ON THE IWASAWA MAIN CONJECTURES FOR MODULAR FORMS AT NON-ORDINARY PRIMES

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Abstract. In recent work, Büyükboduk and Lei formulated new variants of the two-variable Iwasawa main conjecture for modular forms at non-ordinary primes. When the modular form has weight 2, we prove under mild hypotheses some of these main conjectures, and deduce the corresponding one-variable main conjectures of Lei, Loeffler, Zerbes. As a consequence of our results, we deduce the p-part of the Birch and Swinnerton-Dyer formula in analytic ranks 0 and 1 for abelian varieties over $\mathbf{Q}$ of GL_2-type for non-ordinary primes $p > 2$.

Contents

1. Introduction

1.1. **Main results.** Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$ and let $K/\mathbf{Q}$ be an imaginary quadratic field. Let $p > 2$ be a prime of good non-ordinary reduction for f, i.e., $p \nmid N$ and $v_p(a_p) > 0$, and assume that $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K.

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Let Γ_K be the Galois group of the unique $\mathbf{Z}_p^2$-extension of K. Let $\mathcal{O}_L$ be the ring of integers of a finite extension $L/\mathbf{Q}_p$ containing the Fourier coefficients a_n of f, let T_f^* be a fixed Galois stable $\mathcal{O}_L$-lattice in the contragredient of the Galois representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}_L(V_f) \simeq \mathrm{GL}_2(L)$$

associated with f, and let $\Lambda(\Gamma_K) = \mathcal{O}_L[[\Gamma_K]]$ be the two-variable Iwasawa algebra.

In §2.1, we recall the construction in [BL16] of signed Coleman maps

$$\mathrm{Col}_{\mathfrak{q}}^{\sharp},\ \mathrm{Col}_{\mathfrak{q}}^{\flat} : H^1(K_{\mathfrak{q}}, \mathbf{T}) \longrightarrow \Lambda(\Gamma_K) \tag{1.1}$$

for each prime $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, where $\mathbf{T}$ is a two-variable deformation of T_f^*. Similarly as in [LLZ10], [Lei11], and [Spr12] (extending in different degrees of generality an original construction due to Kobayashi [Kob03]) these maps are used to define the local conditions at the places above p cutting out the doubly-signed Selmer groups

$$\mathfrak{Sel}^{\bullet,\circ}(K, \mathbf{A}) \subset H^1(K, \mathbf{A}) \tag{1.2}$$

for $\bullet, \circ \in \{\sharp, \flat\}$, where $\mathbf{A}$ is the Pontrjagin dual of $\mathbf{T}$. In §3.1, we recall the construction of bounded Beilinson–Flach classes

$$\mathcal{BF}^{\sharp},\ \mathcal{BF}^{\flat} \in H^1(K, \mathbf{T}), \tag{1.3}$$

which are obtained from a certain decomposition of the classes constructed by Loeffler–Zerbes in [LZ16]. The doubly-signed main conjectures of [BL16] predict that the Selmer modules (1.2) are $\Lambda(\Gamma_K)$-cotorsion, with characteristic ideals generated by a corresponding doubly-signed p-adic L-function defined by

$$\mathfrak{L}_p^{\bullet,\circ}(f/K) := \mathrm{Col}_{\overline{\mathfrak{p}}}^{\circ}(\mathrm{res}_{\overline{\mathfrak{p}}}(\mathcal{BF}^{\bullet})), \tag{1.4}$$

where $\mathrm{res}_{\overline{\mathfrak{p}}} : H^1(K, \mathbf{T}) \to H^1(K_{\overline{\mathfrak{p}}}, \mathbf{T})$ is the restriction map (*cf.* [BL16, Conj. 4.13]):

Conjecture 1.1. *For each $\bullet, \circ \in \{\sharp, \flat\}$ the module $\mathfrak{Sel}^{\bullet,\circ}(K, \mathbf{A})$ is $\Lambda(\Gamma_K)$-cotorsion,*

$$\mathrm{Char}_{\Lambda(\Gamma_K)}(\mathfrak{Sel}^{\bullet,\circ}(K, \mathbf{A})^{\vee}) = (\mathfrak{L}_p^{\bullet,\circ}(f/K))$$

as ideals in $\Lambda(\Gamma_K)$.

The action of the nontrivial element $\tau \in \mathrm{Gal}(K/\mathbf{Q})$ decomposes $\Gamma_K \simeq \Gamma_K^{\tau=1} \times \Gamma_K^{\tau=-1}$, where $\Gamma_K^{\tau=1}$ is identified with the Galois group Γ^{cyc} of the cyclotomic $\mathbf{Z}_p$-extension of K. One of the main results of this paper is the proof of some cases of Conjecture 1.1 in weight $k = 2$.

Theorem A. *Let $f \in S_2(\Gamma_0(N))$ be a newform, let $p > 2$ be a prime of good non-ordinary reduction for f, and let $K/\mathbf{Q}$ be an imaginary quadratic field in which p splits. Assume that:*

(i) *N is square-free,*

(ii) *$\bar{\rho}_f$ is ramified at every prime $\ell \mid N$ which is nonsplit in K, and there is at least one such prime,*

(iii) *$\bar{\rho}_f|_{\mathrm{Gal}(\overline{\mathbf{Q}}/K)}$ is irreducible.*

If $\bullet, \circ \in \{\sharp, \flat\}$ are such that $\mathfrak{L}_p^{\bullet,\circ}(f/K)$ is nonzero at some character factoring through $\Gamma_K \twoheadrightarrow \Gamma^{\mathrm{cyc}}$, then the module $\mathfrak{Sel}^{\bullet,\circ}(K, \mathbf{A})$ is $\Lambda(\Gamma_K)$-cotorsion, and we have

$$\mathrm{Char}_{\Lambda(\Gamma_K)}(\mathfrak{Sel}^{\bullet,\circ}(K, \mathbf{A})^{\vee}) = (\mathfrak{L}_p^{\bullet,\circ}(f/K))$$

as ideals in $\Lambda(\Gamma_K)$.

Remark 1.2. As we note in Corollary 3.7 in the body of the paper, there is always some pair of elements $\bullet, \circ \in \{\sharp, \flat\}$ for which $\mathfrak{L}_p^{\bullet,\circ}(f/K)$ is not identically zero, even along the "cyclotomic line", and in fact one can always take $\bullet = \circ$ with that property. Thus Theorem A establishes at least one of the equally-signed cases of Conjecture 1.1 in weight 2, and (since one expects that all $\mathfrak{L}_p^{\bullet,\circ}(f/K)$ have nonzero cyclotomic restrictions) it should suffice to establish the four cases of Conjecture 1.1.

In the course of proving Theorem A, we also establish cases of the signed main conjectures of [LLZ10]. To recall the latter, let $\Gamma^{\rm cyc}$ be the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, and set $\Lambda(\Gamma^{\rm cyc}) = \mathcal{O}_L[[\Gamma^{\rm cyc}]]$. In [LLZ10], Lei–Loeffler–Zerbes constructed signed Coleman maps

$$\mathrm{Col}^\sharp,\ \mathrm{Col}^\flat : H^1(\mathbf{Q}_p, \mathbf{T}^{\rm cyc}) \longrightarrow \Lambda(\Gamma^{\rm cyc})$$

for the cyclotomic deformation $\mathbf{T}^{\rm cyc}$ of T_f^*, and formulated Iwasawa main conjectures relating the characteristic ideal of certain signed Selmer groups

$$\mathrm{Sel}^\bullet(\mathbf{Q}, \mathbf{A}^{\rm cyc}) \subset H^1(\mathbf{Q}, \mathbf{A}^{\rm cyc}),$$

where $\mathbf{A}^{\rm cyc}$ is the Pontrjagin dual of $\mathbf{T}^{\rm cyc}$, to corresponding signed p-adic L-functions

$$L_p^\bullet(f/\mathbf{Q}) := \mathrm{Col}_\bullet(\mathrm{res}_p(\mathbf{z}^{\rm Kato})),$$

where $\mathbf{z}^{\rm Kato} \in H^1(\mathbf{Q}, \mathbf{T}^{\rm cyc})$ is constructed from Beilinson–Kato classes. As shown in [LLZ10], one of the divisibilities predicted by either of these signed main conjectures follows from Kato's work [Kat04]. In the direction of these conjectures, we obtain the following result:

Theorem B. *Let $f \in S_2(\Gamma_0(N))$ be a newform of square-free level N, and let $p > 2$ be a prime of good non-ordinary reduction for f. If $\bullet \in \{\sharp, \flat\}$ is such that $L_p^\bullet(f/\mathbf{Q})$ is nonzero, then* $\mathrm{Sel}^\bullet(\mathbf{Q}, \mathbf{A}^{\rm cyc})$ *is $\Lambda(\Gamma^{\rm cyc})$-cotorsion, and we have*

$$\mathrm{Char}_{\Lambda(\Gamma^{\rm cyc})}(\mathrm{Sel}^\bullet(\mathbf{Q}, \mathbf{A}^{\rm cyc})^\vee) = (L_p^\bullet(f/\mathbf{Q}))$$

as ideals in $\Lambda(\Gamma^{\rm cyc})$.

Remark 1.3. If f as in Theorem B has rational Fourier coefficients, so that it corresponds to an elliptic curve by the Eichler–Shimura construction, the main conjectures of [LLZ10] reduce to the signed main conjectures of Kobayashi [Kob03] (when $a_p = 0$) and Sprung [Spr12] (when $p > 2$). Thus Theorem B in the elliptic curve case was first established by X. Wan [Wan16b] (when $a_p = 0$) and by Sprung [Spr16] (when $p > 2$).

1.2. **Outline of the proofs.** The imaginary quadratic field K determines a factorization

$$N = N^+N^-$$

with N^+ (resp. N^-) divisible only by primes which are either split or ramified (resp. inert) in K. Since we assume that N is square-free, the nature of the proof of Theorem A depends on whether:

- N^- is the product of an odd number of primes: the *definite* case, or
- N^- is the product of an even number of primes: the *indefinite* case.

We first outline our proof in the latter case. In §4.1, we construct signed Heegner classes

$$\mathfrak{Z}^\sharp,\ \mathfrak{Z}^\flat \in H^1(K, \mathbf{T}^{\rm ac}),$$

where $\mathbf{T}^{\rm ac}$ is the anticyclotomic deformation of T_f^*. These are shown to land in a certain pro-p Selmer group $\mathfrak{Sel}^\bullet(K, \mathbf{T}^{\rm ac})$ (essentially, the projection of $\mathfrak{Sel}^{\bullet,\bullet}(K, \mathbf{T})$ to the anticyclotomic line), and a natural extension of Perrin-Riou's main conjecture [PR87] leads to the prediction that $\mathfrak{Sel}^\bullet(K, \mathbf{T}^{\rm ac})$ and the Pontrjagin dual $\mathfrak{Sel}^\bullet(K, \mathbf{A}^{\rm ac})^\vee$ have both $\Lambda(\Gamma^{\rm ac})$-rank one, with

$$\mathrm{Char}_{\Lambda(\Gamma^{\rm ac})}(\mathfrak{Sel}^\bullet(K, \mathbf{A}^{\rm ac})^\vee_{\rm tors}) \stackrel{?}{=} \mathrm{Char}_{\Lambda(\Gamma^{\rm ac})}\left(\frac{\mathrm{Sel}^\bullet(K, \mathbf{T}^{\rm ac})}{\Lambda(\Gamma^{\rm ac}) \cdot \mathfrak{Z}^\bullet}\right)^2 \tag{1.5}$$

as ideals in the anticyclotomic Iwasawa algebra $\Lambda(\Gamma^{\rm ac}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ (*cf.* Conjecture 5.3), where the subscript 'tors' denotes the $\Lambda(\Gamma^{\rm ac})$-torsion submodule. In §4.2 we prove an explicit reciprocity law:

$$\mathrm{Log}^\bullet_{\mathfrak{p},\rm ac}(\mathrm{res}_\mathfrak{p}(\mathfrak{Z}^\bullet)) = \mathscr{L}_\mathfrak{p}^{\rm BDP}(f/K)$$

relating the image of the classes $\mathfrak{Z}^\sharp, \mathfrak{Z}^\flat$ under certain signed logarithm maps (whose construction might be of independent interest) to the anticyclotomic p-adic L-function of [CH17a] and

[BDP13]. The square of $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ is easily seen to agree with the restriction to the anticyclotomic line of the p-adic L-function constructed in [Wan16b], which by the Iwasawa–Greenberg main conjectures [Gre94] should generate the characteristic ideal of certain two-variable Selmer group:

$$\mathrm{Char}_{\Lambda(\Gamma_K)}(\mathrm{Sel}_{\mathfrak{p}}(K, \mathbf{A})^\vee) \stackrel{?}{=} (\mathscr{L}_{\mathfrak{p}}(f/K)). \tag{1.6}$$

Combining the divisibility towards (1.6) contained in [Wan16b] with an extension of Howard's Kolyvagin system arguments [How04a] to our non-ordinary setting, we thus arrive at a proof of both (1.5) and (1.6), from where Theorem A in the indefinite case follows easily.

In the definite case, Heegner points are not (directly) available, and so to prove Theorem A in this case we proceed differently. In §4.3, we relate the anticyclotomic restriction of the p-adic L-functions $\mathfrak{L}_p^{\bullet,\circ}(f/K)$ for $\bullet = \circ$ to certain signed theta elements $\Theta_\infty^\bullet \in \Lambda(\Gamma^{\mathrm{ac}})$ extending those introduced by Darmon–Iovita [DI08] when $a_p = 0$. Thus we are able to exploit Vatsal's results on the vanishing of anticyclotomic μ-invariants [Vat03] to deduce again from [Wan16b] on of the divisibilities in Theorem A, which combined with Kato's work (as reformulated in [LLZ10, §6]) leads to the proof of Theorem A in the definite case and the proof of Theorem B.

1.3. **Applications to the p-part of BSD formulae.** We conclude this Introduction by explaining the implications of the preceding results of the p-part of the Birch and Swinnerton-Dyer formula for abelian varieties of $A/\mathbf{Q}$ of GL_2-type; by [KW09, Cor. 10.2] these precisely correspond to the abelian varieties $A_f/\mathbf{Q}$ (up to isogeny) associated with weight 2 eigenforms $f \in S_2(\Gamma_0(N))$ by the Eichler–Shimura construction [Shi94], so that

$$L(A_f, s) = \prod_\sigma L(f^\sigma, s) \tag{1.7}$$

with the product running over all conjugates of f^σ. In particular, letting $r_f := \mathrm{ord}_{s=1}L(f, s)$ we note that

$$\mathrm{ord}_{s=1}L(A_f, s) = [K_f : \mathbf{Q}]r_f = \dim(A_f)r_f,$$

where K_f is the number generated by the Fourier coefficients of f.

The analogue of Theorem B for primes $p > 3$ of good ordinary reduction for f, i.e., Iwasawa's Main Conjecture for $\mathrm{GL}_{2/\mathbf{Q}}$ for good ordinary primes, is one of the main results of [SU14]. Similarly as in [*loc.cit.*, §3.6.1], the interpolation property of the p-adic L-functions $L_p^\bullet(f/\mathbf{Q})$ at the trivial character, together with a variant of Mazur's control theorem for the signed Selmer groups $\mathrm{Sel}^\bullet(\mathbf{Q}, \mathbf{A}^{\mathrm{cyc}})$ yields the following result on the p-part of the Birch and Swinnerton-Dyer formula in analytic rank 0:

Theorem C. *Let $A/\mathbf{Q}$ be a semistable abelian variety of GL_2-type of conductor N. Assume that $L(A, 1) \neq 0$, and let $p > 2$ be a prime of good non-ordinary reduction for A. Then*

$$\mathrm{ord}_p\left(\frac{L(A, 1)}{\Omega_A}\right) = \mathrm{ord}_p\left(\#\text{Ш}(A/\mathbf{Q}) \prod_{\ell \mid N} c_\ell(A/\mathbf{Q})\right),$$

where

- $\Omega_A = \int_{A(\mathbf{R})} |\omega_A|$, *for ω_A a Néron differential, is the real period of A,*
- $\text{Ш}(A/\mathbf{Q})$ *is the Tate–Shafarevich group of A, and*
- $c_\ell(A/\mathbf{Q})$ *is the Tamagawa number of A at the prime ℓ.*

In other words, the p-part of the Birch–Swinnerton-Dyer formula holds for A.

As a key intermediate step in the proof of Theorem A in the indefinite case, we establish the Iwasawa–Greenberg main conjecture for the anticyclotomic p-adic L-function $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ (see Theorem 5.8). A suitable extension of the arguments in [JSW17] then yields the following result on the p-part of the Birch–Swinnerton-Dyer formula in analytic rank 1:

Theorem D. *Let $A/\mathbf{Q}$ be a semistable abelian variety of* GL_2*-type, associated to a newform $f \in S_2(\Gamma_0(N))$. Assume that $\mathrm{ord}_{s=1}L(f,s) = 1$, and let $p > 2$ be a prime of good non-ordinary reduction for f. Then*

$$\mathrm{ord}_p\left(\frac{L^*(A,1)}{\mathrm{Reg}(A/\mathbf{Q})\cdot\Omega_A}\right) = \mathrm{ord}_p\left(\#\text{Ш}(A/\mathbf{Q})\prod_{\ell|N} c_\ell(A/\mathbf{Q})\right),$$

where $L^*(A,1)$ is the leading term of the Taylor expansion of $L(A,s)$ around $s = 1$, $\mathrm{Reg}(A/\mathbf{Q})$ is the discriminant of the Néron–Tate canonical height pairing on $A(\mathbf{Q}) \otimes \mathbf{R}$. In other words, the p-part of the Birch–Swinnerton-Dyer formula holds for A.

Finally, we note that the signed main conjectures of [BL16] are formulated for even weights $k < p$. In the *definite* case, the methods exploited in this paper should generalise, allowing one to deduce from [Wan16a, Thm. 3.9] a higher weight analogue of Theorem A in this case (and yielding, in combination with [LLZ10, Cor. 6.6], a new proof of the main result of [Wan16a]). On the other hand, a higher weight analogue of our Theorem A in the *indefinite* case would seem to require new ideas, as we would like to explore in a future work.

Notations. Throughout the paper, we let $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$ be a newform, $p > 2$ be a prime of good non-ordinary reduction for f, and K be an imaginary quadratic field in which $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits. By choosing an algebraic isomorphism $\mathbf{C} \simeq \mathbf{C}_p$ once and for all, we shall assume that f is defined over a finite extension $L/\mathbf{Q}_p$ with ring of integers $\mathcal{O}_L$. For any p-adic Lie group G, we let $\Lambda(G)$ denote the Iwasawa algebra $\mathcal{O}_L[[G]]$, and set $\Lambda_L(G) := L \otimes_{\mathcal{O}_L} \Lambda(G)$. Finally, for F a finite extension of $\mathbf{Q}$ or $\mathbf{Q}_p$ we let $G_F := \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group.

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2. Preliminaries

2.1. **Signed Coleman maps.** In this section, we briefly recall the two-variable signed Coleman maps introduced in [BL16, §2.3]. In the notations of *loc.cit.*, for our applications in this paper it will suffice to take $F = \mathbf{Q}_p$, and ϑ and η both equal to the trivial character.

Let V_f be the 2-dimensional L-linear $G_{\mathbf{Q}}$-representation associated with f by Deligne, and let $T_f \subset V_f$ be a fixed Galois stable $\mathcal{O}_L$-lattice. As a representation of $G_{\mathbf{Q}_p}$, V_f has Hodge–Tate weights $\{0, -1\}$, with the convention that $\mathbf{Q}_p(1)$ has Hodge–Tate weight $+1$.

