_R)$ is a generator if and only if ξ'_1 is a unit; we call elements of $\mathbf{A}_{\mathrm{inf}}(R)$ having this property *distinguished* (or *primitive*).

In the situation of (a), the top map of (0.3.7) is an isomorphism, hence the same must be true of the bottom map. Under these identifications, the Frobenius $R/\varpi R \to R/\varpi^p R$ is identified with the Frobenius $R^\flat/\varpi^\flat R^\flat \to R^\flat/\varpi^{\flat p} R^\flat$, which is an isomorphism since $R^\flat$ is perfect. It remains to check that ξ is a non-zero-divisor (we have already seen that any other generator of $\mathrm{Ker}(\theta_R)$ differs from ξ by a unit). Pick $b \in \mathbf{A}_{\mathrm{inf}}(R)$ with $(p + [\varpi^\flat]^p x)b = 0$. Then $(p^i + [\varpi^\flat]^{pi} x^i)b = 0$ for odd $i \geq 1$. In particular, if

$$b = [b_0] + [b_1]p + [b_2]p^2 + \cdots$$

is the Teichmüller expansion of b then $\varpi^{\flat pi} \mid b_n$ in $R^\flat$ for each $n \geq 0$. But $R^\flat$ is $\varpi^\flat$-adically separated so $b_n = 0$ for all $n \geq 0$, whence the claim.

Conversely, let us put ourselves in the situation of (b). Then each Frobenius

$$R/\varpi^{1/p^n} R \to R/\varpi^{1/p^{n-1}} R$$

is an isomorphism, implying that the sharp map $R^\flat \to R$ induces an isomorphism

$$R^\flat/\varpi^\flat R^\flat \xrightarrow{\sim} R/\varpi R.$$

Now, consider $x \in \mathrm{Ker}(\theta_R)$. Then then commutativity of (0.3.7) allows us to write $x = \xi y_0 + [\varpi^\flat]x_1$ for some $y_0, x_1 \in \mathbf{A}_{\mathrm{inf}}(R)$. Notice that

$$\varpi\theta_R(x_1) = \theta_R([\varpi^\flat]x_1) = \theta_R(\xi y_0 + [\varpi^\flat]x_1) = \theta_R(x) = 0.$$

Therefore $x_1 \in \mathrm{Ker}(\theta_R)$ since ϖ is not a zero-divisor. We may thus use induction and the fact that $\mathbf{A}_{\mathrm{inf}}(R)$ is $[\varpi^\flat]$-adically complete to write

$$x = \xi(y_0 + y_1[\varpi^\flat] + y_2[\varpi^\flat]^2 + \cdots)$$

as a multiple of ξ. (Note that $\mathbf{A}_{\mathrm{inf}}(R)$ is $[\varpi^\flat]$-complete because it is p-complete and $\mathbf{A}_{\mathrm{inf}}(R)/p\mathbf{A}_{\mathrm{inf}}(R) = R^\flat$ is complete with respect to $[\varpi^\flat] + p\mathbf{A}_{\mathrm{inf}}(R) = \varpi^\flat$, which is a non-zero-divisor.) It follows that $\mathrm{Ker}(\theta_R) = \xi\mathbf{A}_{\mathrm{inf}}(R)$, so that R is perfectoid. □

Remark 0.3.8. Keep the notation of the proof. The first two Witt coordinates of ξ are $\xi_0 = \varpi^\flat x_0$ and $\xi_1 = 1 + \varpi^{\flat p^2} x_1$. The latter is a unit in $R^\flat$, so we can write

$$\xi = p + [\varpi^\flat]^p x = pu + [x_0 \varpi^{\flat p}]$$

for $u \in \mathbf{A}_{\mathrm{inf}}(R)^\times$. Replacing ϖ by $(x_0^{1/p})^\sharp\varpi$, we get an element of R such that:

- R is ϖ-adically complete (by [Sta24, Tag 090T]);
- ϖ^p is a unit multiple of p (since $\theta_R(-u)$ is a unit);
- $(-)^p : R/\varpi R \to R/\varpi^p R$ is an isomorphism (by Lemma 0.3.6);
- ϖ admits a compatible system of p-power roots (since $\varpi \in (R^\flat)^\sharp$);

Any $\varpi \in R$ satisfying these four properties is called a *strict pseudo-uniformiser*.

