A GENTLE INTRODUCTION TO THE TRACE FORMULA

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Abstract. This is a set of notes accompanying my talk at Princeton's GSS on the 19th of March, 2026. Use at your own risk.

Contents

1. Introduction

1.1. **Motivation.** Let M be a square matrix defined over a field. Suppose it has diagonal entries $x_1, ..., x_n$ and eigenvalues $\lambda_1, ..., \lambda_n$ (living in some algebraic closure of the base field). Then, evaluating the trace of M in two different ways, we arrive at the following formula:

$$x_1 + \cdots + x_n = \mathrm{tr}(M) = \lambda_1 + \cdots + \lambda_n.$$

This relates intrinsic information pertaining to the linear map represented by M (as encoded by the eigenvalues) to a superficial set of parameters (describing how the linear map acts on certain vectors).

The *trace formula* takes the idea of 'evaluating the trace in two different ways' to its extreme: it seeks to understand (a particular class of) representations of a unimodular, locally compact group in terms of explicit, but often complicated, integral expressions. The aim of this talk is to informally explain the theory of the trace formula in an easy special case. Time permitting, we should also see a couple of well-known theorems that follow from this formalism.

1.2. **Notation.** As per the above section, there ought to be a group floating around. So let G be a locally compact (in particular Hausdorff) topological group. Then, up to a positive scalar, there is a unique Radon measure on G which is invariant with respect to the right multiplication action of G on itself [Fol95, Theorem 2.10]. We call it a *Haar measure*. Let us fix a particular choice of the latter from now on. An assumption we need to make in order for the theory to run smoothly is that G is *unimodular*, i.e., that the Haar measure is moreover invariant with respect to the left multiplication action of G on itself. Note that, by uniqueness, this is a property of G independent of our choice of the particular Haar measure.

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Example 1.2.1. Any locally compact abelian group is unimodular. Think $(\mathbf{R}, +)$ with the Lebesgue measure. Similarly $(\mathbf{Q}_p, +)$ is unimodular for any prime p.

Example 1.2.2. Any discrete group is unimodular: consider the counting measure.

Example 1.2.3. Any compact group G is unimodular (see [Fol95, Section 2.4]).

Example 1.2.4. Algebraic groups over $\mathbf{R}$ provide a plethora of (counter)examples. To name a few: $\mathrm{GL}_n(\mathbf{R})$, $\mathrm{SL}_n(\mathbf{R})$, $\mathrm{SO}_n(\mathbf{R})$, and $\mathrm{Sp}_{2n}(\mathbf{R})$ are all unimodular; on the other hand, the Borel subgroup of $\mathrm{GL}_2(\mathbf{R})$, consisting of upper-triangular matrices, is not unimodular. Those interested should consult the first few chapters of [GH24].

If $H \subseteq G$ is a closed unimodular subgroup then G acts (on the right) on the quotient space $H\backslash G$ by translation, and our choices of Haar measures on G and H determine a unique G-invariant Radon measure on $H\backslash G$ such that

$$\int_G f(g)dg = \int_{H\backslash G} \left(\int_H f(hg)dh \right) d(Hg)$$

holds for all integrable $f : G \to \mathbf{C}$ [Fol95, Theorem 2.49].

2. Trace formula

2.1. **Setup.** Let $\Gamma \subseteq G$ be a discrete subgroup. Then Γ is in particular unimodular (we normalise the Haar measure so that all singletons have volume 1) and the corresponding invariant measure on $\Gamma\backslash G$ allows us to consider the Hilbert space $L^2(\Gamma\backslash G) = L^2(\Gamma\backslash G, \mathbf{C})$. This admits a unitary representation R of G:

$$(R(g)\phi)(x) = \phi(xg)$$

where $\phi : \Gamma\backslash G \to \mathbf{C}$ is a square-integrable function (or rather, an equivalence class of such functions) and $g, x \in G$. Note that we have allowed ourselves to be sloppy by thinking of ϕ as a G-valued function taking constant values on left cosets of Γ. This sort of thing will persist throughout.