Let $\mathbb{A}^+_{\mathbf{Q}_p} = \mathcal{O}_L[[\pi]]$, where π is a formal variable. Let $\mathbb{N}(T_f)$ be the Wach module associated to T_f (see [Ber04, Lem. II.1.3]), and denote by $\{n_{f,1}, n_{f,2}\}$ the $\mathbb{A}^+_{\mathbf{Q}_p}$-basis for $\mathbb{N}(T_f)$ constructed in [BLZ04, §3]. Since f has trivial nebentypus, we have $T_f^* \simeq T_f(1)$, and therefore letting e_1 be a generator of $\mathbf{Z}_p(1)$ and setting

$$n_i := n_{f,i} \cdot \pi^{-1} e_1, \quad i = 1, 2 \tag{2.1}$$

we obtain an $\mathbb{A}^+_{\mathbf{Q}_p}$-basis $\{n_1, n_2\}$ for $\mathbb{N}(T_f^*)$. By the construction in [Ber04], under the canonical identification

$$\mathbb{N}(T_f^*)/\pi\mathbb{N}(T_f^*) \simeq \mathbb{D}_{\mathrm{cris}}(T_f^*),$$

the reduction mod π of the basis $\{n_1, n_2\}$ yields an $\mathcal{O}_L$-basis $\{v_1, v_2\}$ for $\mathbb{D}_{\mathrm{cris}}(T_f^*)$ satisfying $v_1 \in \mathrm{Fil}^0\mathbb{D}_{\mathrm{cris}}(T_f^*)$, $v_2 = \varphi(v_1)$. In particular, the matrix A_φ of the Frobenius map φ on $\mathbb{N}(T_f^*)$ with respect to $\{v_1, v_2\}$ is given by

$$A_\varphi = p^{-1}\begin{pmatrix} 0 & -1 \\ p & a_p \end{pmatrix}. \tag{2.2}$$

Let $\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p}$ be the subring of $L[[\pi]]$ of power series convergent on the open unit disc in $\mathbf{C}_p$. As explained in [LLZ10, §3.2], there is a change of basis matrix $M \in M_{2\times 2}(\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})$ such that

$$(n_1 \quad n_2) = (v_1 \quad v_2)\, M.$$

Since n_i reduces to v_i mod π, we have

$$M \equiv I_2 \pmod{\pi}, \tag{2.3}$$

where I_2 is the 2×2 identity matrix. Let $\Gamma = \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ be the Galois group of the $\mathbf{Z}_p^\times$-extension of $\mathbf{Q}$. Like $\mathbb{A}^+_{\mathbf{Q}_p}$, the ring $\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p}$ is equipped with $\mathcal{O}_L$-linear actions of φ and $\gamma \in \Gamma$ given by $\pi \mapsto (1+\pi)^p - 1$ and $\pi \mapsto (1+\pi)^{\chi(\gamma)} - 1$, where $\chi : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ is the p-adic cyclotomic character. Let $\mathcal{H}_L(\Gamma)$ the algebra of L-valued locally analytic distributions on Γ. As in [LLZ10], we make the following definition.

Definition 2.1. We define the *logarithmic matrix* $M_{\log} \in M_{2\times 2}(\mathcal{H}_L(\Gamma))$ by

$$M_{\log} := \mathfrak{M}^{-1}((1+\pi)A_\varphi \cdot \varphi(M)),$$

where $\mathfrak{M} : \mathcal{H}_L(\Gamma) \simeq (\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ is the Mellin transform (see e.g. [PR01, Cor. B.2.8]), and ψ is the left inverse of φ.

For any p-adic Lie extension E_∞ of $\mathbf{Q}_p$, set

$$H^1_{\mathrm{Iw}}(E_\infty, T_f^*) := \varprojlim_F H^1(F, T_f^*),$$

where the limit is with respect to corestriction over the finite Galois extensions $F/\mathbf{Q}_p$ contained in E_∞. Let F_∞ be the unramified $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, and set

$$U := \mathrm{Gal}(F_\infty/\mathbf{Q}_p), \quad G := \mathrm{Gal}(F_\infty(\mu_{p^\infty})/\mathbf{Q}_p) \simeq \Gamma \times U.$$

Let $\hat{\mathcal{O}}_{F_\infty}$ be the completion of the ring of integers of F_∞, and let $S_\infty \subset \Lambda_{\hat{\mathcal{O}}_{F_\infty}}(U)$ be the Yager module introduced in [LZ14, §3.2]. With a slight abuse of notation, let $1-\varphi$ be the map

$$1 - \varphi : H^1_{\mathrm{Iw}}(F_\infty(\mu_{p^\infty}), T_f^*) \longrightarrow (\varphi^*\mathbb{N}(T_f^*))^{\psi=0} \hat{\otimes} S_\infty \tag{2.4}$$

constructed in [*op.cit.*, Def. 4.6]. The composition of (2.4) with the $\Lambda(G)$-linear embedding

$$(\varphi^*\mathbb{N}(T_f^*))^{\psi=0} \hat{\otimes} S_\infty \hookrightarrow \mathcal{H}_{\hat{F}_\infty}(G) \otimes \mathbb{D}_{\mathrm{cris}}(T_f^*)$$

deduced from [LLZ11, Prop. 2.11], where $\hat{F}_\infty$ is the completion of F_∞, yields the two-variable regulator map

$$\mathcal{L}_{T_f^*} : H^1_{\mathrm{Iw}}(F_\infty(\mu_{p^\infty}), T_f^*) \longrightarrow \mathcal{H}_{\hat{F}_\infty}(G) \otimes \mathbb{D}_{\mathrm{cris}}(T_f^*)$$

of [LZ14]. On the other hand, by [LLZ10, Thm. 4.24] the $\mathbb{A}^+_{\mathbf{Q}_p}$-basis $\{n_1, n_2\}$ of $\mathbb{N}(T_f^*)$ defined above is such that $\{(1+\pi)\varphi(n_1), (1+\pi)\varphi(n_2))\}$ forms a $\Lambda(\Gamma)$-basis for $(\varphi^*\mathbb{N}(T_f^*))^{\psi=0}$, and so gives rise to a $\Lambda_{\hat{\mathcal{O}}_{F_\infty}}(G)$-linear embedding

$$\mathfrak{J} : (\varphi^*\mathbb{N}(T_f^*))^{\psi=0} \hat{\otimes} S_\infty \hookrightarrow \Lambda_{\hat{\mathcal{O}}_{F_\infty}}(G) \oplus \Lambda_{\hat{\mathcal{O}}_{F_\infty}}(G)$$

allowing us to define (following [BL16, §2.3]) the two-variable signed Coleman maps

$$(\mathrm{Col}^\sharp, \mathrm{Col}^\flat) : H^1_{\mathrm{Iw}}(F_\infty(\mu_{p^\infty}), T_f^*) \longrightarrow \Lambda_{\hat{\mathcal{O}}_{F_\infty}}(G) \oplus \Lambda_{\hat{\mathcal{O}}_{F_\infty}}(G) \tag{2.5}$$

as the composition $\mathfrak{J} \circ (1-\varphi)$.

Let α and β be the roots of the Hecke polynomial $X^2 - a_pX + p$; since f has weight 2, we know that $\alpha \neq \beta$ by [CE98]. By construction, we have the relation

$$\mathcal{L}_{T_f^*} = (v_1 \quad v_2) \cdot M_{\log} \cdot \begin{pmatrix} \mathrm{Col}^\sharp \\ \mathrm{Col}^\flat \end{pmatrix},$$

and letting $\mathcal{L}^\lambda$ be the projection of $\mathcal{L}_{T_f^*}$ onto the λ^{-1}-eigenspace for φ on $\mathbb{D}_{\rm cris}(T_f^*)$ it follows that

$$\begin{pmatrix} \mathcal{L}^\alpha \\ \mathcal{L}^\beta \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log} \cdot \begin{pmatrix} \mathrm{Col}^\sharp \\ \mathrm{Col}^\flat \end{pmatrix}, \tag{2.6}$$

where $Q_{\alpha,\beta} = \left(\begin{smallmatrix} \alpha & -\beta \\ -p & p \end{smallmatrix}\right)$ diagonalizes A_φ.

Let $\Gamma_K := \mathrm{Gal}(K_\infty/K)$ be the Galois group of the $\mathbf{Z}_p^2$-extension of K. As in [BL16, §2.5], we shall apply the above constructions to the G_K-representation

$$\mathbf{T} := T_f^* \hat{\otimes} \Lambda(\Gamma_K)^\iota,$$

where $\Lambda(\Gamma_K)^\iota$ denotes the module $\Lambda(\Gamma_K)$ equipped with the Galois action given by the inverse of the canonical character $G_K \twoheadrightarrow \Gamma_K \hookrightarrow \Lambda(\Gamma_K)^\times$. For each $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, let $D_\mathfrak{q} \subset \Gamma_K$ be the decomposition group of a fixed prime of K_∞ above $\mathfrak{q}$, and let $\gamma_1, \ldots, \gamma_{p^t}$ be a complete set of coset representatives for $\Gamma_K/D_\mathfrak{q}$, so that $\Lambda(\Gamma_K) = \sum_i \Lambda(D_\mathfrak{q}).\gamma_i$. Combined with Shapiro's lemma, we then have natural identifications

$$H^1(K_\mathfrak{q}, \mathbf{T}) \simeq \bigoplus_{i=1}^{p^t} H^1(K_\mathfrak{q}, T_f^* \hat{\otimes} \Lambda(D_\mathfrak{q})^\iota).\gamma_i \simeq \bigoplus_{v \mid \mathfrak{q}} H^1_{\rm Iw}(K_{\infty,v}, T_f^*),$$

where the second sum is over the primes v of K_∞ above $\mathfrak{q}$.

Definition 2.2. Let Δ be the torsion subgroup of Γ, and denote by $e_{\mathbb{1}}$ is the idempotent of $\Lambda(G)$ attached to the trivial character of Δ.

(1) For each $\bullet \in \{\sharp, \flat\}$ and $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, define the *signed Coleman map*

$$\mathrm{Col}^\bullet_\mathfrak{q} : H^1(K_\mathfrak{q}, \mathbf{T}) \longrightarrow \Lambda_{\hat{\mathcal{O}}_{F_\infty}}(\Gamma_K)$$

by $z = \sum_i z_i.\gamma_i \mapsto \sum_i e_{\mathbb{1}} \mathrm{Col}^\bullet(z_i).\gamma_i$, where $\mathrm{Col}^\bullet$ are as in (2.5).

(2) For each $\bullet \in \{\sharp, \flat\}$ and $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, define the *Perrin-Riou regulator map*

$$\mathcal{L}^\lambda_\mathfrak{q} : H^1(K_\mathfrak{q}, \mathbf{T}) \longrightarrow \mathcal{H}_{\hat{F}_\infty}(\Gamma_K)$$

by $z = \sum_i z_i.\gamma_i \mapsto \sum_i e_{\mathbb{1}} \mathcal{L}^\lambda(z_i).\gamma_i$, where $\mathcal{L}^\lambda$ are as in (2.6).

The following determination of the image of the signed Coleman maps will be important later.

Proposition 2.3. *The maps* $\mathrm{Col}^\bullet_\mathfrak{q}$ *have finite cokernel.*

Proof. Since we are working in weight 2 and projecting to the trivial isotypical component for Δ, this follows from [BL16, Lem. 2.10] (see also [*loc.cit.*, Rem. 2.9]). □

2.2. **Signed logarithm maps.** The maps introduced in this section, which generalize the signed logarithm maps introduced in [Wan16b] when $a_p = 0$, will play an important role here. Their construction, which appears to be new, arises naturally after relating the two-variable signed Coleman maps of the §2.1 to a big Perrin-Riou map applied to a certain Hida family.

We maintain the notations introduced in the preceding section, and for each $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$ let $K_\infty^\mathfrak{q}$ be the maximal subfield of K_∞ unramified at $\mathfrak{q}$. Set $\Gamma_\mathfrak{q} := \mathrm{Gal}(K_\infty^\mathfrak{q}/K) \simeq \mathbf{Z}_p$, and let

$$\mathbf{g} = \sum_{n=1}^\infty \mathbf{a}_n q^n \in \Lambda_\mathbf{g}[[q]]$$

be the canonical Hida family of CM forms constructed in [JSW17, §5.2], where $\Lambda_\mathbf{g} := \Lambda(\Gamma_\mathfrak{p})$. The associated Galois representation $M(\mathbf{g})^*$ satisfies

$$M(\mathbf{g})^*|_{G_K} \simeq \Lambda(\Gamma_{\overline{\mathfrak{p}}})^\iota \oplus \Lambda(\Gamma_\mathfrak{p})^\iota, \tag{2.7}$$

where $\Lambda(\Gamma_{\mathfrak{q}})^{\iota}$ denotes the module $\Lambda(\Gamma_{\mathfrak{q}})$ with Galois action given by the inverse of the canonical character $G_K \twoheadrightarrow \Gamma_{\mathfrak{q}} \hookrightarrow \Lambda(\Gamma_{\mathfrak{q}})^{\times}$. Upon restriction to a decomposition group at p, $M(\mathbf{g})^*$ fits into an exact sequence of $\Lambda_{\mathbf{g}}[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow \mathscr{F}^{+}M(\mathbf{g})^* \longrightarrow M(\mathbf{g})^* \longrightarrow \mathscr{F}^{-}M(\mathbf{g})^* \longrightarrow 0, \tag{2.8}$$

with $\mathscr{F}^{\pm}M(\mathbf{g})^* \simeq \Lambda_{\mathbf{g}}$, and with the $G_{\mathbf{Q}_p}$-action on $\mathscr{F}^{-}M(\mathbf{g})^*$ given by the unramified character sending a geometric Frobenius element Fr_p to $\mathbf{a}_p^{-1}$. Letting $\mathbf{k}$ be the weight character of $\mathbf{g}$ (as defined in [KLZ17, §7.1]), the twist $\mathscr{F}^{+}M(\mathbf{g})^*(-1-\mathbf{k})$ is therefore unramified, and we set

$$T_{f,\mathbf{g}}^{+} := T_f^* \otimes \mathscr{F}^{+}M(\mathbf{g})^*(-1-\mathbf{k}), \quad T_{f,\mathbf{g}}^{-} := T_f^* \otimes \mathscr{F}^{-}M(\mathbf{g})^*.$$

The direct sum decomposition (2.7) is compatible upon restriction to a decomposition group at p with the filtration (2.8); in particular, the latter is split, so we may choose identifications[1]

$$\begin{aligned} \jmath_+ &: H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), T_{f,\mathbf{g}}^{+}) \simeq H^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}), \\ \jmath_- &: H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), T_{f,\mathbf{g}}^{-}) \simeq H^1(K_{\mathfrak{p}}, \mathbf{T}). \end{aligned} \tag{2.9}$$

For each $\lambda \in \{\alpha, \beta\}$, denote by f^{λ} the p-stabilization of f with U_p-eigenvalue λ. Let $\mathcal{F}^{\lambda}$ be the Coleman family passing through f^{λ}, and let $\mathcal{L}_{f^{\lambda},\mathbf{g}}$ be the corresponding specialization of the composite map

$$\begin{aligned} \mathcal{L}_{\mathcal{F}^{\lambda},\mathbf{g}} : H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), \mathscr{F}^{-+}D_{V_1\times V_2}(\mathcal{F}^{\lambda}\otimes\mathbf{g})) &\xrightarrow{\mathcal{L}} \mathbf{D}(\mathscr{F}^{-+}M(\mathcal{F}^{\lambda}\otimes\mathbf{g})^*)\hat{\otimes}\mathcal{H}_L(\Gamma) \\ &\longrightarrow I_{\mathcal{F}^{\lambda}}\hat{\otimes}\Lambda_{\mathbf{g}}\hat{\otimes}\mathcal{H}_L(\Gamma), \end{aligned}$$

where the map $\mathcal{L}$ is the Perrin-Riou big logarithm map of [LZ16, Thm. 7.1.4], and the second arrow is given by pairing against the tensor product $\eta_{\mathcal{F}^{\lambda}} \otimes \omega_{\mathbf{g}}$ of the classes constructed in [*op.cit.*, §6.4].

Let $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_f^*)$ and $\mathbf{D}^{\dagger}_{\mathrm{rig}}(M(\mathbf{g})^*)$ be the (φ,Γ)-module associated with f^{λ} and $\mathbf{g}$, respectively, and denote by $\mathscr{F}^{\pm}$ the corresponding triangulations, so that $\mathscr{F}^{-+}D_{V_1\times V_2}(\mathcal{F}^{\lambda}\otimes\mathbf{g}))$ specializes to $\mathscr{F}^{-+}D(f^{\lambda}\otimes\mathbf{g}) := \mathscr{F}^{-}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_f^*)\otimes\mathscr{F}^{+}\mathbf{D}^{\dagger}_{\mathrm{rig}}(M(\mathbf{g})^*)$ by construction.

Proposition 2.4. *Under the identification $\jmath^+$ in* (2.9)*, the $\mathcal{H}_L(\Gamma_{\overline{\mathfrak{p}}})$-linear extension of $\mathcal{L}_{\lambda}^{\overline{\mathfrak{p}}}$ agrees with the composition of $\mathcal{L}_{f^{\lambda},\mathbf{g}}$ with the projection*

$$\begin{aligned} H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), T_{f,\mathbf{g}}^{+})\hat{\otimes}\mathcal{H}_L(\Gamma) &\longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_f^*)\otimes\mathscr{F}^{+}\mathbf{D}^{\dagger}_{\mathrm{rig}}(M(\mathbf{g})^*)) \\ &\longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^{-+}D(f^{\lambda}\otimes\mathbf{g})) \end{aligned}$$

Proof. By [BL16, Prop. 2.16] and [KPX14, Cor. 4.4.11], the first arrow above is an isomorphism, and the claimed agreement follows immediately by comparing the interpolation properties of the maps involved, given by [LZ14, Thm. 4.15] and [LZ16, Thm. 7.1.4], respectively. □

Swapping the roles of $\mathcal{F}^{\lambda}$ and $\mathbf{g}$ in the above discussion, we will arrive at our definition of the signed logarithm maps. Indeed, denote by

$$\mathcal{L}_{\mathbf{g},f^{\lambda}} : H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^{-+}D(\mathbf{g}\otimes f^{\lambda})) \longrightarrow I_{\mathbf{g}}\hat{\otimes}\mathcal{H}_L(\Gamma)$$

the specialization to f^{λ} of the composite map

$$\begin{aligned} \mathcal{L}_{\mathbf{g},\mathcal{F}^{\lambda}} : H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^{-+}D_{V_2\times V_1}(\mathbf{g}\otimes\mathcal{F}^{\lambda})) &\xrightarrow{\mathcal{L}} \mathbf{D}(\mathscr{F}^{-+}M(\mathbf{g}\otimes\mathcal{F}^{\lambda})^*)\hat{\otimes}\mathcal{H}_L(\Gamma) \\ &\longrightarrow I_{\mathbf{g}}\hat{\otimes}\Lambda_{\mathcal{F}^{\lambda}}\hat{\otimes}\mathcal{H}_L(\Gamma), \end{aligned} \tag{2.10}$$

where $I_{\mathbf{g}} \subset \mathrm{Frac}(\Lambda_{\mathbf{g}})$ is the congruence ideal for $\mathbf{g}$, and the second arrow is given by pairing against $\eta_{\mathbf{g}} \otimes \omega_{\mathcal{F}^{\lambda}}$.

[1]Note that the ordering of $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ here is opposite to the one taken in [BL16, §2.3].

Definition 2.5. For each $\lambda \in \{\alpha, \beta\}$, let

$$H^1_\lambda(K_\mathfrak{p}, \mathbf{T}\hat{\otimes}\mathcal{H}_L(\Gamma_\mathfrak{p})) \subset H^1(K_\mathfrak{p}, \mathbf{T}\hat{\otimes}\mathcal{H}_L(\Gamma_\mathfrak{p}))$$

be the kernel of the $\mathcal{H}_L(\Gamma_\mathfrak{p})$-linear extension of the map $\mathcal{L}^\lambda_\mathfrak{p}$, and for each $\bullet \in \{\sharp, \flat\}$, let

$$H^1_\bullet(K_\mathfrak{p}, \mathbf{T}) \subset H^1(K_\mathfrak{p}, \mathbf{T})$$

be the kernel of the map $\mathrm{Col}^\bullet_\mathfrak{p}$.