Examples 0.3.9. Here are some standard examples of perfectoid rings:

(a) Let R be a ring of characteristic p. If R is perfect then it manifestly satisfies conditions (1)–(2) of Definition 0.3.3 with $\varpi = 0$; also, $\theta_R : \mathbf{A}_{\mathrm{inf}}(R) \to R$ is just reduction mod p, so R is perfectoid. Conversely, if R is perfectoid then $p \in \mathbf{A}_{\mathrm{inf}}(R)$ is a distinguished element lying in $\mathrm{Ker}(\theta_R)$, hence

$$R \cong \mathbf{A}_{\mathrm{inf}}(R)/p\mathbf{A}_{\mathrm{inf}}(R) \cong R^\flat$$

by Lemma 0.3.6. So R is perfect.

(b) If R is perfectoid then $\overline{R} = (R/pR)_{\mathrm{red}}$ is a reduced semi-perfect $\mathbf{F}_p$-algebra, hence it is perfect. We call $\overline{R}$ the *crystallisation* of R; it is the universal perfect $\mathbf{F}_p$-algebra receiving a map from R. Note that

$$J_R = \mathrm{Ker}(R \to \overline{R}) = \sqrt{pR}$$

is generated by $\{\varpi, \varpi^{1/p}, \varpi^{1/p^2}, ...\}$ for ϖ a strict pseudo-uniformiser.

(c) Let $K/\mathbf{Q}_p$ be a non-Archimedean extension with a non-discrete value group, and assume $\mathcal{O}_K/p\mathcal{O}_K$ is a semi-perfect $\mathbf{F}_p$-algebra. Then $\mathcal{O}_K$ is perfectoid (cf. Proposition 0.5.2). We remark that any $\varpi \in \mathcal{O}_K$ satisfying

$$0 < \mathrm{val}_p(\varpi) \leq 1/p$$

is a pseudo-uniformiser.

(d) If R is perfectoid then so is the p-adic completion $R[X^{1/p^\infty}]^\wedge$.

0.4. **Perfect prisms.** *Prisms* are commutative algebra gadgets defined in [BS22], generalising perfectoid rings by 'de-perfecting' them. These allow Bhatt-Scholze to define the prismatic site of a p-adic formal scheme (relative to some fixed prism); the resulting cohomology theory interpolates between étale/de Rham/crystalline/... cohomology theories classically considered in the literature. We will not concern ourselves with this today, instead focusing on justifying the following assertion: perfectoid rings are *perfect prisms* in disguise.

Definition 0.4.1. A *perfect prism* is a pair (A, I), where

(1) $A = W(S)$ is the ring of Witt vectors over a perfect $\mathbf{F}_p$-algebra S;

(2) $I \subseteq A$ is a principal ideal admitting a generator $\xi \in I$ whose Witt expansion $\xi = (\xi_0, \xi_1, ...)$ is so that S is ξ_0-adically complete and $\xi_1 \in S^\times$.

We call any ξ as in (2) a *distinguished generator* of I.

Lemma 0.4.2 ([BS22, Lemma 2.34])**.** *Let (A, I) be a perfect prism, and pick a distinguished generator $\xi \in I$. Then:*

(a) *ξ is a non-zero-divisor;*

(b) *$R = A/I$ has bounded p^∞-torsion; in fact, $R[p^\infty] = R[p]$.*

Proof. Part (a) follows via an argument analogous to that employed in the proof of Lemma 0.3.6. For (b), we need only see why $R[p^2] = R[p]$ holds; in view of the fact that A is p-torsion-free, it is enough to show that if $f, g \in A$ satisfy $p^2 f = g\xi$ then $p \mid g$. Comparing Witt coordinates reveals that $g_0\xi_0 = 0$ and $g_0^p\xi_1 + g_1\xi_0^p = 0$. It follows that

$$g_0^{2p}\xi_1 = g_0^{2p}\xi_1 + g_1(g_0\xi_0)^p = g_0^p(g_0^p\xi_1 + g_1\xi_0^p) = 0.$$

Since ξ_1 is a unit, this gives $g_0^{2p} = 0$. We conclude that $g_0 = 0$ using the fact that the Frobenius on $S = A/pA$ is injective. Therefore $p \mid g$ in A, as required. □

Theorem 0.4.3 ([BS22, Theorem 3.10])**.** *The functor*

$$\{\text{perfect prisms}\} \to \{\text{perfectoid rings}\}, \quad (A, I) \mapsto A/I \tag{0.4.4}$$

is a well-defined equivalence of categories.