The overall goal of the trace formula is to understand R; to decompose it into irreducible constituents (whatever this means) and to describe this decomposition in terms of 'orbit integrals'. As it stands, the problem is much too general to be solved completely: one ought to think of the trace formula as more of a guidepost!

Remark 2.1.1. Everything works best when $\Gamma\backslash G$ is compact. This assumption allows one to forget about most (but not all) convergence issues that arise within the purview of the theory in the large. It also means one can avoid doing too much legwork to make sense of what 'the trace' of an operator $L^2(\Gamma\backslash G) \to L^2(\Gamma\backslash G)$ should be (see Definition 2.1.2), because $L^2(\Gamma\backslash G)$ is separable when $\Gamma\backslash G$ is compact. All in all, the analysis one has to contend with becomes significantly less taxing.

On the side of representation theory, the main issue that arises outside of the compact setting is the question of what we might mean by an irreducible constituent of R. When $\Gamma\backslash G$ is compact, the Peter-Weyl theorem guarantees that $L^2(\Gamma\backslash G)$ is (topologically) semi-simple, so an irreducible consistent would just be a unitary irreducible subquotient. We sadly cannot hope for this to work in full generality, and the theory of the direct integral is required to take care of the mess at hand. As luck would have it, however, we have no intention of dwelling on these issues any longer, and will forever be content in assuming that $\Gamma\backslash G$ is compact. (We do offer an exercise, though: use the Riesz representation theorem to prove that the regular representation of $\mathbf{R}$ on $L^2(\mathbf{R})$ does not have any irreducible subquotients.)

As per the above remark, $\Gamma\backslash G$ is assumed to be compact from now on.

We study R using test functions $f \in C_c(G)$. Here, $C_c(G)$ is the space of all compactly-supported continuous complex-valued maps on G. This is a **C**-algebra under convolution: given $f_1, f_2 \in C_c(G)$, we define

$$f_1 * f_2 : G \to \mathbf{C}, \quad x \mapsto \int_G f_1(xy^{-1}) f_2(y) dy.$$

The point is that 'integration against R' gives rise to a **C**-algebra representation of $C_c(G)$ on $L^2(\Gamma\backslash G)$. Explicitly, any $f \in C_c(G)$ determines a bounded operator

$$R(f) : L^2(\Gamma\backslash G) \to L^2(\Gamma\backslash G), \quad \phi \mapsto \left(x \mapsto \int_G f(y)\phi(xy) dy \right)$$

and we have $R(f_1 * f_2) = R(f_1)R(f_2)$ by Fubini's theorem. A more precise goal of the trace formula would be to determine a class of $f \in C_c(G)$ for which $R(f)$ admits a trace, and to evaluate $\mathrm{tr}(R(f))$ in two fundamentally different ways.

Definition 2.1.2. Let V be a separable Hilbert space and let $T : V \to V$ be a bounded operator. We say T is of *trace class* if whenever B is an orthonormal basis for V then the sum

$$\sum_{b \in B} \langle Tb, b \rangle_V$$

converges absolutely. If this is the case, then the above quantity is independent of the particular choice of B; we denote it by $\mathrm{tr}(T)$, and refer to it as the *trace* of T.

2.2. **Geometric side.** We have

$$(R(f)\phi)(x) = \int_G f(y)\phi(xy) dy = \int_G f(x^{-1}y)\phi(y) dy = \int_{\Gamma\backslash G} K_f(x, y)\phi(y) dy$$

where

$$K_f : (\Gamma\backslash G)^2 \to \mathbf{C}, \quad (x, y) \mapsto \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y).$$

Therefore, if we think of $R(f)$ as an infinite matrix, then $K_f(x, y)$ should be thought of as the coefficient of $R(f)$ at the spot corresponding to $(x, y) \in (\Gamma\backslash G)^2$. It would thus seem reasonble to conclude that

$$\mathrm{tr}(R(f)) = \int_{\Gamma\backslash G} K_f(x, x) dx,$$

at least for nice enough $f \in C_c(G)$. This is indeed the case if f is a convolution of two elements of $C_c(G)$, or a finite **C**-linear combination of functions of this type. Let us denote the space of all such $f \in C_c(G)$ by $C_c(G)^{*2}$.