Lemma 2.6. *Under the identification $\jmath^-$ in* (2.9)*, the image of the natural map*

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^{-+}D(\mathbf{g}\otimes f^\lambda)) \longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^-\mathbf{D}^\dagger_{\mathrm{rig}}(M(\mathbf{g})^*)\otimes \mathbf{D}^\dagger_{\mathrm{rig}}(V_f^*))$$

is identified with $H^1_\lambda(K_\mathfrak{p}, \mathbf{T}\hat{\otimes}\mathcal{H}_L(\Gamma_\mathfrak{p}))$.

Proof. Since the regulator map $\mathcal{L}_{T_f^*}$ is injective (see [LZ14, Prop. 4.11]), a class is in the kernel of the $\mathcal{H}_L(\Gamma_\mathfrak{p})$-linear extension of $\mathcal{L}^\lambda_\mathfrak{p}$ if and only if it corresponds under the isomorphism

$$\begin{aligned} H^1(K_\mathfrak{p}, \mathbf{T}\hat{\otimes}\mathcal{H}_L(\Gamma_\mathfrak{p})) &\simeq H^1(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^-M(\mathbf{g})^*\otimes V_f^*\otimes\mathcal{H}_L(\Gamma)) \\ &\simeq H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^-\mathbf{D}^\dagger_{\mathrm{rig}}(M(\mathbf{g})^*)\otimes\mathbf{D}^\dagger_{\mathrm{rig}}(V_f^*)) \end{aligned}$$

induced by $\jmath^-$ and [KPX14, Cor. 4.4.11] to a class projecting trivially onto

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^-\mathbf{D}^\dagger_{\mathrm{rig}}(M(\mathbf{g})^*)\otimes\mathscr{F}^-\mathbf{D}^\dagger_{\mathrm{rig}}(V_f^*)),$$

hence the result. □

For every $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, let $M_{\log,\mathfrak{q}} \in M_{2\times 2}(\mathcal{H}_L(\Gamma_\mathfrak{q}))$ be the logarithmic matrix $M_{\log}$ of Definition 2.1 with $\Gamma_\mathfrak{q}$ in place of Γ.

Lemma 2.7. *If $(\kappa^\sharp, \kappa^\flat)$ is any pair of classes in $H^1_\sharp(K_\mathfrak{p}, \mathbf{T}) \oplus H^1_\flat(K_\mathfrak{p}, \mathbf{T})$, then the pair of classes $(\kappa^\alpha, \kappa^\beta) \in H^1(K_\mathfrak{p}, \mathbf{T}\hat{\otimes}\mathcal{H}_L(\Gamma))^{\oplus 2}$ defined by the relation*

$$\begin{pmatrix} \kappa^\alpha \\ \kappa^\beta \end{pmatrix} = Q^{-1}_{\alpha,\beta} M_{\log,\mathfrak{p}} \cdot \begin{pmatrix} \kappa^\sharp \\ \kappa^\flat \end{pmatrix}$$

lands in $H^1_\alpha(K_\mathfrak{p}, \mathbf{T}\hat{\otimes}\mathcal{H}_L(\Gamma_\mathfrak{p})) \oplus H^1_\beta(K_\mathfrak{p}, \mathbf{T}\hat{\otimes}\mathcal{H}_L(\Gamma_\mathfrak{p}))$.

Proof. As an immediate consequence of (2.6), for each $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$ we have the factorizations

$$\begin{pmatrix} \mathcal{L}^\alpha_\mathfrak{q} \\ \mathcal{L}^\beta_\mathfrak{q} \end{pmatrix} = Q^{-1}_{\alpha,\beta} M_{\log,\mathfrak{q}} \cdot \begin{pmatrix} \mathrm{Col}^\sharp_\mathfrak{q} \\ \mathrm{Col}^\flat_\mathfrak{q} \end{pmatrix}, \tag{2.11}$$

which clearly implies the result. □

For each $\lambda \in \{\alpha, \beta\}$ set

$$\mathcal{L}^\lambda_{\mathbf{g},\mathfrak{p}} := \mathcal{L}_{\mathbf{g},f^\lambda} \circ \jmath_-^{-1}, \tag{2.12}$$

which by Lemma 2.6 is naturally defined on $H^1_\lambda(K_\mathfrak{p}, \mathbf{T}\hat{\otimes}\mathcal{H}_L(\Gamma_\mathfrak{p}))$.

Definition 2.8. The two-variable *signed logarithm maps* are the maps $(\mathrm{Log}^\sharp_\mathfrak{p}, \mathrm{Log}^\flat_\mathfrak{p})$ defined by the composition

$$\begin{aligned} H^1_\sharp(K_\mathfrak{p}, \mathbf{T}) \oplus H^1_\flat(K_\mathfrak{p}, \mathbf{T}) &\overset{Q^{-1}_{\alpha,\beta}M_{\log,\mathfrak{p}}}{\longrightarrow} H^1_\alpha(K_\mathfrak{p}, \mathbf{T}\hat{\otimes}\mathcal{H}_L(\Gamma_\mathfrak{p})) \oplus H^1_\beta(K_\mathfrak{p}, \mathbf{T}\hat{\otimes}\mathcal{H}_L(\Gamma_\mathfrak{p})) \\ &\overset{\mathcal{L}^\alpha_{\mathbf{g},\mathfrak{p}}\oplus\mathcal{L}^\beta_{\mathbf{g},\mathfrak{p}}}{\longrightarrow} I_\mathbf{g}\otimes\mathcal{H}_{\hat{F}_\infty}(\Gamma_K)^{\oplus 2}, \end{aligned}$$

using Lemma 2.6 for the first arrow.

2.3. **Doubly-signed Selmer groups.** Let Σ be a finite set of places of K containing those dividing $Np\infty$, and let $\mathfrak{G}_{K,\Sigma}$ be the Galois group of the maximal extension of F unramified outside the places above Σ. Recall that module $\mathbf{T}$ introduced in §2.1, and set

$$\mathbf{A} := \mathbf{T} \otimes \Lambda(\Gamma_K)^\vee,$$

where $\Lambda(\Gamma_K)^\vee$ is the Pontrjagin dual of $\Lambda(\Gamma_K)$. We shall also need to consider the modules $\mathbf{T}^{\rm ac}$, $\mathbf{A}^{\rm ac}$, $\mathbf{T}^{\rm cyc}$, and $\mathbf{A}^{\rm cyc}$, obtained by replacing Γ_K in the preceding definitions by the Galois group $\Gamma^{\rm ac}$ and $\Gamma^{\rm cyc}$ of the anticyclotomic and the cyclotomic $\mathbf{Z}_p$-extension of K, respectively.

In the following definitions, we let $\mathbf{M}$ denote either of the modules $\mathbf{T}$, $\mathbf{T}^{\rm ac}$, or $\mathbf{T}^{\rm cyc}$.

Definition 2.9. The p-relaxed Selmer group of $\mathbf{M}$ is

$$\mathfrak{Sel}^{\{p\}}(K,\mathbf{M}) := \ker\left\{ H^1(\mathfrak{G}_{K,\Sigma},\mathbf{M}) \longrightarrow \bigoplus_{v\in\Sigma\smallsetminus\{p\}} \frac{H^1(K_v,\mathbf{M})}{H^1_{\rm ur}(K_v,\mathbf{M})}\right\},$$

where

$$H^1_{\rm ur}(K_v,\mathbf{M}) := \ker\{H^1(K_v,\mathbf{M}) \longrightarrow H^1(K_v^{\rm ur},\mathbf{M})\}$$

is the unramified local condition. On the other hand, the p-related *strict* Selmer group of $\mathbf{M}$ is

$$\mathrm{Sel}^{\{p\}}(K,\mathbf{M}) := \ker\left\{ H^1(\mathfrak{G}_{K,\Sigma},\mathbf{M}) \longrightarrow \bigoplus_{v\in\Sigma\smallsetminus\{p\}} H^1(K_v,\mathbf{M})\right\}.$$

Our Selmer groups of interest in this paper are obtained from cutting the p-relaxed ones by various local conditions at the primes above p.

Definition 2.10. For $\mathfrak{q} \in \{\mathfrak{p},\overline{\mathfrak{p}}\}$ and $\mathscr{L}_\mathfrak{q} \in \{\mathrm{rel},\mathrm{str},\sharp,\flat\}$, set

$$H^1_{\mathscr{L}_\mathfrak{q}}(K_\mathfrak{q},\mathbf{M}) = \begin{cases} H^1(K_\mathfrak{q},\mathbf{M}) & \text{if } \mathscr{L}_\mathfrak{q} = \mathrm{rel}, \\ \{0\} & \text{if } \mathscr{L}_\mathfrak{q} = \mathrm{str}, \\ \ker(\mathrm{Col}^\bullet_\mathfrak{q}) & \text{if } \mathscr{L}_\mathfrak{q} = \bullet \in \{\sharp,\flat\}, \end{cases}$$

where $\mathrm{Col}^\bullet_\mathfrak{q}$ is the signed Coleman map of (2.7) (or its anticyclotomic or cyclotomic projection), and for $\mathscr{L} = \{\mathscr{L}_\mathfrak{p},\mathscr{L}_{\overline{\mathfrak{p}}}\}$, define

$$\mathfrak{Sel}^{\mathscr{L}}(K,\mathbf{M}) := \ker\left\{ \mathrm{Sel}^{\{p\}}(K,\mathbf{M}) \longrightarrow \bigoplus_{\mathfrak{q}\in\{\mathfrak{p},\overline{\mathfrak{p}}\}} \frac{H^1(K_\mathfrak{q},\mathbf{M})}{H^1_{\mathscr{L}_\mathfrak{q}}(K_\mathfrak{q},\mathbf{M})}\right\},$$

and similarly for $\mathrm{Sel}^{\mathscr{L}}(K,\mathbf{M})$

Thus, for example, $\mathrm{Sel}^{\mathrm{rel},\mathrm{str}}(K,\mathbf{M})$ is the submodule of $\mathrm{Sel}^{\{p\}}(K,\mathbf{M})$ consisting of classes which are trivial at $\overline{\mathfrak{p}}$ (with no condition at $\mathfrak{p}$).

On the other hand, letting $\mathbf{W}$ denote either of the modules $\mathbf{A}$, $\mathbf{A}^{\rm ac}$, or $\mathbf{A}^{\rm cyc}$, we define $\mathfrak{Sel}^{\{p\}}(K,\mathbf{W})$ and $\mathrm{Sel}^{\{p\}}(K,\mathbf{W})$ just as in Definition 2.9, and set

$$\mathfrak{Sel}^{\mathscr{L}}(K,\mathbf{W}) := \ker\left\{ \mathfrak{Sel}^{\{p\}}(K,\mathbf{W}) \longrightarrow \bigoplus_{\mathfrak{q}\in\{\mathfrak{p},\overline{\mathfrak{p}}\}} \frac{H^1(K_\mathfrak{q},\mathbf{W})}{H^1_{\mathscr{L}_\mathfrak{q}}(K_\mathfrak{q},\mathbf{M})^\perp}\right\},$$

where $H^1_{\mathscr{L}_\mathfrak{q}}(K_\mathfrak{q},\mathbf{M})^\perp$ is the orthogonal complement of $H^1_{\mathscr{L}_\mathfrak{q}}(K_\mathfrak{q},\mathbf{M})$ under local Tate duality, and define $\mathrm{Sel}^{\mathscr{L}}(K,\mathbf{W}) \subset \mathrm{Sel}^{\{p\}}(K,\mathbf{W})$ in the same manner. Finally, we let

$$\mathfrak{X}^{\mathscr{L}}_{K_\infty}(f) := \mathrm{Hom}_{\mathbf{Z}_p}(\mathfrak{Sel}^{\mathscr{L}}(K,\mathbf{A}),\mathbf{Q}_p/\mathbf{Z}_p)$$

be the Pontrjagin dual of $\mathfrak{Sel}^{\mathscr{L}}(K,\mathbf{A})$, and similarly define $\mathfrak{X}^{\mathscr{L}}_{K^{\rm ac}_\infty}(f)$, $\mathfrak{X}^{\mathscr{L}}_{K^{\rm cyc}_\infty}(f)$; and their strict analogues $X^{\mathscr{L}}_{K_\infty}(f)$, $X^{\mathscr{L}}_{K^{\rm ac}_\infty}(f)$, and $X^{\mathscr{L}}_{K^{\rm cyc}_\infty}(f)$.

Remark 2.11. Later in the paper (see esp. §§5.2-3) we shall also need to consider signed Selmer groups $\mathrm{Sel}^\bullet(\mathbf{Q}, \mathbf{T}^{\mathrm{cyc}})$ and $X^\bullet_{\mathbf{Q}_\infty}(f)$ for the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, whose definition (with the strict condition at the places outside p) is the obvious one from the above.

3. Two-variable Iwasawa theory

3.1. **Signed Beilinson–Flach classes.** We keep the notations introduced in the §2. As in [LZ16, §2.2], for each $h \in \mathbf{R}_{\geqslant 0}$ we let $D_h(\Gamma, T_f^*)$ denote the space of T_f^*-valued distributions on Γ of order h, so that in particular we have

$$H^1(\mathbf{Q}, D_0(\Gamma, T_f^*)\hat{\otimes} M(\mathbf{g})^*) \simeq H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T_f^*\hat{\otimes} M(\mathbf{g})^*)[1/p]$$

by [LZ16, Prop. 2.4.2]. For each $\lambda \in \{\alpha, \beta\}$, set $v_\lambda := \mathrm{ord}_p(\lambda)$ and denote by

$$\mathcal{BF}^{\lambda,\mathbf{g}} \in H^1(\mathbf{Q}, D_{v_\lambda}(\Gamma, T_f^*)\hat{\otimes} M(\mathbf{g})^*) \tag{3.1}$$

the class $\mathrm{BF}_m^{\lambda,\mathbf{g}}$ of [BL16, Thm. 3.2] for $m = 1$; up to a p-adic multiplier, this corresponds to the image of the Beilinson–Flach class ${}_c\mathcal{BF}_{1,0}^{\mathcal{F}^\lambda,\mathbf{g}}$ of [LZ16, §5.4] under the map on cohomology induced by the specialization $\mathcal{F}^\lambda \to f^\lambda$, with the auxiliary factor c disposed of.

By Shapiro's lemma and [LZ16, Prop. 2.4.2], the classes (3.1) may be identified with classes

$$\mathcal{BF}^\lambda \in H^1(K, \mathbf{T})\hat{\otimes}\mathcal{H}_{L,v_\lambda}(\Gamma_\mathfrak{p}),$$

where we recall that $\mathbf{T} := T_f^*\hat{\otimes}\Lambda(\Gamma_K)^\iota$.

Theorem 3.1 (Büyükboduk–Lei)**.** *There exist classes*

$$\mathcal{BF}^\sharp,\ \mathcal{BF}^\flat \in H^1(K, \mathbf{T})[1/p]$$

such that

$$\begin{pmatrix} \mathcal{BF}^\alpha \\ \mathcal{BF}^\beta \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log,\mathfrak{p}} \cdot \begin{pmatrix} \mathcal{BF}^\sharp \\ \mathcal{BF}^\flat \end{pmatrix},$$

where $Q_{\alpha,\beta} = \left(\begin{smallmatrix} \alpha & -\beta \\ -p & p \end{smallmatrix}\right)$ *and* $M_{\log,\mathfrak{p}} \in M_{2\times 2}(\mathcal{H}_L(\Gamma_\mathfrak{p}))$ *is the logarithmic matrix of Definition 2.1 with* $\Gamma_\mathfrak{p}$ *in place of* Γ.

Proof. This follows from [BL16, Thm. 3.5] and the discussion in [*loc.cit.*, §3.3]. □

In the following, we fix one for all a nonzero element $c \in \mathcal{O}_L$ such that the classes

$$\mathcal{BF}_c^\bullet := c \cdot \mathcal{BF}^\bullet$$

land in $H^1(K, \mathbf{T})$ for both $\bullet \in \{\sharp, \flat\}$. (The particular choice of c is irrelevant, since it will eventually cancel out with its contribution elsewhere.)

3.2. **Explicit reciprocity laws, I.** In this section, we consider the local images at $\overline{\mathfrak{p}}$ of the Beilinson–Flach classes.

By [LZ16, Thm. 7.1.2], for each $\lambda \in \{\alpha, \beta\}$ the image of $\mathrm{res}_p(\mathcal{BF}^{\lambda,\mathbf{g}})$ under the composite map

$$\begin{aligned} H^1(\mathbf{Q}_p, D_{v_\lambda}(\Gamma, T_f^*)\hat{\otimes} M(\mathbf{g})^*) &\longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathbf{D}^\dagger_{\mathrm{rig}}(V_f^*)\hat{\otimes}\mathbf{D}^\dagger_{\mathrm{rig}}(M(\mathbf{g})^*)) \\ &\longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^-\mathbf{D}^\dagger_{\mathrm{rig}}(V_f^*)\hat{\otimes}\mathbf{D}^\dagger_{\mathrm{rig}}(M(\mathbf{g})^*)), \end{aligned}$$

where the first arrow is given by [LZ16, Cor. 6.1.4] and the second one in the natural projection, lies in the image of the natural map

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^{-+}D(f \otimes \mathbf{g})) \longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^-\mathbf{D}^\dagger_{\mathrm{rig}}(V_f^*)\hat{\otimes}\mathbf{D}^\dagger_{\mathrm{rig}}(M(\mathbf{g})^*)).$$

Thus letting $\mathrm{res}_{\overline{\mathfrak{p}}}(\mathcal{BF}^{\lambda}) \in H^1(K_{\overline{\mathfrak{p}}}, \mathbf{T})\hat{\otimes}\mathcal{H}_{L,v_\lambda}(\Gamma_{\mathfrak{p}})$ be the image of $\mathrm{res}_p(\mathcal{BF}^{\lambda,\mathbf{g}})$ under the identification $\jmath^+$ of (2.9), by Proposition 2.4 we may consider the element

$$\mathfrak{L}_p^{\lambda,\mu}(f/K) := \mathcal{L}_{\overline{\mathfrak{p}}}^{\mu}(\mathrm{res}_{\overline{\mathfrak{p}}}(\mathcal{BF}^{\lambda})) \tag{3.2}$$

in $\mathcal{H}_{L,v_\lambda}(\Gamma_{\mathfrak{p}})\hat{\otimes}\mathcal{H}_{\hat{F}_\infty,v_\mu}(\Gamma_{\overline{\mathfrak{p}}}) \subseteq \mathcal{H}_{\hat{F}_\infty}(\Gamma_K)$ for each $\mu \in \{\alpha, \beta\}$.

Remark 3.2. For $\bullet, \circ \in \{\sharp, \flat\}$, setting

$$\mathfrak{L}_p^{\bullet,\circ}(f/K) := \mathrm{Col}_{\overline{\mathfrak{p}}}^{\circ}(\mathrm{res}_{\overline{\mathfrak{p}}}(\mathcal{BF}_c^{\bullet})), \tag{3.3}$$

the factorizations in (2.11) and Theorem 3.1 imply that

(3.4)

$$\begin{pmatrix} c\mathfrak{L}_p^{\alpha,\alpha}(f/K) & c\mathfrak{L}_p^{\beta,\alpha}(f/K) \\ c\mathfrak{L}_p^{\alpha,\beta}(f/K) & c\mathfrak{L}_p^{\beta,\beta}(f/K) \end{pmatrix} = Q_{\alpha,\beta}^{-1}M_{\log,\overline{\mathfrak{p}}} \cdot \begin{pmatrix} \mathfrak{L}_p^{\sharp,\sharp}(f/K) & \mathfrak{L}_p^{\flat,\sharp}(f/K) \\ \mathfrak{L}_p^{\sharp,\flat}(f/K) & \mathfrak{L}_p^{\flat,\flat}(f/K) \end{pmatrix} \cdot (Q_{\alpha,\beta}^{-1}M_{\log,\mathfrak{p}})^{\intercal},$$

where $A^{\intercal}$ denotes the transpose of a matrix A.

By following explicit reciprocity law of Loeffler–Zerbes [LZ16], the p-adic L-functions (3.2) interpolate critical values for the Rankin–Selberg convolution of f with theta series of Hecke characters of K.