Proof. The strategy is to show that

$$\{\text{perfectoid rings}\} \to \{\text{perfect prisms}\}, \quad R \mapsto (\mathbf{A}_{\mathrm{inf}}(R), \mathrm{Ker}(\theta_R)) \tag{0.4.5}$$

is a quasi-inverse; note that it is well-defined by the proof of Lemma 0.3.6.

Let (A, I) be a perfect prism, and pick a distinguished generator $\xi = (\xi_0, \xi_1, ...)$ of I. Let ϖ be the image of $[\xi_0^{1/p}]$ in $R = A/I$. We show that R is a perfectoid ring with strict pseudo-uniformiser ϖ. By construction, ϖ^p is a unit multiple of p admitting a compatible sequence of p-power roots (cf. Remark 0.3.8). Moreover, the Frobenius on

$$R/pR \cong A/(p, \xi) \cong S/\xi_0 S$$

is surjective since $S = A/pA$ is a perfect $\mathbf{F}_p$-algebra. As to the completeness of R, note that the ϖ-adic and the p-adic topologies on R coincide. We know a fortiori that R is derived p-complete, since A has this property and

$$R = H^0(A \xrightarrow{\xi} A)$$

(see [Sta24, Tag 091P]). But Lemma 0.4.2 implies that $\mathrm{Rlim}_{n \geq 1} R[p^n]$ vanishes in degree -1, and hence that R is (classically) p-complete, say by [Sta24, Tag 0BKG].

It remains to see why $\mathrm{Ker}(\theta_R)$ is principal (then, Lemmas 0.3.5 and 0.3.6 will imply that ϖ is a strict pseudo-uniformiser). In fact, we shall kill two birds with one stone and produce a functorial isomorphism

$$(A, I) \cong (\mathbf{A}_{\mathrm{inf}}(R), \mathrm{Ker}(\theta_R))$$

of perfect prisms. By the universal property of Witt vectors, giving an isomorphism $A \cong \mathbf{A}_{\mathrm{inf}}(R)$ is equivalent to giving an isomorphism $S \cong R^\flat$. Seeing as S is perfect, the map $x \mapsto x^{p^{-n}}$ identifies $S/\xi_0^{p^n} S$ with $S/\xi_0 S = R/pR$ in such a way that reduction mod $\xi_0^{p^{n-1}}$ corresponds to the Frobenius on R/pR. We thus obtain a canonical isomorphism $S^\wedge \xrightarrow{\sim} R^\flat$ (here $S^\wedge$ is the classical ξ_0-adic completion of S) intertwining the obvious S-algebra structures on $S^\wedge$ and $R^\flat$. But S is complete with respect to the ξ_0-adic topology, so $S \to S^\wedge$ is an isomorphism. In view of this, the 'reduction map' $S \to R^\flat$ lifts to a unique isomorphism $A \to \mathbf{A}_{\mathrm{inf}}(R)$. Finally,

$$A \xrightarrow{\sim} \mathbf{A}_{\mathrm{inf}}(R) \xrightarrow{\theta_R} R$$

is just reduction mod I by Proposition 0.2.1, since both of these lift $A/pA \to R/pR$. Therefore, we get an identification

$$I = \mathrm{Ker}(A \to R) \cong \mathrm{Ker}(\theta_R).$$

In particular, $\mathrm{Ker}(\theta_R)$ is generated by the image of ξ, so R is a perfectoid ring.

Putting everything together, we see that (0.4.4) is a well-defined functor with left quasi-inverse (0.4.5). But the latter is also a right quasi-inverse by Example 0.3.2. This concludes the proof. □

Remark 0.4.6. In the proof above, we have used derived completions to conclude that rings lying in the essential image of (0.4.4) are p-complete; a similar strategy can also be used to prove that if (A, I) is a perfect prism then A is $(pA+I)$-complete.