Remark 2.2.1. The point is that if $g \in C_c(G)$ then the kernel $K_g : (\Gamma\backslash G)^2 \to \mathbf{C}$ is square-integrable, so the theory of integral operators on separable Hilbert spaces tells us that $R(g)$ is Hilbert-Schmidt. Crucially, square-integrability fails without the assumptions we imposed. The point is that for each $x, y \in G$, the sum defining $K_f(x, y)$ can be taken over the set $\Gamma \cap x\,\mathrm{supp}(f)y^{-1}$, which is finite as it is both discrete and compact (this is becuase Γ is discrete and f is compactly-supported). Since everything in sight is continuous and $\Gamma\backslash G$ is compact, $K_f \in L^2((\Gamma\backslash G)^2)$.

In particular, if $f = g_1 * g_2$ for $g_i \in C_c(G)$ then $R(f) = R(g_1)R(g_2)$ is a product of two Hilbert-Schmidt operators and hence it must be of trace class by analysis. The explicit computation of the trace is then just a matter of unwinding definitions.

Moving on, let $\{\Gamma\}$ be the set of (representatives for the) conjugacy classes in Γ; if $S \subseteq G$ is a subset and $\gamma \in \Gamma$, put S_γ for the centraliser of γ in S. We compute

$$\begin{aligned}
\operatorname{tr}(R(f)) &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)dx \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\{\Gamma\}} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma\backslash G} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)dx \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{G_\gamma\backslash G} \int_{\Gamma_\gamma\backslash G_\gamma} f(z^{-1}y^{-1}\gamma yz)dydz \\
&= \sum_{\gamma\in\{\Gamma\}} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) O_\gamma(f)
\end{aligned}$$

where

$$O_\gamma(f) = \int_{G_\gamma\backslash G} f(z^{-1}\gamma z)dz$$

is referred to as an orbit integral. This is the geometric side of the trace formula.

Remark 2.2.2. Note that we used the compatibility of invariant measures with respect to quotients (as stated at the end of Section 1.2) when combining integrals. A subtle point here is that the centralisers G_γ are unimodular; this follows from the fact that G is unimodular [DE14, Lemma 9.3.3].

2.3. **Spectral side.** We now go back to representation theory. Let $\widehat{G}$ denote the set of (representatives for the) isomorphism classes of unitary representations of G. The key result here is the Peter-Weyl theorem, which relies in no small part on the assumption that $\Gamma\backslash G$ is compact.

Theorem 2.3.1 (Peter-Weyl)**.** *There is an isomorphism of G-representations*

$$R \cong \widehat{\bigoplus}_{\pi\in\widehat{G}} \pi^{\oplus m_\pi}$$

for some non-negative integers m_π.

Proof. We only sketch the argument. The key lemma is that in any non-zero closed invariant subspace of $L^2(\Gamma\backslash G)$, we can find a closed irreducible invariant subspace (argue as in [Bum97, Lemma 2.3.2]). The desired decomposition, with possibly infinite multiplicities, thus follows by appealing to Zorn's lemma. Indeed, there is a maximal topologically semi-simple subrepresentation R' of R. If $R' \neq R$ then $(R')^\perp$ admits an irreducible subrepresentation π. Then $R' \oplus \pi$ is a topologically semi-simple subrepresentation of R which is strictly larger than R', contradiction. So R decomposes as a completed direct sum of its irreducible subrepresentations.