Theorem 3.3 (Loeffler–Zerbes). *For every $(\alpha_{\mathfrak{p}}, \alpha_{\overline{\mathfrak{p}}}) = (\lambda, \mu) \in \{\alpha, \beta\}^{\oplus 2}$, if $\chi : \Gamma_K \to \mu_{p^\infty}$ is any finite order character of conductor $\mathfrak{c}_\chi$, then*

$$\chi(\mathfrak{L}_p^{\lambda,\mu}(f/K)) = \left(\prod_{\mathfrak{q}|p} \alpha_{\mathfrak{q}}^{-v_{\mathfrak{q}}(\mathfrak{c}_\chi)}\right) \cdot \frac{\mathcal{E}(f,\chi)}{\mathfrak{g}(\chi)\cdot|\mathfrak{c}_\chi|^{1/2}} \cdot \frac{L(f/K,\chi,1)}{(4\pi)^2\cdot\langle f,f\rangle_N},$$

where

$$\mathcal{E}(f,\chi) = \prod_{\mathfrak{q}|p,\ \mathfrak{q}\nmid\mathfrak{c}_\chi} (1-\alpha_{\mathfrak{q}}^{-1}\chi(\mathfrak{q}))(1-\alpha_{\mathfrak{q}}^{-1}\chi^{-1}(\mathfrak{q})).$$

Proof. This follows from [LZ16, Thm. 7.1.5], as complemented by [Loe17]. □

Corollary 3.4. *For each $\lambda, \mu \in \{\alpha, \beta\}$, the p-adic L-function $\mathfrak{L}_p^{\lambda,\mu}(f/K)$ is nonzero.*

Proof. Immediate from Theorem 3.3 and Rohrlich's nonvanishing results (see [Roh88]). □

By the factorization in (3.7), Corollary 3.4 implies that *at least one* of the p-adic L-functions $\mathfrak{L}_p^{\bullet,\circ}(f/K)$ is nonzero. As we shall see belowe, by considering their restriction to the cyclotomic line, it is possible to show that in fact one can always can find some nonzero $\mathfrak{L}_p^{\bullet,\circ}(f/K)$ with $\bullet = \circ$.

Let Γ^{cyc} be the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of K, which we shall also see as the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, and denote by $\mathfrak{L}_p^{\bullet,\circ}(f/K)_{\mathrm{cyc}}$ the image of $\mathfrak{L}_p^{\bullet,\circ}(f/K)$ under the natural projection $\Lambda(\Gamma_K) \to \Lambda(\Gamma^{\mathrm{cyc}})$. By [LLZ10, Thm. 3.25], there exist elements $L_p^{\bullet}(f) \in \Lambda_L(\Gamma^{\mathrm{cyc}})$ such that

$$\begin{pmatrix} L_p^{\alpha}(f) \\ L_p^{\beta}(f) \end{pmatrix} = Q_{\alpha,\beta}^{-1}M_{\log} \cdot \begin{pmatrix} L_p^{\sharp}(f) \\ L_p^{\flat}(f) \end{pmatrix}, \tag{3.5}$$

where $L_p^{\lambda}(f) \in \mathcal{H}_{L,v_\lambda}(\Gamma^{\mathrm{cyc}})$ are the p-adic L-functions of Amice–Vélu and Vishik (see [MTT86] or [Pol03, §2]).

Proposition 3.5. *For each pair $\bullet, \circ \in \{\sharp, \flat\}$ we have*

$$\mathfrak{L}_p^{\bullet,\circ}(f/K)_{\mathrm{cyc}} = \frac{1}{2}(L_p^{\bullet}(f)\cdot L_p^{\circ}(f_K) + L_p^{\circ}(f)\cdot L_p^{\bullet}(f_K)),$$

where $f_K := f \otimes \epsilon_K$ is the twist of f by the quadratic character associated to K.

Remark 3.6. In particular, the equality in Proposition 3.5 implies (the otherwise clear fact) that $\mathfrak{L}_p^{\sharp,\flat}(f/K)_{\rm cyc} = \mathfrak{L}_p^{\flat,\sharp}(f/K)_{\rm cyc}$.

Sketch of proof. Let $\mathfrak{L}_p^{\lambda,\mu}(f/K)_{\rm cyc}$ denote the image of the p-adic L-function $\mathfrak{L}_p^{\lambda,\mu}(f/K)$ of (3.2) under the map induced by the natural projection $\Gamma_K \twoheadrightarrow \Gamma^{\rm cyc}$. From their respective interpolation properties, we see that

$$\mathfrak{L}_p^{\lambda,\mu}(f/K)_{\rm cyc} = \frac{1}{2}(L_p^{\lambda}(f)\cdot L_p^{\mu}(f_K) + L_p^{\mu}(f)\cdot L_p^{\lambda}(f_K)). \tag{3.6}$$

For $\lambda=\mu$, this is shown by proving the analogous factorization for the two-variable p-adic L-functions attached to the Coleman family passing through the λ-stabilization of f, using that for higher weights we have more critical values to compare the two sides, and specializing back to f. (See [Wan16a, Lem. 4.22], noting that the constant in *loc.cit.* is given by the ratio $\Omega_f^{\rm can}/\Omega_f^+\Omega_f^-$, which is a p-adic unit by [SZ14, Lem. 9.5] in our weight 2 case.)

For $\lambda\neq\mu$, one uses the existence of an element (constructed in ongoing work of Büyükboduk–Lei–Loeffler–Venkat) in $\bigwedge^2 H^1_{\rm Iw}(K_\infty, V_f^*)$ whose restriction to

$$\bigwedge^2 H^1_{\rm Iw}(K_\infty^{\rm cyc}, V_f^*) \simeq H^1_{\rm Iw}(\mathbf{Q}^{\rm cyc}, V_f^*)\otimes H^1_{\rm Iw}(\mathbf{Q}^{\rm cyc}, V_{f_K}^*)$$

agrees (by (3.6) in the cases $\lambda=\mu$) with the tensor product $\mathbf{z}_f\otimes\mathbf{z}_{f_K}$ of Kato's zeta elements. This agreement, together with Kato's explicit reciprocity law yields the asymmetric cases of (3.6).

Letting

$$\mathcal{L} := \begin{pmatrix} L_p^{\alpha}(f)L_p^{\alpha}(f_K) & L_p^{\alpha}(f)L_p^{\beta}(f_K) \\ L_p^{\beta}(f)L_p^{\alpha}(f_K) & L_p^{\beta}(f)L_p^{\beta}(f_K)\end{pmatrix},$$

by the decomposition (3.7), we thus have

$$Q_{\alpha,\beta}^{-1}M_{\log}\cdot\begin{pmatrix} \mathfrak{L}_p^{\sharp,\sharp}(f/K)_{\rm cyc} & \mathfrak{L}_p^{\flat,\sharp}(f/K)_{\rm cyc} \\ \mathfrak{L}_p^{\sharp,\flat}(f/K)_{\rm cyc} & \mathfrak{L}_p^{\flat,\flat}(f/K)_{\rm cyc}\end{pmatrix}\cdot(Q_{\alpha,\beta}^{-1}M_{\log})^{\intercal} = \frac{1}{2}(\mathcal{L}+\mathcal{L}^{\intercal}). \tag{3.7}$$

On the other hand, since p splits in K, we have $\epsilon_K(p)=1$, so the roots of the p-th Hecke of polynomial of f_K are the same as that of f. Thus taking the product of the matrices in the factorization (3.5) and the analogous factorization for f_K:

$$\begin{pmatrix} L_p^{\alpha}(f_K) \\ L_p^{\beta}(f_K)\end{pmatrix}^{\intercal} = \begin{pmatrix} L_p^{\sharp}(f_K) \\ L_p^{\flat}(f_K)\end{pmatrix}^{\intercal}\cdot(Q_{\alpha,\beta}^{-1}M_{\log})^{\intercal}, \tag{3.8}$$

we see that the above matrix $\mathcal{L}$ is also given by

$$\mathcal{L} = Q_{\alpha,\beta}^{-1}M_{\log}\cdot\begin{pmatrix} L_p^{\sharp}(f)L_p^{\sharp}(f_K) & L_p^{\sharp}(f)L_p^{\flat}(f_K) \\ L_p^{\flat}(f)L_p^{\sharp}(f_K) & L_p^{\flat}(f)L_p^{\flat}(f_K)\end{pmatrix}\cdot(Q_{\alpha,\beta}^{-1}M_{\log})^{\intercal}, \tag{3.9}$$

from where the result follows from substituting the expression for $\mathcal{L}$ in (3.9) into (3.7). ☐

Corollary 3.7. *If $\bullet\in\{\sharp,\flat\}$ is such that $L_p^{\bullet}(f)$ is nonzero, then the restriction $\mathfrak{L}_p^{\bullet,\bullet}(f/K)_{\rm cyc}$ is nonzero. In particular, $\mathfrak{L}_p^{\bullet,\bullet}(f/K)$ is nonzero.*

Proof. The interpolation property of $L_p^{\alpha}(f_K)$ and $L_p^{\beta}(f_K)$ and Rohrlich's nonvanishing results [Roh84] imply that $L_p^{\alpha}(f_K)$ and $L_p^{\beta}(f_K)$ are both nonzero; from (3.8), it follows that $L_p^{\circ}(f_K)$ is nonzero for at least one $\circ\in\{\sharp,\flat\}$. For the sake of the argument, say $\bullet=\sharp$ in the statement, i.e. $L_p^{\sharp}(f)\neq 0$, and for the sake of contradiction suppose that $L_p^{\sharp}(f_K)=0$. Then $L_p^{\flat}(f_K)\neq 0$, and Proposition 3.5 yields

$$\mathfrak{L}_p^{\sharp,\flat}(f/K)_{\rm cyc} = \frac{1}{2}(L_p^{\sharp}(f)\cdot L_p^{\flat}(f_K) + 0), \tag{3.10}$$

and both sides are nonzero. However, the left-hand side of (3.10) is unchanged by swapping $\sharp$ and $\flat$ (see Remark 3.6), whereas the right-hand side becomes zero. This contradiction shows that if $L_p^\sharp(f) \neq 0$, we must also have $L_p^\sharp(f_K) \neq 0$, and the result follows from Proposition 3.5. $\square$

3.3. **Explicit reciprocity laws, II.** In this section, we consider the images of the Beilinson–Flach classes under the restriction map at $\mathfrak{p}$.

As shown in [BL16, Cor. 3.12], for each $\bullet \in \{\sharp, \flat\}$ we have the inclusion

$$\mathrm{res}_{\mathfrak{p}}(\mathcal{BF}_c^\bullet) \in \ker(\mathrm{Col}_{\mathfrak{p}}^\bullet); \tag{3.11}$$

in particular, the signed logarithm map $\mathrm{Log}_{\mathfrak{p}}^\bullet$ introduced in §2.2 may be applied to this class.

From work of Hida–Tilouine [HT94] and Rubin [Rub91] (see e.g. [Cas17b, Thm. 2.7]) one can show that the congruence ideal $I_{\mathbf{g}}$ appearing in (2.10) is generated by $\frac{h_K}{w_K} \cdot L_{\mathfrak{p}}^{\mathrm{ac}}(K)$ up to "exceptional primes", where h_K is the class number of K, w_K the number of roots of unity of K, and $L_{\mathfrak{p}}^{\mathrm{ac}}(K)$ an anticyclotomic Katz p-adic L-function. Thus for each $\bullet \in \{\sharp, \flat\}$ the map

$$\widetilde{\mathrm{Log}}_{\mathfrak{p}}^\bullet := \frac{h_K}{w_K} \cdot L_{\mathfrak{p}}^{\mathrm{ac}}(K) \times \mathrm{Log}_{\mathfrak{p}}^\bullet$$

is integral up to exceptional primes.

For the next result, let $\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_L)$ be the residual representation associated with f, where κ_L is the residue field of L.

Theorem 3.8. *Assume that $\bar{\rho}_f$ is ramified at every prime $\ell \mid N$ which is nonsplit in K. Then for each $\bullet \in \{\sharp, \flat\}$ we have*

$$\widetilde{\mathrm{Log}}_{\mathfrak{p}}^\bullet(\mathrm{res}_{\mathfrak{p}}(\mathcal{BF}_c^\bullet)) = c\mathscr{L}_{\mathfrak{p}}(f/K)$$

up to exceptional primes, where $\mathscr{L}_{\mathfrak{p}}(f/K)$ is the two-variable Rankin–Selberg p-adic L-function constructed in [Wan16b, §4.6].

Proof. For each $\lambda \in \{\alpha, \beta\}$, let $L_{p,\lambda}(f/K) \in \mathrm{Frac}(\mathcal{H}_L(\Gamma_K))$ be the image of Urban's three-variable p-adic L-function $L_p(\mathbf{g}, \mathcal{F}^\lambda, 1+\mathbf{j})$ under the specialization map $\mathcal{F}^\lambda \to f^\lambda$, where as always $\mathcal{F}^\lambda$ denotes the Coleman family passing through to f^λ. By the explicit reciprocity law of [LZ16, Thm. 7.1.5] and [Loe17] we have the relation

$$\mathcal{L}_{\mathbf{g},\mathfrak{p}}^\lambda(\mathrm{res}_{\mathfrak{p}}(\mathcal{BF}_c^\lambda)) = cL_{p,\lambda}(f/K), \tag{3.12}$$

where $\mathcal{L}_{\mathbf{g},\mathfrak{p}}^\lambda$ is as in (2.12). By Definition 2.8 and the factorization of Theorem 3.1, the pair of equalities (3.12) for $\lambda = \alpha$ and β amounts to the equality

$$\begin{pmatrix} \mathrm{Log}_{\mathfrak{p}}^\sharp(\mathrm{res}_{\mathfrak{p}}(\mathcal{BF}_c^\sharp)) \\ \mathrm{Log}_{\mathfrak{p}}^\flat(\mathrm{res}_{\mathfrak{p}}(\mathcal{BF}_c^\flat)) \end{pmatrix} = \begin{pmatrix} cL_{p,\alpha}(f/K) \\ cL_{p,\beta}(f/K) \end{pmatrix}. \tag{3.13}$$

On the other hand, by the calculations in [JSW17, §5.3] we have the relation

$$\mathscr{L}_{\mathfrak{p}}(f/K) = \frac{h_K}{w_K} \cdot L_{\mathfrak{p}}^{\mathrm{ac}}(K) \cdot L_{p,\lambda}(f/K), \tag{3.14}$$

possibly up to powers of p coming the constant $\alpha(f, f_B)$ appearing in [*loc.cit.*, §5.1]. Since by the discussion in [Pra06, p. 912] our ramification hypothesis on $\bar{\rho}_f$ forces $\alpha(f, f_B)$ to be a p-adic unit, the result follows from (3.13) and (3.14). $\square$

3.4. **Main conjectures.** In this section, building on the explicit reciprocity laws of the preceding sections, we relate different variants of the two-variable Iwasawa main conjectures for modular forms at non-ordinary primes.

Theorem 3.9. *The following three statements are equivalent, where the equalities of characteristic ideals are up to exceptional primes.*

(1) *Both* $\mathrm{Sel}^{\mathrm{str},\mathrm{rel}}(K,\mathbf{T})$ *and* $X^{\mathrm{rel},\mathrm{str}}_{K_\infty}(f)$ *are* $\Lambda(\Gamma_K)$*-torsion, and*
$$\mathrm{Char}_{\Lambda(\Gamma_K)}(X^{\mathrm{rel},\mathrm{str}}_{K_\infty}(f)) = (\mathscr{L}_{\mathfrak{p}}(f/K))$$
as ideals in $\Lambda_{R_0}(\Gamma_K)$.

(2) *For all* $\bullet\in\{\sharp,\flat\}$, $X^{\bullet,\mathrm{str}}_{K_\infty}(f)$ *is* $\Lambda(\Gamma_K)$*-torsion,* $\mathrm{Sel}^{\bullet,\mathrm{rel}}(K,\mathbf{T})$ *has* $\Lambda(\Gamma_K)$*-rank* 1*, and*
$$c\cdot\mathrm{Char}_{\Lambda(\Gamma_K)}(X^{\bullet,\mathrm{str}}_{K_\infty}(f)) = \mathrm{Char}_{\Lambda(\Gamma_K)}\left(\frac{\mathrm{Sel}^{\bullet,\mathrm{rel}}(K,\mathbf{T})}{\Lambda(\Gamma_K)\cdot\mathcal{BF}^\bullet_c}\right)$$
as ideals in $\Lambda(\Gamma_K)$.

(3) *If* $\bullet,\circ\in\{\sharp,\flat\}$ *are that* $\mathfrak{L}^{\bullet,\circ}_p(f/K)$ *is nonzero, then* $X^{\bullet,\circ}_{K_\infty}(f)$ *is* $\Lambda(\Gamma_K)$*-torsion, and*
$$c\cdot\mathrm{Char}_{\Lambda(\Gamma_K)}(X^{\bullet,\circ}_{K_\infty}(f)) = (\mathfrak{L}^{\bullet,\circ}_p(f/K))$$
as ideals in $\Lambda(\Gamma_K)$.

Proof. We shall just prove the implications (1) ⇒ (2) ⇒ (3), since the converse implications are shown in the same way. Poitou–Tate duality gives rise to the exact sequence
(3.15)
$$0\longrightarrow \mathrm{Sel}^{\mathrm{str},\mathrm{rel}}(K,\mathbf{T})\longrightarrow \mathrm{Sel}^{\bullet,\mathrm{rel}}(K,\mathbf{T})\xrightarrow{\mathrm{res}_{\mathfrak{p}}} H^1_\bullet(K_{\mathfrak{p}},\mathbf{T})\longrightarrow X^{\mathrm{rel},\mathrm{str}}_{K_\infty}(f)\longrightarrow X^{\bullet,\mathrm{str}}_{K_\infty}(f)\longrightarrow 0.$$

Assume that (1) holds. Since $\mathscr{L}_{\mathfrak{p}}(f/K)$ is nonzero (see [Cas17b, Rem. 1.3]), by Theorem 3.8 the image of $\mathrm{res}_{\mathfrak{p}}$ is not $\Lambda(\Gamma_K)$-torsion, and since $H^1_\bullet(K_{\mathfrak{p}},\mathbf{T})$ has $\Lambda(\Gamma_K)$-rank 1, it follows from (3.15) that $\mathrm{Sel}^{\bullet,\mathrm{rel}}(K,\mathbf{T})$ has also $\Lambda(\Gamma_K)$-rank 1. Since $\bar{\rho}_f$ is irreducible by [Edi92], the $\Lambda(\Gamma_K)$-torsion in $H^1(K,\mathbf{T})$ is zero (see e.g. [Cas17b, Lem. 2.6]), and so our assumption implies that $\mathrm{Sel}^{\mathrm{str},\mathrm{rel}}(K,\mathbf{T})=\{0\}$. From (3.15) we thus arrive at the exact sequence

$$0\longrightarrow\frac{\mathrm{Sel}^{\bullet,\mathrm{rel}}(K,\mathbf{T})}{\Lambda(\Gamma_K)\cdot\mathcal{BF}^\bullet_c}\longrightarrow\frac{\mathrm{Im}(\widetilde{\mathrm{Log}}^\bullet_{\mathfrak{p}})}{(c\cdot\mathscr{L}_{\mathfrak{p}}(f/K))}\longrightarrow X^{\mathrm{rel},\mathrm{str}}_{K_\infty}(f)\longrightarrow X^{\bullet,\mathrm{str}}_{K_\infty}(f)\longrightarrow 0, \tag{3.16}$$

using Theorem 3.8 for the second term; in particular, it follows that $X^{\bullet,\mathrm{str}}_{K_\infty}(f)$ is $\Lambda(\Gamma_K)$-torsion. Since the map $\widetilde{\mathrm{Log}}^\bullet_{\mathfrak{p}}$ has pseudo-null cokernel, the conclusion in part (2) follows after taking characteristic ideals in (3.16).