Indeed, the fact that A is p-complete immediately reduces the task at hand to showing that each A/p^n is I-complete (cf. [Sta24, Tag 05GG]). In fact, more is true: A/p^n is derived ξ-complete and $(A/p^n)[\xi^\infty] = (A/p^n)[\xi^n]$, for $\xi \in I$ a choice of a distinguished generator. It suffices to consider the case $n = 1$, as the general case then follows by contemplating the exact sequence

$$0 \to A/p^{n-1}A \xrightarrow{p} A/p^n A \to A/pA \to 0$$

(the buzzword here is *dévissage*). But this is easy: $S = A/pA$ is derived ξ-complete by assumption, while the fact that $S[\xi^\infty] = S[\xi]$ follows from S being perfect.

Example 0.4.7. We say a perfect prism (A, I) is *crystalline* if we have $I = pA$. Any perfect prism (A, I) has a *crystallisation* $(\overline{A}, \overline{I})$, defined as follows. Consider the ideal $J = (I + pA)/pA$ of $S = A/pA$. Then $\overline{A} = A/[J]A$ and $\overline{I} = I\overline{A} = p\overline{A}$. We remark that $(\overline{A}, \overline{I})$ is the universal crystalline perfect prism over (A, I). On the other hand, the *tilt* of (A, I) is simply defined as (A, pA).

Under the equivalence of categories from Theorem 0.4.3, crystalline perfect prisms correspond to perfect $\mathbf{F}_p$-algebras. Furthermore, the above equivalence intertwines 'tilting' for perfect prisms and perfectoid rings, and similarly for 'crystallisation'.

Corollary 0.4.8 (Tilting equivalence)**.** *Let R be a perfectoid ring with a strict pseudouniformiser $\varpi \in R$. Then*

$$\{\text{perfectoid } R\text{-algebras}\} \xrightarrow{\sim} \{\varpi^\flat\text{-complete perfectoid } R^\flat\text{-algebras}\}, \quad R' \mapsto R'^\flat$$

is an equivalence of categories, with quasi-inverse $S \mapsto \mathbf{A}_{\mathrm{inf}}(S) \otimes_{\mathbf{A}_{\mathrm{inf}}(R)} R$.

Proof. Let $(A, I) = (\mathbf{A}_{\mathrm{inf}}(R), \mathrm{Ker}(\theta_R))$ and let $\xi = [\varpi^\flat]^p + pu$ be a distinguished generator as in Remark 0.3.8. Theorem 0.4.3 tells us that perfectoid $R^\flat$-algebras are the same thing as a perfect prism (B, J) together with a map $\phi : (A, I) \to (B, J)$. If $\phi(\xi) = (\xi_0', \xi_1', ...)$ is the Witt expansion of $\phi(\xi)$, then we have $\xi_1' \in (B/pB)^\times$ by the functoriality of Teichmüller characters (see Remark 0.2.2). But $\phi(\xi) \in J$, so the proof of Lemma 0.3.6 implies that $\phi(\xi)$ generates J. Observe that B/pB is complete with respect to $\xi_0' = \phi([\varpi^\flat])$.

The content of the above discussion is that the functor

$$\{\text{perfect prisms over } (A, I)\} \to \{A\text{-algebras}\}, \quad (B, J) \mapsto B$$

is an equivalence onto the subcategory of those A-algebras B such that there is a $\varpi^\flat$-complete perfect $\mathbf{F}_p$-algebra S equipped with a ring isomorphism $B \cong W(S)$. We conclude by noting that the latter is equivalent to the category of $\varpi^\flat$-complete perfect $\mathbf{F}_p$-algebras receiving a map from $A/pA \cong R^\flat$, by Proposition 0.2.1. □

Remark 0.4.9. Consider the perfectoid rings $R_1 = \mathbf{Z}_p[p^{1/p^\infty}]^\wedge$ and $R_2 = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$. Then $R_1^\flat$ and $R_2^\flat$ are both isomorphic to $\mathbf{F}_p[[X^{1/p^\infty}]]$, with X corresponding to $(p, p^{1/p}, p^{1/p^2}, ...)$ and $(1, \zeta_p, \zeta_{p^2}, ...) - 1$, respectively. We thus cannot hope for an absolute analogue of the tilting equivalence for p-torsion-free perfectoid rings.