The matter of the multiplicities can also be approached with our key lemma: in the course of its proof, one constructs for any given closed invariant subspace $V \subseteq L^2(\Gamma\backslash G)$ a function $f \in C_c(G)$ such that $R(f)|_V \neq 0$. Hence the claim is a consequence of the fact that $R(f)$ has finite Hilbert-Schmidt norm for all $f \in C_c(G)$ (for which see [Sch60, Theorem II.4]). □

Theorem 2.3.1 readily supplies the spectral side of the trace formula: we have

$$\operatorname{tr}(R(f)) = \sum_{\pi \in \widehat{G}} m_\pi \operatorname{tr}(\pi(f))$$

for any $f \in C_c(G)^{*2}$. Here, $\pi(f)$ is the operator on V_π (this is our notation for the Hilbert space acted on by π) sending $\phi \in V_\pi$ to

$$\pi(f)(\phi) = \int_G f(g)\pi(g)(\phi)dg.$$

The above expression involves the usage of the Gelfand-Pettis integral, which is a vector-valued integral; the relevant theory is nicely laid out in [Gar18, Section 14].

Remark 2.3.2. Implicit in the spectral side of the trace formula is the following surprising assertion: if $\pi \in \widehat{G}$ occurs in R then $\pi(f)$ is of trace class for $f \in C_c(G)^{*2}$. A priori, there is no good reason to expect this to be true for an arbitrary $\pi \in \widehat{G}$.

Our labourings have finally paid off: we have seen that if $f \in C_c(G)$ is a finite linear combination of convolutions then $R(f)$ is of trace class, with trace equal to

$$\sum_{\gamma \in \{\Gamma\}} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) O_\gamma(f) = \sum_{\pi \in \widehat{G}} m_\pi \operatorname{tr}(\pi(f)).$$

This is the trace formula for G relative to Γ, at least in the case $\Gamma \backslash G$ compact.

3. Examples

It is unlikely that there will be time to meanigfully cover any examples during the talk. Since knowing them does make for a good human being, we include the following discussion for completeness.

3.1. **Poisson summation.** Let $G = \mathbf{R}$ (with the Lebesgue measure) and $\Gamma = \mathbf{Z}$ (with the counting measure). Then $\Gamma \backslash G \cong S^1$ is the circle group, which is compact. We may thus apply the trace formula and see what falls out of it.

Recall that, since $\mathbf{R}$ is abelian, we have $\widehat{\mathbf{R}} = \operatorname{Hom}_{\mathrm{cts}}(\mathbf{R}, S^1)$. In turn, we can parametrise all continuous homomorphisms $\mathbf{R} \to S^1$ by $\mathbf{R}$: the map

$$\mathbf{R} \to \operatorname{Hom}_{\mathrm{cts}}(\mathbf{R}, S^1), \quad y \mapsto (e_y : x \mapsto \exp(2\pi i x y))$$

is a topological and algebraic isomorphism [Tat67, Lemma 2.2.1]. Moreover, it is fairly easy to see that e_y embeds into $L^2(S^1)$ if and only if $y \in \mathbf{Z}$, whence $m_{e_y} = 1$. Indeed, if $y \in \mathbf{Z}$ then e_y factors through S^1 and we automatically have $e_y \in L^2(S^1)$ because S^1 is compact. Conversely, given $y \in \mathbf{R}$ such that there is some non-zero $\phi \in L^2(S^1)$ in the e_y-isotypic subspace, we immediately see that

$$\phi(x) = (R(x)\phi)(0) = e_y(x)\phi(0)$$

since $e_y(x)$ is a scalar. Hence e_y factors through S^1, which can happen only if $y \in \mathbf{Z}$; finally, the fact that ϕ lies in the $\mathbf{C}$-span of e_y implies that $m_{e_y} = 1$.

In view of this, the spectral side of the trace formula reads

$$\operatorname{tr}(R(f)) = \sum_{y \in \mathbf{Z}} \operatorname{tr}(e_y(f)) = \sum_{y \in \mathbf{Z}} \int_G f(x) \exp(2\pi i x y) dx = \sum_{y \in \mathbf{Z}} \widehat{f}(y)$$

for $f \in C_c(\mathbf{R})^{*2}$. Here of course $\widehat{f} : \mathbf{R} \to \mathbf{C}$ denotes the Fourier transform of f.