Assume now that the hypotheses in (2) hold, let $\bullet,\circ\in\{\sharp,\flat\}$ be such that $\mathfrak{L}^{\bullet,\circ}_p(f/K)$ is nonzero, and consider the exact sequence
(3.17)
$$0\longrightarrow\mathrm{Sel}^{\bullet,\circ}(K,\mathbf{T})\longrightarrow\mathrm{Sel}^{\bullet,\mathrm{rel}}(K,\mathbf{T})\xrightarrow{\mathrm{res}_{\overline{\mathfrak{p}}}}\frac{H^1(K_{\overline{\mathfrak{p}}},\mathbf{T})}{H^1_\circ(K_{\overline{\mathfrak{p}}},\mathbf{T})}\longrightarrow X^{\bullet,\circ}_{K_\infty}(f)\longrightarrow X^{\bullet,\mathrm{str}}_{K_\infty}(f)\longrightarrow 0.$$

Since $\mathcal{BF}^\bullet$ lands in $\mathrm{Sel}^{\bullet,\mathrm{rel}}(K,\mathbf{T})$ by [BL16, Cor. 3.12] and $\mathfrak{L}^{\bullet,\circ}_p(f/K)$ is nonzero by hypothesis, the image of $\mathrm{res}_{\overline{\mathfrak{p}}}$ is not $\Lambda(\Gamma_K)$-torsion (see (3.3)), and since $H^1(K_{\overline{\mathfrak{p}}},\mathbf{T})/H^1_\circ(K_{\overline{\mathfrak{p}}},\mathbf{T})$ has $\Lambda(\Gamma_K)$-rank 1, we deduce from (3.17) that both $\mathrm{Sel}^{\bullet,\circ}(K,\mathbf{T})$ and $X^{\bullet,\circ}_{K_\infty}(f)$ are $\Lambda(\Gamma_K)$-torsion. Since $\mathrm{Sel}^{\bullet,\circ}(K,\mathbf{T})$ is then forced to vanish similarly as before, we arrive at the exact sequence

$$0\longrightarrow\frac{\mathrm{Sel}^{\bullet,\mathrm{rel}}(K,\mathbf{T})}{\Lambda(\Gamma_K)\cdot\mathcal{BF}^\bullet_c}\longrightarrow\frac{\mathrm{Im}(\mathrm{Col}^\circ_{\overline{\mathfrak{p}}})}{(\mathfrak{L}^{\bullet,\circ}_p(f/K))}\longrightarrow X^{\bullet,\circ}_{K_\infty}(f)\longrightarrow X^{\bullet,\mathrm{str}}_{K_\infty}(f)\longrightarrow 0, \tag{3.18}$$

using (3.3) for the second term. Since $\mathrm{Col}^\circ_{\overline{\mathfrak{p}}}$ has pseudo-null cokernel by Proposition 2.3, the conclusion in part (3) now follows after taking characteristic ideals in (3.18). □

4. Anticyclotomic Iwasawa theory

Throughout this section, we let $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$ be a newform, and $p > 2$ be a prime of good non-ordinary reduction for f as before. The imaginary quadratic field K (in which we continue to assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ split) determines a factorization:

- $N = N^+N^-$ with $(N^+, N^-) = 1$,
- $\ell \mid N^+$ if and only if ℓ is split or ramified in K,
- $\ell \mid N^-$ if and only if ℓ is inert in K.

We shall assume that

$$N^- \text{ is square-free,}$$

and say that the pair (f, K) is *indefinite* (resp. *definite*) if N^- is the product of an even (resp. odd) number of primes; if π is the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated with f and $\epsilon(\pi, K, s)$ is the epsilon-factor of the base change of π to $\mathrm{GL}_2(\mathbb{A}_K)$, we then have $\epsilon(\pi, K, \frac{1}{2}) = -1$ (resp. $\epsilon(\pi, K, \frac{1}{2}) = +1$).

4.1. **Signed Heegner points.** Assume that (f, K) is an indefinite pair in the above sense, so that N^- is the square-free product of an *even* number of primes. Let $X = X_{N^+,N^-}$ be the Shimura curve (with the cusps added if $N^- = 1$) over $\mathbf{Q}$ attached to the quaternion algebra $B/\mathbf{Q}$ of discriminant N^- and an Eichler order $R \subset \mathcal{O}_B$ of level N^+. We embed X into its Jacobian $J(X)$ by choosing an auxiliary prime $\ell_0 \nmid Np$ and defining

$$\iota_{\ell_0} : X \longrightarrow J(X)$$

by $x \mapsto (T_{\ell_0} - \ell_0 - 1)[x]$, where T_{ℓ_0} is the usual Hecke correspondence on X, and $[x] \in \mathrm{Div}(X)$ is the divisor class of $x \in X$. Attached to f is an isogeny class of abelian variety quotients:

$$\pi_f : J(X) \longrightarrow A_f. \tag{4.1}$$

Let κ_L be the residue field of L, and denote by $\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_L)$ the reduction of ρ_f modulo the maximal ideal of $\mathcal{O}_L$. Letting $K_f \subseteq \mathbf{C}$ be the Hecke field of f, and $\mathcal{O}_f \subseteq K_f$ be its ring of integers, we may assume $L = K_{f,\mathfrak{P}}$, where $\mathfrak{P}$ is the prime of K_f above p induced by our fixed isomorphism $\mathbf{C} \simeq \mathbf{C}_p$. Then T_f^* corresponds to the $\mathfrak{P}$-adic Tate module of A_f, and up to changing A_f within its isogeny class, we may and will assume that $\mathcal{O}_f \hookrightarrow \mathrm{End}_{\mathbf{Q}}(A_f)$.

For each integer $m > 0$, let $\mathcal{O}_m = \mathbf{Z} + m\mathcal{O}_K$ be the order of K of conductor m, and denote by $K[m]$ the ring class field of K of that conductor, so that $\mathrm{Gal}(K[m]/K) \simeq \mathrm{Pic}(\mathcal{O}_m)$ under the Artin reciprocity map.

Proposition 4.1. *There is a collection of Heegner points $h[m] \in A_f(K[m])$, indexed by the positive integers m prime to ND_K, such that*

$$[\mathcal{O}_m^\times : \mathcal{O}_{m\ell}^\times] \cdot \mathrm{Norm}_{K[m]}^{K[m\ell]}(h[m\ell]) = \begin{cases} a_\ell \cdot h[m] & \text{if } \ell \nmid m \text{ is inert in } K, \\ (a_\ell - \sigma_{\mathfrak{l}} - \sigma_{\bar{\mathfrak{l}}}) \cdot h[m] & \text{if } \ell = \mathfrak{l}\bar{\mathfrak{l}} \nmid m \text{ splits in } K, \\ a_\ell \cdot h[m] - h[m/\ell] & \text{if } \ell \mid m, \end{cases}$$

where $\sigma_{\mathfrak{l}}, \sigma_{\bar{\mathfrak{l}}} \in \mathrm{Gal}(K[m]/K)$ are the Frobenius elements at $\mathfrak{l}, \bar{\mathfrak{l}}$, respectively.

Proof. This is standard, letting $h([m]))$ be the image under the composition

$$X \xrightarrow{\iota_{\ell_0}} J(X) \xrightarrow{\pi_f} A_f$$

of certain canonical CM points $\tilde{h}([m]) \in X(K[m])$, see e.g. [How04b, Prop. 1.3.2]. □

From now on, we choose the prime ℓ_0 above so that $a_{\ell_0} - \ell_0 - 1$ is a unit in $\mathcal{O}_L^\times$, and let

$$y[m] \in H^1(K[m], T_f^*)$$

be the image of $h[m] \otimes (a_{\ell_0}(f) - \ell_0 - 1)^{-1} \in A_f(K[m]) \otimes \mathcal{O}_L$ under the Kummer map. (Note that $y[m]$ is independent of the choice of ℓ_0 by construction.)

For every integer $m > 0$ prime to NDp, let $K_n^{\mathrm{ac}}[m]$ denote the compositum $K_n^{\mathrm{ac}} K[m]$, and define

$$\mathfrak{Y}_n[m] := \mathrm{cor}_{K_n^{\mathrm{ac}}[m]}^{K[mp^{k(n)}]}(y[mp^{k(n)}]), \tag{4.2}$$

where $k(n) := \min\{k \mid K_n^{\mathrm{ac}} \subset K[p^k]\}$.

Definition 4.2. Let $\lambda \in \{\alpha, \beta\}$ be a root of $X^2 - a_p X + p$, and let $m > 0$ be an integer prime to NDp. The *λ-stabilized Heegner class*

$$\mathfrak{Z}_n[m]^\lambda \in H^1(K_n^{\mathrm{ac}}[m], T_f^*)$$

is defined by

$$\mathfrak{Z}_n[m]^\lambda := \begin{cases} \mathfrak{Y}_n[m] - \frac{1}{\lambda} \cdot \mathfrak{Y}_{n-1}[m] & \text{if } n > 0, \\ \frac{1}{u_K}\left(1 - \frac{1}{\lambda}\sigma_{\mathfrak{p}}\right)\left(1 - \frac{1}{\lambda}\sigma_{\overline{\mathfrak{p}}}\right) \cdot \mathfrak{Y}_0[m] & \text{if } n = 0, \end{cases}$$

where $u_K := |\mathcal{O}_K^\times|/2$.

Letting cor_{n-1}^n denote the corestriction map for the extension $K_n^{\mathrm{ac}}[m]/K_{n-1}^{\mathrm{ac}}[m]$, an immediate calculation using Proposition 4.1 reveals that

$$\mathrm{cor}_{n-1}^n(\mathfrak{Z}_n[m]^\lambda) = \lambda \cdot \mathfrak{Z}_{n-1}[m]^\lambda \tag{4.3}$$

for all $n > 0$. Let $\Gamma^{\mathrm{ac}} = \mathrm{Gal}(K_\infty^{\mathrm{ac}}/K)$ be the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of K, and set

$$\mathbf{T}^{\mathrm{ac}} := T_f^* \hat{\otimes} \Lambda(\Gamma^{\mathrm{ac}})^\iota,$$

where $\Lambda(\Gamma^{\mathrm{ac}})^\iota$ is the module $\Lambda(\Gamma^{\mathrm{ac}})$ equipped with the G_K-action given by the inverse of the tautological character $G_K \twoheadrightarrow \Gamma^{\mathrm{ac}} \hookrightarrow \Lambda(\Gamma^{\mathrm{ac}})^\times$.

Proposition 4.3. *For each $\lambda \in \{\alpha, \beta\}$ there exists a unique element*

$$\mathfrak{Z}[m]^\lambda \in H^1(K[m], \mathbf{T}^{\mathrm{ac}}) \hat{\otimes} \mathcal{H}_{L,v_\lambda}(\Gamma^{\mathrm{ac}}),$$

where $v_\lambda := \mathrm{ord}_p(\lambda)$, whose image in $H^1(K_n^{\mathrm{ac}}[m], V_f^*)$ is $\lambda^{-n} \cdot \mathfrak{Z}_n[m]^\lambda$ for all $n \geqslant 0$.

Proof. Since f has weight 2 and p is non-ordinary for f, we have $v_\lambda < 1$, and the result follows from (4.3) and [LLZ14, Prop. A.2.10]. □

In order to construct from the pair of unbounded Heegner classes of Proposition 4.3 a pair of "signed" Heegner classes with bounded growth over the anticyclotomic tower, we will need the following variant of a useful lemma from [BL16]. Let $\gamma_{\mathrm{ac}} \in \Gamma^{\mathrm{ac}}$ be a topological generator, and denote by $M_{\log,\mathrm{ac}} \in M_{2\times 2}(\mathcal{H}_L(\Gamma^{\mathrm{ac}}))$ the logarithmic matrix $M_{\log}$ of Definition 2.1 with Γ^{ac} in place of Γ.

Lemma 4.4. *Suppose that for every $\lambda \in \{\alpha, \beta\}$ there exists $F^\lambda \in \mathcal{H}_{L,v_\lambda}(\Gamma^{\mathrm{ac}})$ such that for all finite order character ϕ of Γ^{ac} of conductor $p^n > 1$, we have*

$$\phi(F^\lambda) = \lambda^{-n} \cdot c_\phi$$

for some $c_\phi \in \overline{\mathbf{Q}}_p$ independent of λ. Then there exist elements $F^\sharp, F^\flat \in \Lambda_L(\Gamma^{\mathrm{ac}})$ such that

$$\begin{pmatrix} F^\alpha \\ F^\beta \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log,\mathrm{ac}} \cdot \begin{pmatrix} F^\sharp \\ F^\flat \end{pmatrix}.$$

Moreover, if there exists a sequence of polynomials $P_{n,\lambda} \in \mathcal{O}_L[(\gamma_{\mathrm{ac}} - 1)]$ such that

$$F_\lambda \equiv \lambda^{-n} \cdot P_{\lambda,n} \pmod{\gamma_{\mathrm{ac}}^{p^n} - 1},$$

then there exists a nonzero $C \in \mathcal{O}_L$ depending only on λ such that $F^\sharp, F^\flat \in C^{-1}\Lambda(\Gamma^{\mathrm{ac}})$.

Remark 4.5. For the sake of clarity, let us remark that the last statement in Lemma 4.4 means that if G^α and G^β are *any* other pair of elements as in the statement, the corresponding $G^\sharp, G^\bullet$ produced by the lemma will be in $C^{-1}\Lambda(\Gamma^{\mathrm{ac}})$ for *the same* nonzero $C \in \mathcal{O}_L$.

Proof. See [BL16, Prop. 2.6], whose proof with Γ^{ac} in place of Γ is the same. □

Theorem 4.6. *Assume that $\bar{\rho}_f|_{G_K}$ is irreducible. Then for each positive integer m prime to NDp there exist bounded classes*

$$\mathfrak{Z}[m]^\sharp,\ \mathfrak{Z}[m]^\flat \in H^1(K[m], \mathbf{T}^{\mathrm{ac}})[1/p]$$

such that

$$\begin{pmatrix} \mathfrak{Z}[m]^\alpha \\ \mathfrak{Z}[m]^\beta \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log,\mathrm{ac}} \cdot \begin{pmatrix} \mathfrak{Z}[m]^\sharp \\ \mathfrak{Z}[m]^\flat \end{pmatrix}.$$

Moreover, there is a nonzero $C \in \mathcal{O}_L$ such that $\mathfrak{Z}[m]^\sharp, \mathfrak{Z}[m]^\flat \in C^{-1}H^1(K[m], \mathbf{T}^{\mathrm{ac}})$ for all m.

Proof. This follows from a straightforward adjustment of the argument in [BL16, Thm. 3.5]. Indeed, by [CW16, Lem. 4.3] the assumption on $\bar{\rho}_f$ implies that $H^1(K[m], \mathbf{T}^{\mathrm{ac}})$ is free over $\Lambda(\Gamma^{\mathrm{ac}})$, and so $H^1(K[m], \mathbf{T}^{\mathrm{ac}}) \hat{\otimes} \mathcal{H}_{L,v_\lambda}(\Gamma^{\mathrm{ac}})$ is free over $\mathcal{H}_{L,v_\lambda}(\Gamma^{\mathrm{ac}})$. Writing

$$\mathfrak{Z}[m]^\lambda = \sum_i F_{i,m}^\lambda \cdot \mathfrak{Z}[m]_i$$

with $F_{i,m}^\lambda \in \mathcal{H}_{L,v_\lambda}(\Gamma^{\mathrm{ac}})$ and $\mathfrak{Z}[m]_i \in H^1(K, \mathbf{T}^{\mathrm{ac}})$, we see from Proposition 4.3 and the definition of $\mathfrak{Z}_n[m]^\lambda$ that if $\phi : \Gamma^{\mathrm{ac}} \to \mu_{p^\infty}$ is any finite order character of Γ^{ac} of conductor $p^n > 1$, then

$$\phi(F_{i,m}^\lambda) = \lambda^{-n} \cdot c_\phi^\lambda$$

for some $c_\phi^\lambda \in \overline{\mathbf{Q}}_p$ with $c_\phi^\alpha = c_\phi^\beta$. By Lemma 4.4 applied to the coefficients $F_{i,m}^\lambda$, the existence of classes $\mathfrak{Z}[m]^\bullet$ decomposing the unbounded $\mathfrak{Z}[m]^\lambda$ as indicated follows. Moreover, since $\mathfrak{Z}_n[m]^\lambda$ clearly lands in $\lambda^{-1}H^1(K_n^{\mathrm{ac}}, T_f^*)$, the existence of polynomials $P_{n,\lambda}$ as in Lemma 4.4 for each $F_{i,m}^\lambda$ follows again from Proposition 4.3, and therefore the last property of the classes $\mathfrak{Z}[m]^\bullet$ follows from the last claim in Lemma 4.4. □

From now on, we fix a nonzero $c \in \mathcal{O}_L$ as in Theorem 4.6, and set

$$\mathfrak{Z}_c[m]^\bullet := c \cdot \mathfrak{Z}[m]^\bullet \in H^1(K[m], \mathbf{T}^{\mathrm{ac}})$$

for each m prime to NDp and $\bullet \in \{\sharp, \flat\}$.

4.2. **Explicit reciprocity law.** We keep the hypotheses from the preceding section, let R_0 denote the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$.

Theorem 4.7. *There exists an element $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K) \in \Lambda_{R_0}(\Gamma^{\mathrm{ac}})$ such that if $\psi : \Gamma^{\mathrm{ac}} \to \mathbf{C}_p^\times$ has trivial conductor and infinity type $(-m, m)$ with $m > 0$, then*

$$\left(\frac{\psi(\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K))}{\Omega_p^{2m}}\right)^2 = \Gamma(m)\Gamma(m+1) \cdot (1 - p^{-1}\psi(\mathfrak{p})\alpha)^2 (1 - p^{-1}\psi(\mathfrak{p})\beta)^2 \cdot \frac{L(f/K, \psi, 1)}{\pi^{2m+1} \cdot \Omega_K^{4m}},$$

where $(\Omega_p, \Omega_K) \in R_0^\times \times \mathbf{C}^\times$ are CM periods attached to K. Moreover, $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ is nonzero, and if $\bar{\rho}_f|_{G_K}$ is absolutely irreducible, the μ-invariant of $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ vanishes.

Proof. The construction of $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ with the stated interpolation property follows from the results in [CH17a, §3.3] (see the proof of Theorem 4.10 below for the precise relation between $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ the construction in *loc.cit.*). The nontriviality of $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ is deduced in [CH17a, Thm. 3.7] as a consequence of [Hsi14, Thm. C], and the vanishing of its μ-invariant similarly follows from [Hsi14, Thm. B] (and from [Bur17, Thm. B], in the cases where $N^- \neq 1$). □

As in the proof of [CH17a, Thm. 5.1], one may deduce from the two-variable regulator maps $\mathcal{L}_{\mathfrak{p}}^{\lambda}$ in Definition 2.2 the construction of linear maps

$$\mathcal{L}_{\mathfrak{p},\mathrm{ac}}^{\lambda} : H^1(K_{\mathfrak{p}}, \mathbf{T}^{\mathrm{ac}}) \longrightarrow \mathcal{H}_{\hat{F}_\infty}(\Gamma^{\mathrm{ac}}) \tag{4.4}$$

such that for any class $\mathbf{z} \in H^1(K_{\mathfrak{p}}, \mathbf{T}^{\mathrm{ac}})$ and χ a finite order character of Γ^{ac}, we have

$$\chi(\mathcal{L}_{\mathfrak{p},\mathrm{ac}}^{\lambda}(\mathbf{z})) \doteq \langle \exp^*(\mathbf{z}^{\chi^{-1}}), \eta_{f^\lambda}\rangle, \tag{4.5}$$

where $\exp^*$ is the Bloch–Kato dual exponential map for the twisted representation $V_f^*(\chi^{-1})$, $\mathbf{z}^{\chi^{-1}}$ is the natural image of $\mathbf{z}$ in $H^1(K_{\mathfrak{p}}, V_f^*(\chi^{-1}))$, and η_{f^λ} is the specialization of the class $\eta_{\mathcal{F}^\lambda}$ appeared right before Proposition 2.4. The same construction applied to the two-variable Coleman maps $\mathrm{Col}_{\mathfrak{p}}^{\bullet}$ in Definition 2.2 yields $\Lambda(\Gamma^{\mathrm{ac}})$-linear maps

$$\mathrm{Col}_{\mathfrak{p},\mathrm{ac}}^{\bullet} : H^1(K_{\mathfrak{p}}, \mathbf{T}^{\mathrm{ac}}) \to \Lambda_{\hat{\mathcal{O}}_{F_\infty}}(\Gamma^{\mathrm{ac}})$$

satisfying

$$\begin{pmatrix} \mathcal{L}_{\mathfrak{p},\mathrm{ac}}^{\alpha} \\ \mathcal{L}_{\mathfrak{p},\mathrm{ac}}^{\beta} \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log,\mathrm{ac}} \cdot \begin{pmatrix} \mathrm{Col}_{\mathfrak{p},\mathrm{ac}}^{\sharp} \\ \mathrm{Col}_{\mathfrak{p},\mathrm{ac}}^{\flat} \end{pmatrix} \tag{4.6}$$

as an immediate consequence of (2.11).