0.5. **Tate perfectoid rings.** Perfectoid rings in the sense of Definition 0.3.3 were first introduced by Bhatt-Morrow-Scholze [BMS18]. It has since become standard to refer to these as *integral* perfectoid rings, partially in order to juxtapose them with the pre-existing notion of perfectoid rings due to Fontaine [Fon13] (who himself expanded on the seminal work of Scholze [Sch12]). The purpose of the discussion to follow is to introduce this alternative sense in which a ring can be perfectoid, and to see how it relates to Definition 0.3.3.

Before we begin, we need some vocabulary from the world of topological rings.

Definition 0.5.1. Let R be a topological ring. A subset $S \subseteq R$ is said to be *bounded* if whenever $U \subseteq R$ is an open neighbourhood of 0, there is an open neighbourhood $V \subseteq R$ of 0 such that $VS \subseteq U$, where VS is the $\mathbf{Z}$-module generated by $\{vs\}_{v \in V, s \in S}$. Write $R^\circ \subseteq R$ for the subring of all those $f \in R$ such that $\{f^n\}_{n \geq 1}$ is bounded. Finally, we say R is *uniform* if R° is itself bounded.

Consider a *complete Tate ring* R, i.e., a complete topological ring R containing an open subring $R_0 \subseteq R$ and an element $\pi_0 \in R_0$ such that:

(1) the topology on R_0 is the π_0-adic topology;
(2) $R = R_0[1/\pi_0]$.

Such an R is said to be a *Tate perfectoid ring* if it is uniform and if there is a topologically nilpotent unit $\pi \in R$ such that:

(3) π^p divides p in R°;
(4) $R^\circ/\pi^p R^\circ$ is a semi-perfect $\mathbf{F}_p$-algebra.

This is the definition given in [Fon13]. The following result shows how to translate between the two notions of a 'perfectoid ring'.

Proposition 0.5.2 ([BMS18, Lemma 3.20])**.** *Let R be a complete Tate ring. Pick a* ring of integral elements $R^+ \subseteq R$*; this means that R^+ is an open, integrally-closed subring of R°.*

(a) *Assume R is a Tate perfectoid ring. Then R^+ is an integral perfectoid ring.*
(b) *Assume R^+ is an integral perfectoid ring and that R^+ is bounded in R. Then R is a Tate perfectoid ring.*

Proof. Suppose R is a Tate perfectoid. We first show that R° is perfectoid. Well, the fact that R is uniform implies that R° is π-complete [Hub93, Proposition 1]. So, by Lemma 0.3.6, it is enough to show that the Frobenius

$$R^\circ/\pi R^\circ \to R^\circ/\pi^p R^\circ$$

is a injective (it is surjective since $R^\circ/\pi^p R^\circ$ is semi-perfect). Let $x \in R^\circ$ be so that $x^p = \pi^p y$ for some $y \in R^\circ$. Then the p-th power of $z = x/\pi \in R$ is power-bounded, so z must be power-bounded itself. This implies that x dies in $R^\circ/\pi R^\circ$, as required.

We now attend to R^+. For this, we may assume that π^{2p} divides p in R°; indeed, R° is an integral perfectoid ring and so some unit multiple of π admits a p-th root, by Remark 0.3.4. Now, since R^+ is open and integrally-closed, it must contain πR°. In particular, $\pi^p \mid p$ in R^+ so, arguing as above, we need only check that

$$R^+/\pi R^+ \to R^+/\pi^p R^+$$

is surjective. Let $x \in R^+$. Then $x = y^p + p\pi z$ for $y, z \in R^\circ$ by (4) and Lemma 0.3.5. But $\pi z \in R^+$ so $y^p \in R^+$, whence in fact $y \in R^+$. So R^+ is integral perfectoid.