On the other hand, the geometric side is simply

$$\mathrm{tr}(R(f)) = \sum_{x\in\mathbf{Z}} \mathrm{vol}(\mathbf{Z}\backslash\mathbf{R}) f(x) = \sum_{x\in\mathbf{Z}} f(x).$$

Note that the circle has volume 1 with the way we set things up because the interval $[0,1]$ is a fundamental domain for the action of $\mathbf{Z}$ on $\mathbf{R}$, and $\mathrm{vol}([0,1]) = 1$.

The trace formula thus specialises to the Poisson summation formula: we have

$$\sum_{x\in\mathbf{Z}} f(x) = \sum_{y\in\mathbf{Z}} \widehat{f}(y) \tag{3.1.1}$$

for all $f \in C_c(G)^{*2}$. This is one of the many reasons why we should care about Poisson summation: it falls out from the representation theory of S^1.

Remark 3.1.2. Let $C_c^\infty(\mathbf{R})$ denote the space of all smooth compactly-supported complex-valued functions on $\mathbf{R}$. Then $C_c^\infty(\mathbf{R}) \subseteq C_c(\mathbf{R})^{*2}$ (see [DM78]) so (3.1.1) holds for all $f \in C_c^\infty(\mathbf{R})$. Now, recall that Schwartz functions can be approximated arbitrarily well with respect to the L^1-norm by such f. Staring at (3.1.1) for a bit, we can thus convince ourselves that it holds whenever f is a Schwartz functions. This is perhaps the form of Poisson summation most familiar to analysts.

3.2. **Riemann-Roch.** This is kind of fun. Let X be a smooth projective curve defined over $\mathbf{F}_q$, the finite field with q elements (q is a power of a prime number). Write F for the function field of X. We suddenly find ourselves overwhelmed by the pressing desire to apply the trace formula with F floating around somewhere. It would thus be a good idea to introduce some topology into the picture. Here is one way to do this. Each closed point $p \in X$ defines a norm $|\cdot|_p$ on F as follows: if $\alpha \in F$ then we can consider the 'order of vanishing' $v_p(\alpha)$ of α at p, and then

$$|\alpha|_p = |k(p)|^{-v_p(\alpha)}.$$

where $k(p)$ is the residue field of X at p. One possible way to think about the order of vanishing is that α can be expressed as a Laurent series on some small enough 'formal neighbourhood' of p, and $v_p(\alpha)$ is the lowest possible degree which appears (see also Remark 3.2.1).

The good news is that the valuations we defined exhaust all possible ways to topologise F in a reasonable way: the assignment $p \mapsto |\cdot|_p$ gives a bijection

$$\{\text{closed points of } X\} \xrightarrow{\sim} \{\text{equivalence classes of non-trivial valuations on } F\}.$$

(See [Cas67, Section 1] for the definitions.) This is *Ostrowski's theorem.*

Remark 3.2.1. We ought to give a few words on the assumptions. The fact that X is smooth allows us to talk about the order of vanishing in a meaningful way. This is because the local ring of a smooth curve at any of its closed points is a DVR, being a one-dimensional Noetherian regular local domain [AM69, Proposition 9.2]. The order of vanishing is then simply the number of times the uniformiser of the local ring divides the given rational function.

On the other hand, projectivity of X ensures that Ostrowski's theorem holds. Indeed, if X is projective then it is in particular proper, and the valuative criterion for properness implies that the only valuation rings contained in F are: F itself (corresponding to the trivial valuation) and the local rings of X at the closed points. Of course, conversely, proper curves are projective so our assumptions are 'minimal'.

Let F_p be the completion of F with respect to $|\cdot|_p$ and let

$$\mathfrak{o}_p = \{\alpha \in F_p \mid v_p(\alpha) \geq 0\} = \{\alpha \in F_p \mid |\alpha|_p \leq 1\}$$

be the subring of F_p consisting of those 'functions' which do not have a pole at p. Since our base field $\mathbf{F}_q$ has finite cardinality, $\mathfrak{o}_p$ is a compact open subring of F_p (see [Cas67, Section 7]). In particular, F_p is a locally compact topological field. From the perspective of representation theory, $(F_p, +)$ behaves much like $(\mathbf{R}, +)$: any non-trivial continuous character $\chi_p : F_p \to S^1$ gives rise to an identification $F_p \cong \widehat{F}_p$ (see [Wei95, Section II.5] for an explicit construction of such a χ_p).