Lemma 4.8. *For each $\bullet \in \{\sharp, \flat\}$ and m a positive integer prime to NDp, we have*

$$\mathrm{Col}_{\mathfrak{p},\mathrm{ac}}^{\bullet}(\mathrm{res}_{\mathfrak{p}}(\mathfrak{Z}_c[m]^{\bullet})) = 0.$$

Proof. The following argument is inspired by the proof of [BL16, Cor. 3.12]. We only prove the case $\bullet = \sharp$, since the proof for the other case is essentially the same. By their construction as Kummer images of Heegner points, if $\mathfrak{P}$ is any prime of $K_n^{\mathrm{ac}}[m]$ above $\mathfrak{p}$, the λ-stabilized Heegner classes $\mathfrak{Z}_n[m]^{\lambda}$ of Definition 4.2 have $\mathrm{res}_{\mathfrak{P}}(\mathfrak{Z}_n[m]^{\lambda})$ in the kernel of the Bloch–Kato dual exponential map. By (4.5) and Proposition 4.3, it follows that

$$\mathcal{L}_{\mathfrak{p},\mathrm{ac}}^{\mu}(\mathrm{res}_{\mathfrak{p}}(\mathfrak{Z}[m]^{\lambda})) = 0 \tag{4.7}$$

for all $\lambda, \mu \in \{\alpha, \beta\}$. Write $Q_{\alpha,\beta}^{-1} M_{\log,\mathrm{ac}} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and set

$$\begin{aligned} \hat{\mathfrak{Z}}[m]^{\sharp} &:= d \cdot \mathfrak{Z}[m]^{\alpha} - b \cdot \mathfrak{Z}[m]^{\beta} \\ &= \det(Q_{\alpha,\beta}^{-1} M_{\log,\mathrm{ac}}) \cdot \mathfrak{Z}[m]^{\sharp} \in H^1(K[m], \mathbf{T}^{\mathrm{ac}}) \hat{\otimes} \mathcal{H}_L(\Gamma^{\mathrm{ac}}), \end{aligned} \tag{4.8}$$

where the second equality follows from the decomposition in Theorem 4.6. Extending (4.4) by $\mathcal{H}_L(\Gamma^{\mathrm{ac}})$-linearity, we deduce from (4.7) that

$$\mathcal{L}_{\mathfrak{p},\mathrm{ac}}^{\mu}(\mathrm{res}_{\mathfrak{p}}(\hat{\mathfrak{Z}}[m]^{\sharp})) = 0$$

for all $\mu \in \{\alpha, \beta\}$, and similarly extending the maps $\mathrm{Col}_{\mathfrak{p},\mathrm{ac}}^{\bullet}$ we conclude that

$$\mathrm{Col}_{\mathfrak{p},\mathrm{ac}}^{\sharp}(\mathrm{res}_{\mathfrak{p}}(\hat{\mathfrak{Z}}[m]^{\sharp})) = 0 \tag{4.9}$$

by the decomposition (4.6). Combining (4.8) and (4.9), the result follows. □

Proposition 4.9. *For each $\lambda \in \{\alpha, \beta\}$, there exists an injective linear map*

$$\mathscr{L}_{\mathfrak{p},\mathrm{ac}}^{\lambda} : H^1(K_{\mathfrak{p}}, \mathbf{T}^{\mathrm{ac}} \hat{\otimes} \mathcal{H}_L(\Gamma^{\mathrm{ac}})) \longrightarrow \mathcal{H}_{\hat{F}_\infty}(\Gamma^{\mathrm{ac}})$$

with finite cokernel such that for every $\mathbf{z} \in H^1(K_{\mathfrak{p}}, \mathbf{T}^{\mathrm{ac}} \hat{\otimes} \mathcal{H}_L(\Gamma^{\mathrm{ac}}))$ and $\phi : \Gamma^{\mathrm{ac}} \to \overline{\mathbf{Q}}_p^{\times}$ of infinity type $(-\ell, \ell)$ with $\ell > 0$, we have

$$\phi(\mathscr{L}_{\mathfrak{p},\mathrm{ac}}^{\lambda}(\mathbf{z})) = \frac{\mathfrak{g}(\phi^{-1})\phi^{-1}(p^n)}{\lambda^n p^n} \cdot \frac{(-1)^{\ell-1}}{(\ell-1)!} \cdot \langle \log_{V_f}(\mathbf{z}^{\phi^{-1}}), \omega_{f^\lambda} \otimes t^{\ell} \rangle,$$

where $\mathfrak{g}(\phi^{-1})$ is a Gauss sum, $\log_{V_f}$ is the Bloch–Kato logarithm, and $t \in \mathbf{B}_{\mathrm{dR}}$ is Fontaine's $2\pi i$.

Proof. Similar to the proof [CH17a, Thm. 5.1], replacing the appeal to [LZ14, Thm. 4.7] by the analogous construction based on [LZ16, (6.2.1)]. □

Let $H^1_\bullet(K_\mathfrak{p}, \mathbf{T}^{\rm ac}) \subset H^1(K_\mathfrak{p}, \mathbf{T}^{\rm ac})$ be the kernel of the map $\mathrm{Col}^\bullet_{\mathfrak{p},\rm ac}$, so that in particular by Lemma 4.8 we have

$$\mathrm{res}_\mathfrak{p}(\mathfrak{Z}_c[1]^\bullet) \in H^1_\bullet(K_\mathfrak{p}, \mathbf{T}^{\rm ac})$$

for every $\bullet \in \{\sharp, \flat\}$. Letting $H^1_\lambda(K_\mathfrak{p}, \mathbf{T}^{\rm ac}\hat{\otimes}\mathcal{H}_L(\Gamma^{\rm ac}))$ be the kernel of the $\mathcal{H}_L(\Gamma^{\rm ac})$-linear extension of the map $\mathcal{L}^\lambda_{\mathfrak{p},\rm ac}$, we clearly have an anticyclotomic analogue of Lemma 2.6, and following Definition 2.8, we define the anticyclotomic signed logarithm maps $(\mathrm{Log}^\sharp_{\mathfrak{p},\rm ac}, \mathrm{Log}^\flat_{\mathfrak{p},\rm ac})$ by the composition

$$\begin{aligned} H^1_\sharp(K_\mathfrak{p}, \mathbf{T}^{\rm ac}) \oplus H^1_\flat(K_\mathfrak{p}, \mathbf{T}^{\rm ac}) &\overset{Q^{-1}_{\lambda,\mu}M_{\log,\rm ac}}{\longrightarrow} H^1_\alpha(K_\mathfrak{p}, \mathbf{T}^{\rm ac}\hat{\otimes}\mathcal{H}_L(\Gamma^{\rm ac})) \oplus H^1_\beta(K_\mathfrak{p}, \mathbf{T}^{\rm ac}\hat{\otimes}\mathcal{H}_L(\Gamma^{\rm ac})) \\ &\overset{\mathscr{L}^\lambda_{\mathfrak{p},\rm ac}\oplus\mathscr{L}^\mu_{\mathfrak{p},\rm ac}}{\longrightarrow} \mathcal{H}_L(\Gamma^{\rm ac})^{\oplus 2}, \end{aligned}$$

where $\mathscr{L}^\lambda_{\mathfrak{p},\rm ac}$ and $\mathscr{L}^\mu_{\mathfrak{p},\rm ac}$ are as in Proposition 4.9.

Theorem 4.10. *For each $\bullet \in \{\sharp, \flat\}$, the class $\mathfrak{Z}^\bullet_c := \mathfrak{Z}_c[1]^\bullet$ is such that*

$$\mathrm{Log}^\bullet_{\mathfrak{p},\rm ac}(\mathrm{res}_\mathfrak{p}(\mathfrak{Z}^\bullet_c)) = c \cdot \mathscr{L}^{\sf BDP}_\mathfrak{p}(f/K)$$

up to a unit, where $\mathscr{L}^{\sf BDP}_\mathfrak{p}(f/K)$ is as in Theorem 4.7. In particular, the class $\mathfrak{Z}^\bullet_c$ is nontorsion over $\Lambda(\Gamma^{\rm ac})$.

Proof. Let ψ be an anticyclotomic Hecke character of K of infinity type $(1, -1)$ and conductor prime to p, and let $\mathscr{L}_{\mathfrak{p},\psi}(f) \in \Lambda_{R_0}(\Gamma^{\rm ac})$ be as in [CH17a, Def. 3.7]. The p-adic L-function $\mathscr{L}^{\sf BDP}_\mathfrak{p}(f/K)$ of Theorem 4.7 is then given by

$$\mathscr{L}^{\sf BDP}_\mathfrak{p}(f/K) = \mathrm{Tw}_{\psi^{-1}}(\mathscr{L}_{\mathfrak{p},\psi}(f)),$$

where $\mathrm{Tw}_{\psi^{-1}} : \Lambda_{R_0}(\Gamma^{\rm ac}) \to \Lambda_{R_0}(\Gamma^{\rm ac})$ is the R_0-linear isomorphism given by $\gamma \mapsto \psi^{-1}(\gamma)\gamma$ for $\gamma \in \Gamma^{\rm ac}$. Let $\phi : \Gamma^{\rm ac} \to \mu_{p^\infty}$ be a nontrivial character of conductor p^n. Following the calculations in [CH17a, Thm. 4.9], we see that

$$\phi(\mathscr{L}^{\sf BDP}_\mathfrak{p}(f/K)) = \mathfrak{g}(\phi)\phi^{-1}(p^n)p^{-n} \sum_{\sigma\in\Gamma^{\rm ac}/p^n} \phi^{-1}(\sigma) \cdot \langle \log_{V_f}(\mathrm{res}_\mathfrak{p}(\mathfrak{Y}_n[1]^\sigma)), \omega_f \otimes t\rangle, \tag{4.10}$$

where $\mathfrak{Y}_n[1]$ is as in (4.2). Since $n > 0$, we may replace $\mathfrak{Y}_n[1]$ by any of its 'p-stabilizations' $\mathfrak{Z}_n[1]^\lambda$; combined with Proposition 4.3 and Proposition 4.9, we thus conclude from (4.10) that for each $\lambda \in \{\alpha, \beta\}$ we have

$$\mathscr{L}^\lambda_{\mathfrak{p},\rm ac}(\mathrm{res}_\mathfrak{p}(\mathfrak{Z}^\lambda)) = \mathscr{L}^{\sf BDP}_\mathfrak{p}(f/K), \tag{4.11}$$

where $\mathfrak{Z}^\lambda := \mathfrak{Z}[1]^\lambda$. By the decomposition in Theorem 4.6 and the construction of $\mathrm{Log}^\bullet_{\mathfrak{p},\rm ac}$, the result follows. □

4.3. **Signed theta elements.** Assume now that the pair (f, K) is definite, so that N^- is the square-free product of an *odd* number of primes, and continue to assume that $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K.

For each $n \geqslant 0$ let $\mathcal{G}_n := \mathrm{Gal}(K[p^n]/K)$ be the Galois group of the ring class field of K of conductor p^n; thus in particular, $\mathcal{G}_0$ is the Galois group of the Hilbert class field of K.

Proposition 4.11. *There is a system of theta elements $\Theta_n \in \mathcal{O}_L[\mathcal{G}_n]$ such that*

$$\mathrm{cor}^n_{n-1}(\Theta_n) = \begin{cases} a_p \cdot \Theta_{n-1} - \Theta_{n-2} & \text{if } n \geqslant 2, \\ \frac{1}{u_K}(a_p - \sigma_\mathfrak{p} - \sigma_{\overline{\mathfrak{p}}}) \cdot \Theta_0 & \text{if } n = 1, \end{cases}$$

where $u_K := |\mathcal{O}^\times_K|/2$, $\sigma_\mathfrak{p}, \sigma_{\overline{\mathfrak{p}}} \in \mathcal{G}_0$ are the Frobenius elements at $\mathfrak{p}$, $\overline{\mathfrak{p}}$, respectively, and $\mathrm{cor}^n_{n-1} : \mathcal{O}_L[\mathcal{G}_n] \to \mathcal{O}_L[\mathcal{G}_{n-1}]$ is the natural projection.

Proof. See [BD96, §2]. □

Letting $\lambda \in \{\alpha, \beta\}$ be any of the roots of $X^2 - a_pX + p$ (and after possibly enlarging L so that it contains these roots), the λ-*regularized* theta-elements $\Theta_n^\lambda \in L[\mathcal{G}_n]$ defined by

$$\Theta_n^\lambda := \begin{cases} \Theta_n - \frac{1}{\lambda} \cdot \Theta_{n-1} & \text{if } n \geqslant 1, \\ \frac{1}{u_K}\left(1 - \frac{1}{\lambda}\sigma_{\mathfrak{p}}\right)\left(1 - \frac{1}{\lambda}\sigma_{\overline{\mathfrak{p}}}\right) \cdot \Theta_0 & \text{if } n = 0, \end{cases}$$

are immediately seen (in light of Proposition 4.11) to satisfy the norm-compatibility

$$\mathrm{cor}_{n-1}^n(\Theta_n^\lambda) = \lambda \cdot \Theta_{n-1}^\lambda$$

for all $n > 0$. Thus there exists a unique element

$$\Theta_\infty^\lambda \in \mathcal{H}_{L,v_\lambda}(\Gamma^{\mathrm{ac}})$$

mapping to $\lambda^{-n} \cdot \Theta_n^\lambda$ under the natural projection $\mathcal{H}_L(\Gamma^{\mathrm{ac}}) \to L[\Gamma_n^{\mathrm{ac}}]$ for all $n \geqslant 0$ (see [Pol03, Prop. 2.8] and the references therein, which readily adapt to our anticyclotomic setting).

Let X_{N^+,N^-} be the Shimura set attached to a quaternion algebra $B/\mathbf{Q}$ of discriminant N^- and an Eichler order of level N^+. The module $\mathrm{Pic}(X_{N^+,N^-})$ is equipped with a natural action of the quotient of the Hecke algebra of level N acting faithfully on subspace of N^--new forms in $S_2(\Gamma_0(N))$, and in the construction of Θ_n (as explained in [PW11, §2.1], for example) one chooses a generator ϕ_f of its f-isotypical component.

Taking ϕ_f to be normalized as in [PW11, Lem. 2.1], we define the *Gross period* Ω_{f,N^-} by

$$\Omega_{f,N^-} := \frac{(4\pi)^2\langle f, f\rangle_N}{\langle \phi_f, \phi_f\rangle}, \tag{4.12}$$

where $\langle\ ,\ \rangle$ is the intersection pairing on $\mathrm{Pic}(X_{N^+,N^-})$.

Proposition 4.12. *For each $\lambda \in \{\alpha, \beta\}$, the elements Θ_∞^λ satisfy the following interpolation property: If χ is a finite order character of Γ^{ac} of conductor $p^n > 1$, then*

$$\chi(\Theta_\infty^\lambda)^2 = \frac{1}{\lambda^{2n}} \cdot \frac{L(f/K, \chi, 1)}{\Omega_{f,N^-}} \cdot \sqrt{D}p^n \cdot u_K^2. \tag{4.13}$$

Moreover, there exist elements $\Theta_\infty^\sharp, \Theta_\infty^\flat \in \Lambda_L(\Gamma^{\mathrm{ac}})$ such that

$$\begin{pmatrix} \Theta_\infty^\alpha \\ \Theta_\infty^\beta \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log,\mathrm{ac}} \cdot \begin{pmatrix} \Theta_\infty^\sharp \\ \Theta_\infty^\flat \end{pmatrix}.$$

Proof. Note that by construction we have

$$\chi(\Theta_\infty^\lambda)^2 = \lambda^{-2n} \cdot \chi(\Theta_n^\lambda)^2 = \lambda^{-2n} \cdot \chi(\Theta_n)^2,$$

using that χ has conductor $p^n > 1$ for the second equality. Letting $\psi_f : \mathrm{Pic}(X_{N^+,N^-}) \to \mathcal{O}_L$ denote the function given by $\psi_f(x) = \langle x, \phi_f\rangle$, the theta element $\Theta_n \in \mathcal{O}_L[\mathcal{G}_n]$ is defined by

$$\Theta_n := \sum_{\sigma\in\mathcal{G}_n} \psi_f(P_n^\sigma)\sigma,$$

where $P_n \in X_{N^+,N^-}$ is a certain "Gross point" of conductor p^n. As in [PW11, Lem. 2.2] and [CH17b, Prop. 4.3], the interpolation property (4.13) then follows from Gross's special value formula; and with (4.13) in hand, the last claim in the proposition follows from Lemma 4.4. □

The following application of Vatsal's results [Vat03] will play an key role in our proof of Theorem A in the definite setting.

Proposition 4.13. *For each $\bullet \in \{\sharp, \flat\}$, we have $\mu(\Theta_\infty^\bullet) = 0$.*

Proof. Following [LLZ17], for each $n \geqslant 1$ we set

$$\mathscr{H}_{n,\mathrm{ac}} := \mathfrak{M}^{-1}((1+\pi)\varphi^{n-1}(P^{-1})\cdots\varphi(P^{-1})) \in M_{2\times 2}(\mathcal{H}_L(\Gamma^{\mathrm{ac}})),$$

where

$$P = \begin{pmatrix} 0 & -q^{-1} \\ 1 & a_p\delta q^{-1} \end{pmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})$$

is the matrix of φ on $\mathbb{N}(T_f^*)$ with respect to the basis $\{n_1, n_2\}$ in (2.1), with $q = \varphi(\pi)/\pi \in \mathbb{A}^+_{\mathbf{Q}_p}$ and $\delta \in (\mathbb{A}^+_{\mathbf{Q}_p})^\times$. Letting $\gamma_{\mathrm{ac}} \in \Gamma^{\mathrm{ac}}$ be a fixed topological generator, by [*loc.cit.*, Lem. 3.7] and the defining interpolating property of Θ^λ_∞ we have the congruence

$$\begin{pmatrix} \alpha^{-n}\cdot\Theta^\alpha_n \\ \beta^{-n}\cdot\Theta^\beta_n \end{pmatrix} \equiv Q^{-1}_{\alpha,\beta}\cdot A^n_\varphi \cdot \mathscr{H}_{n,\mathrm{ac}} \cdot \begin{pmatrix} \Theta^\sharp_n \\ \Theta^\flat_n \end{pmatrix} \quad (\mathrm{mod}\ \gamma_{\mathrm{ac}}^{p^n}-1),$$

where $\Theta^\bullet_n$ is the image of $\Theta^\bullet_\infty$ under the natural projection $\Lambda(\Gamma^{\mathrm{ac}}) \to \mathcal{O}_L[\Gamma^{\mathrm{ac}}_n]$ and A_φ is as in (2.2). Since the matrix $Q_{\alpha,\beta}$ diagonalizes A_φ:

$$A_\varphi \cdot Q_{\alpha,\beta} = Q_{\alpha,\beta}\cdot \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix},$$

using the definition of Θ^λ_n we see that (4.14) reduces to

$$(\alpha-\beta)\begin{pmatrix} \Theta_n \\ \Theta_{n-1} \end{pmatrix} \equiv \mathscr{H}_{n,\mathrm{ac}} \cdot \begin{pmatrix} \Theta^\sharp_n \\ \Theta^\flat_n \end{pmatrix} \quad (\mathrm{mod}\ \gamma_{\mathrm{ac}}^{p^n}-1).$$

Now let ζ_{p^n} be a primitive p^n-th root of unity, and set $\epsilon_n := \zeta_{p^n}-1$. Viewing $\Theta^\bullet_\infty$ as elements in $\mathcal{O}_L[[T]]$, let $\lambda^\bullet$ and $\mu^\bullet$ be the corresponding λ- and μ-invariants, and set $\lambda := \min(\lambda^\sharp, \lambda^\flat)$. Then from (4.14) and [LLZ17, Cor. 4.7] we obtain the estimates

$$\mathrm{ord}_p(\Theta_n(\epsilon_n)) \geqslant \begin{cases} \lambda + (p^n - p^{n-1})\big(\frac{\mu^\bullet}{e} + \sum_{i=1}^{\frac{n-1}{2}} p^{1-2i}\big) & \text{if } n \text{ is odd}, \\ \lambda + (p^n - p^{n-1})\big(\frac{\mu^\bullet}{e} + \sum_{i=1}^{\frac{n}{2}} p^{-2i}\big) & \text{if } n \text{ is even}, \end{cases} \tag{4.14}$$

for each of the signs $\bullet \in \{\sharp, \flat\}$, where e is the absolute ramification degree of L. Since on the other hand the arguments in [Vat03, §5.9] imply that $\mu(\Theta_n) = 0$ for n sufficiently large (see also [PW11, Thm. 2.5]), we conclude from (4.14) and [PR03, Lem. 5.5] that $\mu^\bullet = 0$, as was to be shown. □

5. Main results

In this section we conclude the proof of the results stated in the Introduction. The proof of Theorem A naturally splits into two cases according to the sign $\epsilon(\pi, K, \frac{1}{2})$ (see the discussion at the beginning of §4), but in both cases it builds on the following result toward the Iwasawa–Greenberg main conjecture for the p-adic L-function $\mathscr{L}_{\mathfrak{p}}(f/K)$ appearing in Theorem 3.8:

Theorem 5.1. *Assume that:*

- *N is square-free,*
- *$\bar\rho_f$ is ramified at some prime at every prime $q \mid N$ which is nonsplit in K, and there is at least one such prime,*
- *$\bar\rho_f|_{G_K}$ is irreducible.*

Then we have the divisibility

$$\mathrm{Char}_{\Lambda(\Gamma_K)}(\mathfrak{X}^{\mathrm{rel,str}}_{K_\infty}(f)) \subseteq (\mathscr{L}_{\mathfrak{p}}(f/K))$$

as ideals in $\Lambda_{R_0}(\Gamma_K)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$.