Let us move on to (b). Since π_0 is a topologically nilpotent unit, $\pi_0 R^\circ \subseteq R^+$. The fact that R^+ is bounded hence implies that R° is bounded. We are free to assume that π_0 admits a compatible system of p-power roots inside R^+. Indeed, we may pick a θ_{R^+}-preimage $a \in \mathbf{A}_{\inf}(R^+)$ of $\pi_0 \in R^+$ by Example 0.3.2; we thus obtain topologically nilpotent unit having all p-power roots on replacing π_0 by $a_0^\sharp$. Moving on, there are some $n \geq 0$ and $r \in R^+$ such that

$$p^{p^n} = \pi_0 r.$$

(Note that $\pi_0 R^+ \subseteq R$ is open since $\pi_0 \in R^\times$.) Then r admits a p^n-th root in R, which necessarily lies in R^+ by virtue of R^+ being integrally-closed. Now, replace π_0 with $\pi_0^{1/p^{n+1}}$, thus guaranteeing that π_0^p divides p in $R^+ \subseteq R^\circ$.

The last thing to check is that $R^\circ/\pi_0^p R^\circ$ is semi-perfect. Pick $x \in R^\circ$. Then $\pi_0 x \in R^+$ so $\pi_0 x = y^p + p\pi_0 z$ for $y, z \in R^+$ by Lemma 0.3.5. Note that $\widetilde{y} = y/\pi_0^{1/p}$, which a priori is merely an element of R, must in fact lie in R^+ since

$$\widetilde{y}^p = x - pz \in R^+.$$

This implies that R°/pR° is semi-perfect. But $R^\circ/\pi_0^p R^\circ$ is a quotient of R°/pR°, hence semi-perfect. We are done. $\square$

Example 0.5.3 ([BMS18, Lemma 3.21])**.** Consider an integral perfectoid ring R_0 such that the pseudo-uniformiser $\varpi \in R_0$ from Definition 0.3.3 can be chosen to be a non-zero-divisor. Then $R = R_0[1/\varpi]$, when equipped with the ϖ-adic topology on R_0, is a complete Tate ring. In fact, it is a Tate perfectoid. For this, we would like to apply Proposition 0.5.2. It is easy to see that R_0 is an open and bounded subring of R contained in R°; the problem is that it may not be integrally-closed. However, examining the proof of Proposition 0.5.2, we see that all we need is that R_0 is closed under the taking of p-th roots in R.

We follow the argument of [Sch12, Lemma 5.7]. Let $x \in R$ be so that $x^p \in R_0$. There is a least integer $m \geq 0$ such that $y = \varpi^m x$ lies in R_0. If $m > 0$ then the p-th power of y lies in $\varpi^p R_0$. But Lemma 0.3.6 implies that the Frobenius

$$R_0/\varpi R_0 \to R_0/\varpi^p R_0$$

is an isomorphism, so in fact $y = \varpi \widetilde{y}$ for some $\widetilde{y} \in R_0$. Then $\varpi^{m-1} x = \widetilde{y} \in R_0$ since ϖ is a non-zero-divisor, contradicting our choice of m. So $m = 0$ and $x \in R_0$.

0.6. **Acknowledgment.** I would like to thank Inés Borchers Arias for many helpful discussions about perfectoid rings.

References

[BMS18] B. Bhatt, M. Morrow, and P. Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 219–397.

[BS22] B. Bhatt and P. Scholze, *Prisms and prismatic cohomology*, Ann. of Math. (2) **196** (2022), no. 3, 1135–1275.

[Col19] P. Colmez, *Le programme de Fontaine*, Enseign. Math. **65** (2019), no. 3-4, 487–531.

[Fon13] J.-M. Fontaine, *Perfectoïdes, presque pureté et monodromie-poids (d'après Peter Scholze)*, Astérisque (2013), no. 352, Exp. No. 1057, x, 509–534, Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058.

[Hub93] R. Huber, *Continuous valuations*, Math. Z. **212** (1993), no. 3, 455–477.

[Sch12] P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313.

[Ser79] J.-P. Serre, *Local fields*, Grad. Texts in Math., vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated by M. J. Greenberg.

[Sta24] The Stacks Project Authors, *Stacks Project*, `https://stacks.math.columbia.edu`, 2024.

Department of Mathematics, Princeton University, Fine Hall, Washington Rd, Princeton, New Jersey 08540

Email address: `fk4805@math.princeton.edu`