One issue remains, however: even though we have produced a steady supply of locally compact fields, there is no good reason to pick one of them over another. We therefore try to combine them all into one ring. The naïve guess of taking the product of the F_p is no good, because the resulting space is not locally compact. The correct thing to do is to define

$$\mathbf{A} = \Big\{(\alpha_p) \in \prod_p F_p \,\Big|\, \alpha_p \in \mathfrak{o}_p \text{ for all but a finite number of } p\Big\}.$$

This is called the *adele ring* of X, and its underlying space is locally compact because of the way topology works [Cas67, Section 13]. Since any rational function on X has at most a finite number of poles, we can realise F as a subring of $\mathbf{A}$ via the diagonal embedding. What is amazing is that F inherits the discrete topology from $\mathbf{A}$ and the quotient $F\backslash\mathbf{A}$ is compact [Cas67, Section 14]. This is precisely the setting of the trace formula.

The characters $\chi_p : F_p \to S^1$ can be chosen in such a way that $\chi_p(\mathfrak{o}_p) = 1$ for all but a finite number of p, so that we get a global character $\chi : \mathbf{A} \to S^1$; moreover, we can arrange for χ to be trivial on F [Wei95, Section IV.2]. Life is fair and therefore χ furnishes an identification of $\mathbf{A}$ (the additive group) with its dual $\widehat{\mathbf{A}}$, so re-tracing our steps from Section 3.1 we find that for any $f \in C_c(\mathbf{A})^{*2}$ there is an equality

$$\text{vol}(F\backslash\mathbf{A}) \sum_{\alpha \in F} f(\alpha) = \sum_{\beta \in F} \widehat{f}(\beta) \tag{3.2.2}$$

of absolutely convergent infinite sums. Here

$$\widehat{f} : \mathbf{A} \to \mathbf{C}, \quad \beta \mapsto \int_{\mathbf{A}} f(\alpha)\chi(\alpha\beta)d\alpha$$

is the Fourier transform of f. We have thus arrived at an adelic version of the Poisson summation formula.

Remark 3.2.3. Since we have not endeavoured to normalise the Haar measure on $\mathbf{A}$ in any way, we believe it best to leave the precise value of $\text{vol}(F\backslash\mathbf{A})$ ambiguous for now. The trace formula will help us relate it to some other important quantities.

We are only a few definitions away from the punchline. Consider

$$\mathfrak{o} = \prod_p \mathfrak{o}_p.$$

This is a compact open subring of $\mathbf{A}$, called the *integral adele ring* of X. We have a natural identification of the group of $\mathfrak{o}$-lattices in $\mathbf{A}$ (i.e., free $\mathfrak{o}$-submodules $\mathfrak{m} \subseteq \mathbf{A}$

such that $\mathfrak{m}\mathbf{A} = \mathbf{A}$) with the group of Weil divisors on X, seeing as

$$\mathbf{A}^\times/\mathfrak{o}^\times \cong \bigoplus_p F_p^\times/\mathfrak{o}_p^\times \cong \bigoplus_p \mathbf{Z}.$$

Explicitly, given a Weil divisor $D = \sum_p n_p \cdot p$, we let $\mathfrak{m}_D \subseteq \mathbf{A}$ be the $\mathfrak{o}$-submodule generated by $(\varpi_p^{-n_p})_p \in \mathbf{A}^\times$, where $\varpi_p \in \mathfrak{o}_p$ is a uniformiser.