Proof. See part (2) of [Wan16b, Thm. 5.3]. □

The following results will be used in to descend to the cyclotomic line.

Proposition 5.2. *For each* $\bullet \in \{\sharp, \flat\}$ *there are* $\Lambda(\Gamma^{\rm cyc})$*-module isomorphisms*

$$X^{\bullet,\bullet}_{K_\infty}(f)/(\gamma_{\rm ac}-1)X^{\bullet,\bullet}_{K_\infty}(f) \simeq X^{\bullet,\bullet}_{K_\infty^{\rm cyc}}(f)$$
$$\simeq X^{\bullet}_{\mathbf{Q}_\infty}(f) \oplus X^{\bullet}_{\mathbf{Q}_\infty}(f_K).$$

In particular, letting $\mathcal{L}_p^{\bullet,\bullet}(f/K)_{\rm cyc}$ *be the image of a generator of* $\mathrm{Char}_{\Lambda(\Gamma_K)}(X^{\bullet,\bullet}_{K_\infty}(f))$ *under the natural projection* $\Lambda(\Gamma_K) \twoheadrightarrow \Lambda(\Gamma^{\rm cyc})$*, we have the divisibility*

$$(\mathcal{L}_p^{\bullet,\bullet}(f/K)_{\rm cyc}) \supseteq \mathrm{Char}_{\Lambda(\Gamma^{\rm cyc})}(X^{\bullet}_{\mathbf{Q}_\infty}(f)) \cdot \mathrm{Char}_{\Lambda(\Gamma^{\rm cyc})}(X^{\bullet}_{\mathbf{Q}_\infty}(f_K)).$$

Proof. This follows from a straightforward extension of the arguments in Proposition 3.9 and Lemma 3.6 of [SU14]. □

5.1. **Indefinite case.** In this section, we finish the proof of Theorem A in the case where N^- is the product of an *even* number of primes. This extends to general non-ordinary modular forms of weight 2 the main results of [CW16] on B.-D. Kim's doubly-signed main conjectures, which assumed $a_p = 0$.

A key ingredient in the proof will be the relation between Conjecture 5.3 and Conjecture 5.4 below, corresponding to the analogue of Perrin-Riou's main conjecture [PR87] for the signed Heegner points constructed in §4.1, and Greenberg's main conjecture [Gre94] for $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$, respectively.

Recall that attached to f is an isogeny class of abelian variety quotients

$$\pi_f : J(X) \longrightarrow A_f, \tag{5.1}$$

where $J(X)$ is the Jacobian variety of a Shimura curve X_{N^+,N^-} attached to an indefinite rational quaternion algebra of discriminant N^-. We assume that (5.1) is an optimal quotient, i.e., $\ker(\pi_f)$ is connected, and set $\delta(N^+,N^-) = \pi \circ \pi^\vee$. Replacing (5.1) by a classical modular parametrization

$$\varphi_f : J_0(N) \longrightarrow A_f$$

we similarly define $\delta(N) := \delta(N,1)$, and set

$$\delta_{N^+,N^-} := \frac{\delta(N^+,N^-)}{\delta(N)}.$$

Finally, let c_f be the Manin constant associated with φ_f (see e.g [ARS06]) and $u_K := |\mathcal{O}_K^\times|/2$.

Conjecture 5.3. *For each* $\bullet \in \{\sharp, \flat\}$*, both* $\mathfrak{X}^{\bullet,\bullet}_{K_\infty^{\rm ac}}(f)$ *and* $\mathfrak{Sel}^{\bullet,\bullet}(K, \mathbf{T}^{\rm ac})$ *have* $\Lambda(\Gamma^{\rm ac})$*-rank* 1*, and*

$$c^2 \cdot \mathrm{Char}_{\Lambda(\Gamma^{\rm ac})}(\mathfrak{X}^{\bullet,\bullet}_{K_\infty^{\rm ac}}(f)_{\rm tors}) = \frac{\delta_{N^+,N^-}}{c_f^2 u_K^2} \cdot \mathrm{Char}_{\Lambda(\Gamma^{\rm ac})}\left(\frac{\mathfrak{Sel}^{\bullet,\bullet}(K,\mathbf{T}^{\rm ac})}{\Lambda(\Gamma^{\rm ac}) \cdot \mathfrak{Z}_c^{\bullet}}\right)^2$$

as ideals in $\Lambda(\Gamma^{\rm ac})$*, where the subscript* tors *denotes the* $\Lambda(\Gamma^{\rm ac})$*-torsion submodule.*

Conjecture 5.4. *The module* $X^{\rm rel,str}_{K_\infty^{\rm ac}}(f)$ *is* $\Lambda(\Gamma^{\rm ac})$*-torsion, and*

$$\mathrm{Char}_{\Lambda(\Gamma^{\rm ac})}(X^{\rm rel,str}_{K_\infty^{\rm ac}}(f)) = (\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)^2)$$

as ideals in $\Lambda_{R_0}(\Gamma^{\rm ac})$.

We record the following auxiliary results for our later use.

Lemma 5.5. *Let* $\bullet \in \{\sharp, \flat\}$*, and assume that* $\mathfrak{Sel}^{\bullet,\bullet}(K, \mathbf{T}^{\rm ac})$ *has* $\Lambda(\Gamma^{\rm ac})$*-rank* 1*. Then* $\mathfrak{X}^{\bullet,\bullet}_{K_\infty^{\rm ac}}(f)$ *has* $\Lambda(\Gamma^{\rm ac})$*-rank* 1*, and the following statements hold:*

(1) $\mathfrak{X}^{\bullet,\rm str}_{K_\infty^{\rm ac}}(f)$ *is* $\Lambda(\Gamma^{\rm ac})$*-torsion and the inclusion*

$$\mathfrak{Sel}^{\bullet,\bullet}(K, \mathbf{T}^{\rm ac}) \subseteq \mathfrak{Sel}^{\bullet,\rm rel}(K, \mathbf{T}^{\rm ac})$$

is an equality.

(2) $\mathfrak{X}^{\rm rel,str}_{K^{\rm ac}_\infty}(f)$ *is* $\Lambda(\Gamma^{\rm ac})$*-torsion, and for any height one prime of* $\Lambda(\Gamma^{\rm ac})$*, we have*

$$\mathrm{length}_{\mathfrak{P}}(\mathfrak{X}^{\rm rel,str}_{K^{\rm ac}_\infty}(f)) = \mathrm{length}_{\mathfrak{P}}(\mathfrak{X}^{\bullet,\bullet}_{K^{\rm ac}_\infty}(f)_{\rm tors}) + 2\,\mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{res}_{\mathfrak{p}}))$$

and

$$\mathrm{ord}_{\mathfrak{P}}(c\cdot\mathscr{L}^{\rm BDP}_{\mathfrak{p}}(f/K)) = \mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{res}_{\mathfrak{p}})) + \mathrm{length}_{\mathfrak{P}}\left(\frac{\mathfrak{Sel}^{\bullet,\bullet}(K,\mathbf{T}^{\rm ac})}{\Lambda(\Gamma^{\rm ac})\cdot\mathfrak{Z}^{\bullet}_c}\right),$$

where $\mathrm{res}_{\mathfrak{p}}:\mathfrak{Sel}^{\bullet,\bullet}(K,\mathbf{T}^{\rm ac})\to H^1_\bullet(K_{\mathfrak{p}},\mathbf{T}^{\rm ac})$ *is the natural restriction map.*

Proof. The proof of the corresponding three lemmas in [CW16, §4.3] applies almost verbatim, using the explicit reciprocity law of Theorem 4.10 in place of [*loc.cit.*, Thm. 4.6] for the proof of the second equality in part (2). □

Fix a finite set of places Σ of K containing those dividing $Np\infty$. For each $\bullet\in\{\sharp,\flat\}$, define the Selmer structure $\mathcal{F}^\bullet$ on $\mathbf{T}^{\rm ac}$ (in the sense of [How04a, §1.1]) by taking the unramified local condition

$$H^1_{\rm ur}(K_v,\mathbf{T}^{\rm ac}) := \ker\left\{H^1(K_v,\mathbf{T}^{\rm ac})\longrightarrow H^1(K_v^{\rm ur},\mathbf{T}^{\rm ac})\right\}$$

at the places in $v\in\Sigma$ not dividing p, and the local condition $H^1_\bullet(K_v,\mathbf{T}^{\rm ac})\subset H^1(K_v,\mathbf{T}^{\rm ac})$ at the primes v dividing p. (Thus the Selmer group denoted $\mathfrak{Sel}^{\bullet,\bullet}(K,\mathbf{T}^{\rm ac})$ in §2.3 corresponds to $H^1_{\mathcal{F}^\bullet}(K,\mathbf{T}^{\rm ac})$ in the notations of [How04a].)

For the statement of the next result, we refer the reader to [How04a, §1.2] for the definition of the module of Kolyvagin systems $\mathbf{KS}(\mathbf{T}^{\rm ac},\mathcal{F},\mathcal{L})$ attached to a Selmer structure $\mathcal{F}$ on $\mathbf{T}^{\rm ac}$ and a certain set $\mathcal{L}$ of primes inert in K.

Proposition 5.6. *For each* $\bullet\in\{\sharp,\flat\}$*, there is a Kolyvagin system* $\kappa^\bullet\in\mathbf{KS}(\mathbf{T}^{\rm ac},\mathcal{F}^\bullet,\mathcal{L})$ *with* $\kappa^\bullet_1=\mathfrak{Z}^\bullet_c$.

Proof. As in the proof of the corresponding result in [CW16, Thm. 4.14] (to which we refer the reader for the definition of the ideal $I_m\subset p\mathcal{O}_L$, noting that S is *loc.cit.* corresponds to m here), this is reduced to showing that the classes

$$\kappa^\bullet_m\in H^1(K,\mathbf{T}^{\rm ac}/I_m\mathbf{T}^{\rm ac}),$$

obtained by applying the 'Kolyvagin's derivative' construction in [How04a, §1.7] to the classes $\mathfrak{Z}[m]^\bullet$ of Theorem 4.6, are such that

$$\mathrm{res}_v(\kappa^\bullet_m)\in\mathrm{im}\left(H^1_\bullet(K_v,\mathbf{T}^{\rm ac})\longrightarrow H^1(K_v,\mathbf{T}^{\rm ac}/I_m\mathbf{T}^{\rm ac})\right) \tag{5.2}$$

for all $v\mid p$. To show this, let $G(m)=\prod_\ell G(\ell)$ be the Galois group of the extension $K[m]/K[1]$, and recall that Kolyvagin's derivative operator $D_m=\prod_\ell D_\ell\in\mathbf{Z}[G(m)]$ is defined by $D_\ell=\sum_{i=1}^{\ell}i\sigma_\ell^i$, where $\sigma_\ell\in G(\ell)$ is any generator. Setting

$$Z[m]^\bullet := \sum_\sigma \sigma D_m\mathfrak{Z}_c[m]\in H^1(K[m],\mathbf{T}^{\rm ac}), \tag{5.3}$$

where the sum is over a complete set of representatives of the quotient of $\mathcal{G}(m):=\mathrm{Gal}(K[m]/K)$ by $G(m)$, one readily checks that the natural image $\bar{Z}[m]^\bullet$ of $Z[m]^\bullet$ in $H^1(K[m],\mathbf{T}^{\rm ac}/I_m\mathbf{T}^{\rm ac})$ is fixed by $\mathcal{G}(m)$; the derivative class $\kappa^\bullet_m$ is then determined by the condition that

$$\mathrm{res}(\kappa^\bullet_m)=\bar{Z}[m]^\bullet \tag{5.4}$$

under the isomorphism $\mathrm{res}:H^1(K,\mathbf{T}^{\rm ac}/I_m\mathbf{T}^{\rm ac})\simeq H^1(K[m],\mathbf{T}^{\rm ac}/I_m\mathbf{T}^{\rm ac})^{\mathcal{G}(m)}$.

Now, if w is any place of $K[m]$ above v, by Lemma 4.8 we have the inclusion

$$\mathrm{res}_w(\mathfrak{Z}_c[m]^\bullet)\in H^1_\bullet(K[m]_w,\mathbf{T}^{\rm ac}),$$

and hence from (5.3) and (5.4), the inclusion (5.2) follows. □

Equipped with the Kolyvagin system of Proposition 5.6 (whose nontriviality is guaranteed by Theorem 4.10) the following result towards Conjecture 5.3 follows easily.

Theorem 5.7. *Let* $\bullet \in \{\sharp, \flat\}$*, and assume that* $\bar{\rho}_f|_{G_K}$ *is irreducible. Then both* $\mathfrak{X}^{\bullet,\bullet}_{K^{\rm ac}_\infty}(f)$ *and* $\mathfrak{Sel}^{\bullet,\bullet}(K, \mathbf{T}^{\rm ac})$ *have* $\Lambda(\Gamma^{\rm ac})$*-rank* 1*, and we have the divisibility*

$$c^2 \cdot \mathrm{Char}_{\Lambda(\Gamma^{\rm ac})}(\mathfrak{X}^{\bullet,\bullet}_{K^{\rm ac}_\infty}(f)_{\rm tors}) \supseteq \mathrm{Char}_{\Lambda(\Gamma^{\rm ac})}\left(\frac{\mathfrak{Sel}^{\bullet,\bullet}(K, \mathbf{T}^{\rm ac})}{\Lambda(\Gamma^{\rm ac}) \cdot \mathfrak{Z}^\bullet_c}\right)^2.$$

Proof. This follows from a straightforward adaptation of the arguments in the last paragraph of [CW16, §4.3], with the self-duality of the plus/minus local conditions quoted from [Kim07] in *loc.cit.* replaced by the corresponding results for the signed local conditions of §2.3 above (see e.g. Lemma 2.4 and Remark 2.5 in [HL16]). □

Now we assemble all the pieces, yielding the follow result on Conjecture 5.4.

Theorem 5.8. *Let* $f \in S_2(\Gamma_0(N))$ *be a newform, let* $p \nmid N$ *be an odd prime, and let* $K/\mathbf{Q}$ *be an imaginary quadratic field in which* $p = \mathfrak{p}\bar{\mathfrak{p}}$ *splits. Assume that* f *is non-ordinary at* p*, and that*

- N *is square-free,*
- N^- *is divisible by an even number of primes,*
- $\bar{\rho}_f$ *is ramified at every prime* $\ell \mid N$ *which is nonsplit in* K*, and there is at least one such prime,*
- $\bar{\rho}_f|_{G_K}$ *is irreducible.*

Then $X^{\rm rel,str}_{K^{\rm ac}_\infty}(f)$ *is* $\Lambda(\Gamma^{\rm ac})$*-cotorsion, and*

$$Ch_{\Lambda(\Gamma^{\rm ac})}(X^{\rm rel,str}_{K^{\rm ac}_\infty}(f)) = (\mathscr{L}^{\mathsf{BDP}}_{\mathfrak{p}}(f/K)^2)$$

as ideals in $\Lambda_{R_0}(\Gamma^{\rm ac})$*. That is, under the stated hypotheses Conjecture 5.4 holds.*

Proof. We shall adapt the arguments in [CW16], indicating the necessary adjustments. From Theorem 5.7 and Lemma 5.5, we know that $\mathfrak{Sel}^{\bullet,\bullet}(K, \mathbf{T}^{\rm ac})$ and $\mathfrak{X}^{\bullet,\bullet}_{K^{\rm ac}_\infty}(f)$ both have $\Lambda(\Gamma^{\rm ac})$-rank 1, $\mathfrak{X}^{\rm rel,str}_{K^{\rm ac}_\infty}(f)$ is $\Lambda(\Gamma^{\rm ac})$-torsion, and for any height one prime $\mathfrak{P}$ of $\Lambda(\Gamma^{\rm ac})$, the inequalities

$$\mathrm{length}_{\mathfrak{P}}(c^2 \cdot \mathfrak{X}^{\bullet,\bullet}_{K^{\rm ac}_\infty}(f)_{\rm tors}) \leqslant 2\ \mathrm{length}_{\mathfrak{P}}\left(\frac{\mathfrak{Sel}^{\bullet,\bullet}(K, \mathbf{T}^{\rm ac})}{\Lambda(\Gamma^{\rm ac}) \cdot \mathfrak{Z}^\bullet_c}\right) \tag{5.5}$$

and

$$\mathrm{length}_{\mathfrak{P}}(\mathfrak{X}^{\rm rel,str}_{K^{\rm ac}_\infty}(f)) \leqslant 2\ \mathrm{length}_{\mathfrak{P}}(\mathscr{L}^{\mathsf{BDP}}_{\mathfrak{p}}(f/K)) \tag{5.6}$$

hold. On the other hand, the divisibility in Theorem 5.1 combined with the comparison of p-adic L-functions in [CW16, Cor. 1.12] and a standard control theorem (as in [SU14, Prop. 3.9] and [Wan14, §3.2]) implies that

$$\mathrm{Char}_{\Lambda(\Gamma^{\rm ac})}(\mathfrak{X}^{\rm rel,str}_{K^{\rm ac}_\infty}(f)) \subseteq (\mathscr{L}^{\mathsf{BDP}}_{\mathfrak{p}}(f/K)^2) \tag{5.7}$$

as ideals in $\Lambda_{R_0}(\Gamma^{\rm ac}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Since $\mathscr{L}^{\mathsf{BDP}}_{\mathfrak{p}}(f/K)$ has trivial μ-invariant by Theorem 4.7, the divisibility (5.7) holds integrally, which combined with (5.6) yields the equality in Conjecture 5.4 with $\mathfrak{X}^{\rm rel,str}_{K^{\rm ac}_\infty}(f)$ in place of $X^{\rm rel,str}_{K^{\rm ac}_\infty}(f)$; since by [PW11, §3] (see also [Cas17a, Prop. 2.5]) both Selmer modules have the same characteristic ideal under our ramification hypotheses on $\bar{\rho}_f$, the result follows. □

In particular, Theorem 5.8 yields the divisibility opposite to Theorem 5.1 on the Iwasawa–Greenberg main conjecture for $\mathscr{L}_{\mathfrak{p}}(f/K)$:

Corollary 5.9. *Under the hypotheses of Theorem 5.8, the module* $\mathfrak{X}^{\rm rel,str}_{K_\infty}(f)$ *is* $\Lambda(\Gamma_K)$*-torsion, and we have*

$$\mathrm{Char}_{\Lambda(\Gamma_K)}(X^{\rm rel,str}_{K_\infty}(f)) = (\mathscr{L}_{\mathfrak{p}}(f/K))$$

as ideals in $\Lambda_{R_0}(\Gamma_K)$*,* *Proof.* This follows by combining Theorem 5.8 and the divisibility in Theorem 5.1 as in [CW16, Thm. 5.2]. □

We can now conclude the proof of our first main result.