Note that $\mathbf{F}_q$-vector space of global sections of the line bundle on X associated to D is simply given by

$$L(D) = \mathfrak{m}_D \cap F.$$

Being both compact and discrete, the above is finite in cardinality (aha!) and hence it makes sense to contemplate its dimension

$$l(D) = \dim_{\mathbf{F}_q}(L(D)).$$

Another important numerical invariant attached to a Weil divisor D is its degree. We can recover it from $\mathfrak{m}_D$ as follows:

$$\frac{\mathrm{vol}(\mathfrak{m}_D)}{\mathrm{vol}(\mathfrak{o})} = \prod_p |\varpi_p^{-n_p}|_p = q^{\deg(D)}.$$

The point is that the present setting's incarnation of the trace formula implies the following celebrated result:

Theorem 3.2.4 (Riemann-Roch)**.** *There is an integer $g \geq 0$ and a Weil divisor K on X such that*

$$l(D) - l(K-D) = \deg(D) + 1 - g$$

holds for all Weil divisors D on X.

Proof. The plan is to apply the Poisson summation formula to the indicator function $[\mathfrak{m}_D]$ of $\mathfrak{m}_D$. It is a pleasant exercise to convince oneself that $[\mathfrak{m}_D]$ lies in $C_c(\mathbf{A})^{*2}$. (Hint: first compute $[\mathfrak{o}] * [\mathfrak{o}]$.) The geometric side reads

$$\mathrm{vol}(F\backslash\mathbf{A}) \sum_{\alpha\in F} [\mathfrak{m}_D](\alpha) = \mathrm{vol}(F\backslash\mathbf{A})|L(D)|.$$

We now wish to determine the Fourier transform of $[\mathfrak{m}_D]$. To do this, we appeal to a very helpful fact of life: whenever one deals with the adeles, it is often the case that things can be done 'one closed point at a time'. Indeed,

$$\widehat{[\mathfrak{m}_D]}(\beta) = \int_{\mathfrak{m}_D} \chi(\alpha\beta)d\alpha = \prod_p \int_{\mathfrak{m}_{D,p}} \chi_p(\alpha_p\beta_p)d\alpha_p$$

for $\beta \in \mathbf{A}$. Here $\mathfrak{m}_{D,p} = \varpi_p^{-n_p}\mathfrak{o}_p$ is the component of $\mathfrak{m}_D$ at p. Now

$$\int_{\mathfrak{m}_{D,p}} \chi_p(\alpha_p\beta_p)d\alpha_p = \begin{cases} \mathrm{vol}(\mathfrak{m}_{D,p}) & \text{if } \chi_p(\mathfrak{m}_{D,p}\beta_p) = 1 \\ 0 & \text{if } \chi_p(\mathfrak{m}_{D,p}\beta_p) \neq 1 \end{cases}$$

by the invariance of the Haar measure on F_p with respect to translation. Let e_p be the largest integer such that $\varpi_p^{-e_p}\mathfrak{o}_p \subseteq \mathrm{Ker}(\chi_p)$ (which exists since χ_p is non-trivial and continuous). We have $e_p = 0$ for almost all p [Wei95, Section IV.2], so if K is the Weil divisor on X with component e_p at p then we find that

$$\widehat{[\mathfrak{m}_D]}(\beta) = \prod_p \mathrm{vol}(\mathfrak{m}_{D,p})[\mathfrak{m}_{K-D,p}](\beta_p) = \mathrm{vol}(\mathfrak{m}_D)[\mathfrak{m}_{K-D}](\beta).$$

On the spectral side, we therefore have

$$\sum_{\beta \in F} \widehat{[\mathfrak{m}_D]}(\beta) = \mathrm{vol}(\mathfrak{m}_D)|L(K-D)| = q^{\deg(D)}\,\mathrm{vol}(\mathfrak{o})|L(K-D)|.$$

Comparing the two sides of the trace formula and taking q-logarithms shows that

$$g = \deg(D) + 1 + l(K-D) - l(D)$$

is an integer independent of D. To see why g is non-negative, simply take $D = 0$ in the above equation: what comes out is that $g = l(K)$. □

Remark 3.2.5. A neat consequence of the proof above is that

$$\frac{\mathrm{vol}(\mathfrak{o}\backslash\mathbf{A})}{\mathrm{vol}(\mathfrak{o})} = q^{g-1}.$$

This is not unrelated to the fact that $\deg(K) = 2g-2$.

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