Proof of Theorem A in the indefinite case. Let $\bullet, \circ \in \{\sharp, \flat\}$ be such that $\mathfrak{L}_p^{\bullet,\circ}(f/K)_{\rm cyc}$ is nonzero. (Note that this is always possible by Corollary 3.7.) By Theorem 3.9, the conclusion of Corollary 5.9 then implies that the module $\mathfrak{X}_{K_\infty}^{\bullet,\circ}(f)$ is $\Lambda(\Gamma_K)$-torsion and we have the equality

$$\mathrm{Char}_{\Lambda(\Gamma_K)}(\mathfrak{X}_{K_\infty}^{\bullet,\circ}(f)) = (\mathfrak{L}_p^{\bullet,\circ}(f/K)) \tag{5.8}$$

as ideals in $\Lambda(\Gamma_K)$, up to exceptional primes. Thus to conclude the proof it suffices to show that neither side in equality (5.8) is divisible by exceptional primes. For the right-hand side, this follows from the assumption that the restriction $\mathfrak{L}_p^{\bullet,\circ}(f/K)_{\rm cyc}$ is nonzero; for the left-hand side, we argue as follows. If $\bullet = \circ$ say with $\bullet = \sharp$, so that our assumption is that $\mathfrak{L}_p^{\sharp,\sharp}(f/K)_{\rm cyc}$ is nonzero, then Corollary 3.7 shows that both $L_p^\sharp(f)$ and $L_p^\sharp(f_K)$ are nonzero, and so both $X_{\mathbf{Q}_\infty}^\sharp(f)$ and $X_{\mathbf{Q}_\infty}^\sharp(f_K)$ are $\Lambda(\Gamma^{\rm cyc})$-cotorsion by [LLZ10, Thm. 6.5]. Combined with Proposition 5.2, this shows that $\mathrm{Char}_{\Lambda(\Gamma_K)}(\mathfrak{X}_{K_\infty}^{\sharp,\sharp}(f))$ has no exceptional prime divisors, concluding the proof of Theorem A follows in this case.

On the other hand, suppose $\bullet \neq \circ$, so that our assumption is that $\mathfrak{L}_p^{\sharp,\flat}(f/K)_{\rm cyc}$ is nonzero. Then to conclude the proof in this case we need to show that $\mathfrak{X}_{K_\infty}^{\sharp,\flat}(f)/(\gamma_{\rm ac}-1)\mathfrak{X}_{K_\infty}^{\sharp,\flat}(f)$ is $\Lambda(\Gamma^{\rm cyc})$-torsion. As shown in the proof of Corollary 3.7, the nonvanishing of $\mathfrak{L}_p^{\sharp,\flat}(f/K)_{\rm cyc}$ implies that either both of $L_p^\sharp(f)$ and $L_p^\flat(f)$ are nonzero, or both of $L_p^\sharp(f_K)$ and $L_p^\flat(f_K)$ are nonzero; suppose is the former case (otherwise, switch f and f_K). Letting $\mathbf{A}_g^{\rm cyc}$ be the module $\mathbf{A}^{\rm cyc}$ in the notations of §2.3 with $g = f$ or f_K, we shall use that

$$H^1(K, \mathbf{A}_f^{\rm cyc}) \simeq H^1(\mathbf{Q}, \mathbf{A}_f^{\rm cyc}) \oplus H^1(\mathbf{Q}, \mathbf{A}_{f_K}^{\rm cyc}), \tag{5.9}$$

where the terms in the right-hand side correspond by the restriction map with the eigenspaces of $H^1(K, \mathbf{A}_f^{\rm cyc})$ under the action of the nontrivial element in $\mathrm{Gal}(K/\mathbf{Q})$ (*cf.* [SU14, Lem. 3.1]). Suppose $(a, b) \in \mathrm{Sel}^{\sharp,\flat}(K, \mathbf{A}_f^{\rm cyc})$. Then we have the inclusions

$$\mathrm{res}_{\mathfrak{p}}(a) + \mathrm{res}_{\mathfrak{p}}(b) \in H^1_\sharp(K_{\mathfrak{p}}, \mathbf{A}_f^{\rm cyc}), \quad \mathrm{res}_{\bar{\mathfrak{p}}}(a) + \mathrm{res}_{\bar{\mathfrak{p}}}(b) \in H^1_\flat(K_{\bar{\mathfrak{p}}}, \mathbf{A}_f^{\rm cyc});$$

but applying the nontrivial element in $\mathrm{Gal}(K/\mathbf{Q})$ the latter inclusion becomes the relation $\mathrm{res}_{\mathfrak{p}}(a) - \mathrm{res}_{\mathfrak{p}}(b) \in H^1_\flat(K_{\mathfrak{p}}, \mathbf{A}_f^{\rm cyc})$, so we have $a \in \mathrm{Sel}^\sharp(\mathbf{Q}, \mathbf{A}_f) + \mathrm{Sel}^\flat(\mathbf{Q}, \mathbf{A}_f)$. Now consider the projection

$$\mathrm{Sel}^{\sharp,\flat}(K, \mathbf{A}_f^{\rm cyc}) \longrightarrow \mathrm{Sel}^\sharp(\mathbf{Q}, \mathbf{A}_f) + \mathrm{Sel}^\flat(\mathbf{Q}, \mathbf{A}_f) \tag{5.10}$$

given by $(a, b) \mapsto a$. Under the identification under (5.9), the kernel of (5.10) is the subgroup of elements of the form $(0, b)$ in $\mathrm{Sel}^{\sharp,\flat}(K, \mathbf{A}_f^{\rm cyc})$, which is just $\mathrm{Sel}^{\rm str}(\mathbf{Q}, \mathbf{A}_{f_K})$ and so is cotorsion, since at least one of $\mathrm{Sel}^\sharp(\mathbf{Q}, \mathbf{A}_{f_K}^{\rm cyc})$ and $\mathrm{Sel}^\flat(\mathbf{Q}, \mathbf{A}_{f_K}^{\rm cyc})$ are (by [LLZ10, Thm. 6.5], since at least one of $L_p^\sharp(f_K)$ and $L_p^\flat(f_K)$ is nonzero, as noted in the proof of Corollary 3.7). At the same time, $\mathrm{Sel}^\sharp(\mathbf{Q}, \mathbf{A}_f^{\rm cyc}) + \mathrm{Sel}^\flat(\mathbf{Q}, \mathbf{A}_f^{\rm cyc})$ is a quotient of the cotorsion group $\mathrm{Sel}^\sharp(\mathbf{Q}, \mathbf{A}_f^{\rm cyc}) \oplus \mathrm{Sel}^\flat(\mathbf{Q}, \mathbf{A}_f^{\rm cyc})$ (this is cotorsion by another application of [LLZ10, Thm. 6.5], since both $L_p^\sharp(f)$ and $L_p^\flat(f)$ are non-zero), so $\mathrm{Sel}^\sharp(\mathbf{Q}, \mathbf{A}_f^{\rm cyc}) + \mathrm{Sel}^\flat(\mathbf{Q}, \mathbf{A}_f^{\rm cyc})$ is also cotorsion, and therefore $\mathrm{Sel}^{\sharp,\flat}(K, \mathbf{A}_f^{\rm cyc})$ is $\Lambda(\Gamma^{\rm cyc})$-cotorsion, so $\mathfrak{X}_{K_\infty}^{\sharp,\flat}(f)/(\gamma_{\rm ac}-1)\mathfrak{X}_{K_\infty}^{\sharp,\flat}(f)$ is $\Lambda(\Gamma^{\rm cyc})$-torsion, concluding the proof of Theorem A in the indefinite case. □

5.2. **Definite case.** In this section we complete the proof of Theorem A in the case in which N^- has an *odd* number of prime factors, and give the proof of Theorem B, our main result on the signed main conjectures of Lei–Loeffler–Zerbes [LLZ10].

Key to both proofs will be the following intermediate result:

Theorem 5.10. *Assume that:*

- *N is square-free,*
- *N^- has an odd number of prime factors,*
- *$\bar{\rho}_f$ is ramified at every prime $\ell \mid N$ which is nonsplit in K,*
- *$\bar{\rho}_f|_{G_K}$ is irreducible.*

If $\bullet, \circ \in \{\sharp, \flat\}$ are such that the restriction $\mathfrak{L}_p^{\bullet,\circ}(f/K)_{\rm cyc}$ is nonzero, then the module $\mathfrak{X}_{K_\infty}^{\bullet,\circ}(f)$ is $\Lambda(\Gamma_K)$-torsion, and we have the divisibility

$$\mathrm{Char}_{\Lambda(\Gamma_K)}(\mathfrak{X}_{K_\infty}^{\bullet,\circ}(f)) \subseteq (\mathfrak{L}_p^{\bullet,\circ}(f/K))$$

as ideals in $\Lambda(\Gamma_K)$.

Proof. The arguments in the proof of Theorem 3.9 show that Theorem 5.1 and the nonvanishing of $\mathfrak{L}_p^{\bullet,\circ}(f/K)$ implies that $\mathfrak{X}_{K_\infty}^{\bullet,\circ}(f)$ is $\Lambda(\Gamma_K)$-torsion, and that the following divisibility holds:

$$\mathrm{Char}_{\Lambda(\Gamma_K)}(\mathfrak{X}_{K_\infty}^{\bullet,\circ}(f)) \subseteq (\mathfrak{L}_p^{\bullet,\circ}(f/K))$$

as ideals in $\Lambda(\Gamma_K)$, up to powers of p and P, indeterminacies that can be removed thanks to Proposition 4.13 and the nonvanishing of the restriction $\mathfrak{L}_p^{\bullet,\circ}(f/K)_{\rm cyc}$, respectively. □

The proof of Theorem A in the definite case will build on Theorem B, so we begin by proving the latter.

Proof of Theorem B. We shall apply Theorem 5.10 for a suitable choice of K, and descend to the cyclotomic line. Indeed, Ribet's level-lowering result [Rib90, Thm. 1.1] forces the residual representation $\bar{\rho}_f$ to be ramified at at least one prime q, which we fix. Let K be an imaginary quadratic field such that:

(a) q is inert in K,
(b) every prime dividing N/q splits in K,
(c) p splits in K,
(d) $L(f_K, 1) \neq 0$.

(The existence of such K is guaranteed by [FH95, Thm. B].) By [Edi92], the residual representation $\bar{\rho}_f$ is irreducible, and by [Ski14, Lem. 2.8.1] (using that $\bar{\rho}_f$ is ramified at the prime q), the restriction $\bar{\rho}_f|_{G_K}$ remains irreducible. Thus the triple (f, K, p) satisfies the conditions of Theorem 5.10, whose divisibility combined with Proposition 3.5 and Proposition 5.2 yields the divisibility

$$(5.11) \qquad (L_p^{\bullet}(f) \cdot L_p^{\bullet}(f_K)) \supseteq \mathrm{Char}_{\Lambda(\Gamma^{\rm cyc})}(X_{\mathbf{Q}_\infty}^{\bullet}(f)) \cdot \mathrm{Char}_{\Lambda(\Gamma^{\rm cyc})}(X_{\mathbf{Q}_\infty}^{\bullet}(f_K)),$$

where we used the fact that we may replace $X_{K_\infty}^{\bullet,\bullet}(f)$ by $\mathfrak{X}_{K_\infty}^{\bullet,\bullet}(f)$ in Proposition 5.2 (since $\bar{\rho}_f$ is ramified at every prime dividing $N^- = q$).

On the other hand, the function $L_p^{\bullet}(f)$ is nonzero by hypothesis, while the nonvanishing of $L_p^{\bullet}(f_K)$ follows from condition (d) above and the interpolation property of $L_p^{\bullet}(f_K)$ at the trivial character (see [LLZ10, Prop. 3.28]). Hence by [LLZ10, Thm. 6.5][2] we have the divisibilities

$$(L_p^{\bullet}(f)) \subseteq \mathrm{Char}_{\Lambda(\Gamma^{\rm cyc})}(X_{\mathbf{Q}_\infty}^{\bullet}(f)) \quad \text{and} \quad (L_p^{\bullet}(f_K)) \subseteq \mathrm{Char}_{\Lambda(\Gamma^{\rm cyc})}(X_{\mathbf{Q}_\infty}^{\bullet}(f_K)).$$

Since having strict inclusion in either of these would contradict (5.11), the proof of Theorem B follows. □

[2]Note that Assumption (A) in *loc.cit.* is only used to guarantee the nonvanishing of the p-adic L-function.

Proof of Theorem A in the definite case. This now follows immediately from the combination of Theorem 5.10 and Theorem B. Indeed, let $\bullet \in \{\sharp, \flat\}$ be such that $\mathfrak{L}_p^{\bullet,\bullet}(f/K)_{\rm cyc}$ is nonzero (see Corollary 3.7), and define the ideals $X, Y \subseteq \Lambda(\Gamma_K)$ by

$$X := \mathrm{Char}_{\Lambda(\Gamma_K)}(\mathfrak{X}^{\bullet,\bullet}_{K_\infty}(f)), \quad Y := (\mathfrak{L}_p^{\bullet,\bullet}(f/K)),$$

and let $I^{\rm ac} = (\gamma_{\rm ac} - 1)$ be the kernel of the projection $\Lambda(\Gamma_K) \twoheadrightarrow \Lambda(\Gamma^{\rm cyc})$. By Theorem 5.10 we have the divisibility $X \subseteq Y$, which combined with Proposition 5.2 yields the divisibilities

$$\begin{aligned}\mathrm{Char}_{\Lambda(\Gamma^{\rm cyc})}(X^{\bullet}_{\mathbf{Q}_\infty}(f)) \cdot \mathrm{Char}_{\Lambda(\Gamma^{\rm cyc})}(X^{\bullet}_{\mathbf{Q}_\infty}(f_K)) &\subseteq (X \bmod I^{\rm ac}) \\ &\subseteq (Y \bmod I^{\rm ac}) \\ &= (L_p^{\bullet}(f) \cdot L_p^{\bullet}(f_K)),\end{aligned}$$

where the last equality is given by Proposition 3.5. Since the extremes of this string are equal by Theorem B applied to f and f_K, we conclude that $(X \bmod I^{\rm ac}) = (Y \bmod I^{\rm ac})$, and hence $X = Y$ by [SU14, Lem. 3.2], finishing the proof of Theorem A. □

5.3. **BSD formulae.** In this section we deduce the applications of the previous results to the p-part of the Birch–Swinnerton-Dyer formula for abelian varieties over $\mathbf{Q}$ of GL_2-type.

Proof of Theorem C. Let $A/\mathbf{Q}$ be an abelian variety of GL_2-type as in the statement of Theorem C. By [KW09, Cor. 10.2], we have $L(A, s) = L(f, s)$ for some $f \in S_2(\Gamma_0(N))$, and by [Rib90, Thm. 1.1] the residual representation $\bar{\rho}_f$ is ramified at some prime q. Let L be the completion of the Heck field of f at the prime $\mathfrak{P}$ above p inducted by our fixed isomorphism $\mathbf{C} \simeq \mathbf{C}_p$, and set $T_f^* := T_{\mathfrak{P}}A$, the $\mathfrak{P}$-adic Tate module of A.

Take a sign $\bullet \in \{\sharp, \flat\}$, and define $H^1_\bullet(\mathbf{Q}_p, A[\mathfrak{P}^\infty])$ to be the image of $H^1_\bullet(\mathbf{Q}, \mathbf{T}^{\rm cyc})$ under the natural map

$$H^1_\bullet(\mathbf{Q}_p, \mathbf{T}^{\rm cyc}) \longrightarrow H^1(\mathbf{Q}_p, T_f^*) \longrightarrow H^1(\mathbf{Q}_p, A[\mathfrak{P}^\infty]),$$

and similarly define the unramified local condition $H^1_{\rm ur}(\mathbf{Q}_\ell, A[\mathfrak{P}^\infty])$ for primes $\ell \neq p$. Letting $\mathrm{Sel}^\bullet_{\mathbf{Q}}(f) \subseteq H^1(\mathbf{Q}, A[\mathfrak{P}^\infty])$ be defined by the same recipe as $\mathrm{Sel}^\bullet(\mathbf{Q}, \mathbf{A}^{\rm cyc})$ (see Remark 2.11), we then have

$$\mathrm{Sel}^\bullet_{\mathbf{Q}}(f) = \mathrm{Sel}_{\mathfrak{P}^\infty}(A/\mathbf{Q})$$

(see [HL16, Prop. 2.14]). The result then follows from Theorem B, the interpolation property satisfied by $L_p^\bullet(f)$ at the trivial character (see [LLZ10, Prop. 3.28]), and a variant of [Gre99, Thm. 4.1] for signed Selmer groups (see [Spr16, §4.1]). □

Proof of Theorem D. A straightforward adaptation of the arguments in [JSW17, §7.4]; in fact, in light of Theorem 5.8, it suffices to adapt the slightly simpler argument in [Cas17a, §5]. We briefly recall the details for the convenience of the reader.

Let $A/\mathbf{Q}$ be an abelian variety of GL_2-type as in the statement of Theorem D. Then, just as before, we have $L(A, s) = L(f, s)$ for some $f \in S_2(\Gamma_0(N))$, and the residual representation $\bar{\rho}_f$ is ramified at some prime q, which we fix. With notations as in the proof of Theorem C, let $\mathcal{O}_L$ be the ring of integers of L. Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field satisfying:

(a) q is ramified in K,
(b) every prime factor $\ell \neq q$ of N splits in K,
(c) p splits in K,
(d) $L(f^D, 1) \neq 0$.

By (1.7) and (d), we have $\mathrm{ord}_{s=1} L(A/K, s) = [K_f : \mathbf{Q}]$ and hence $A(K) \otimes_{\mathcal{O}_f} \mathcal{O}_L$ has $\mathcal{O}_L$-rank 1 and Ш(A/K) is finite by the work of Gross–Zagier and Kolyvagin. As in [Cas17a, Thm. 2.3], by the results of [JSW17, §3.3] the latter two conditions imply that $X^{\rm rel,str}_{K^{\rm ac}_\infty}(A[\mathfrak{P}^\infty])$ is $\Lambda(\Gamma^{\rm ac})$-torsion, and letting $f_{\rm ac}(T) \in \Lambda(\Gamma^{\rm ac}) \simeq \mathcal{O}_L[[T]]$ be a generator of $\mathrm{Char}_{\Lambda(\Gamma^{\rm ac})}(X^{\rm rel,str}_{K^{\rm ac}_\infty}(A[\mathfrak{P}^\infty]))$, we have

$$\#\mathcal{O}_L/f_{\rm ac}(0) = \#\text{Ш}(A/K)[\mathfrak{P}^\infty] \cdot \left(\frac{\#\mathcal{O}_L/((\frac{1-a_p+p}{p})\log_{\omega_f} P)}{[A(K)\otimes_{\mathcal{O}_f}\mathcal{O}_L : \mathcal{O}_L.P]}\right)^2 \cdot \prod_w c_w(A/K)_{\mathfrak{P}},$$

where $P \in A(K)$ is any point of infinite order, and $c_w(A/K)_{\mathfrak{P}}$ is the $\mathfrak{P}$-part of the Tamagawa number of A/K at w. Here the product is over primes w dividing N/q, but since $\bar{\rho}_f$ is ramified at q, by [Zha14, Lem. 6.3] there is no contribution from the prime above q. Taking for P the Heegner point appearing in [JSW17, Prop. 5.1.7] (which has infinite order by the Gross–Zagier formula [YZZ13]), the result follows just as in the aforementioned references, using Theorem C to descend from K to $\mathbf{Q}$. □

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