ON THE p-ADIC VARIATION OF HEEGNER POINTS

FRANCESC CASTELLA

Abstract. In this paper, we prove a new "explicit reciprocity law" relating Howard's system of big Heegner points to a two-variable p-adic L-function (constructed here) interpolating the p-adic Rankin L-series of Bertolini–Darmon–Prasanna in Hida families. As applications, we obtain a direct relation between classical Heegner cycles and the higher weight specializations of big Heegner points, refining earlier work of the author, and prove the vanishing of Selmer groups of CM elliptic curves twisted by 2-dimensional Artin representations in cases predicted by the equivariant Birch and Swinnerton-Dyer conjecture.

Contents

1. Introduction

Let $f = \sum_{n\geqslant 1} a_n(f)q^n \in S_{2r}(\Gamma_0(N))$ be a newform of weight $2r \geqslant 2$, fix a prime $p \nmid 6N$, and let L be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathfrak{O}$ containing the image the Fourier coefficients $a_n(f)$ under a fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Denote by

$$\rho_f : G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}_L(V_f(r)) \simeq \mathrm{GL}_2(L)$$

the Kummer self-dual twist of the p-adic Galois representation associated with f. Let K be an imaginary quadratic field of odd discriminant $-D_K < -3$ and ring of integers $\mathcal{O}_K$ satisfying the classical *Heegner hypothesis* relative to N:

$$\text{(heeg)} \qquad \text{there is an integral ideal } \mathfrak{N} \text{ of } K \text{ with } \mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z};$$

equivalently, every prime $q \mid N$ either splits or ramifies in K, with $q^2 \nmid N$ in the latter case.

The first purpose of this paper is to complete earlier work of the author [Cas13] comparing two natural constructions of a cohomology class of "Heegner-type" attached to the pair (f, K). For the first of these, let $\mathrm{Sel}(K, V_f(r)) \subseteq H^1(G_K, V_f(r))$ be the Bloch–Kato Selmer group for the representation $V_f(r)|_{\mathrm{Gal}(\overline{\mathbf{Q}}/K)}$. By [Nek00], the image under the p-adic étale Abel–Jacobi map of classical Heegner cycles [Nek95] on the $(2r-1)$-dimensional Kuga–Sato variety of level N give rise to a class

$$\Phi^{\text{ét}}_{f,K}(\Delta_r^{\text{heeg}}) \in \mathrm{Sel}(K, V_f(r)).$$

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For the second class, assume that f is *ordinary at* $\imath_p$, i.e.:

$$\text{(ord)} \qquad a_p(f) \in \mathfrak{O}^\times.$$

Fix a Galois-stable $\mathfrak{O}$-lattice $T_f \subseteq V_f$, let $\bar\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_L)$ be the associated semisimple residual representation, where κ_L is the residue field of L, and assume that

$$\text{(irred)} \qquad \bar\rho_f \text{ is irreducible.}$$

Let $D_p \subseteq G_{\mathbf{Q}}$ be a fixed decomposition group at p. By hypothesis (ord), the restriction $\rho_f|_{D_p}$ can be made upper-triangular, and we shall assume in addition that

$$\text{(dist)} \qquad \bar\rho_f \text{ is } D_p\text{-distinguished;}$$

i.e., the semisimplification of $\bar\rho_f|_{D_p}$ is the direct sum of two *distinct* characters. Suppose that $r \equiv 1 \pmod{p-1}$, and let $\mathbf{f} = \sum_{n\geqslant 1} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$ be the Hida family passing though f. Thus $\mathbb{I}$ is a finite flat extension of $\mathfrak{O}[[X]]$, and for every continuous $\mathfrak{O}$-algebra homomorphism $\nu : \mathbb{I} \to \overline{\mathbf{Q}}_p$ satisfying $\nu(1+X) = (1+p)^{k_\nu - 2}$ for some integer $k_\nu \geqslant 2$ with $k_\nu \equiv 2 \pmod{p-1}$, the q-series $\mathbf{f}_\nu := \sum_{n\geqslant 1} \nu(\mathbf{a}_n) q^n$ is such that

$$\mathbf{f}_\nu = f_\nu(q) - \frac{p^{k_\nu - 1}}{\nu(\mathbf{a}_p)} f_\nu(q^p),$$

for some newform $f_\nu \in S_{k_\nu}(\Gamma_0(N))$, with $f = f_\nu$ for a unique ν with $k_\nu = 2r$. (More generally, we may consider the image of $\mathbf{f}$ under maps $\nu : \mathbb{I} \to \overline{\mathbf{Q}}_p$ with arbitrary $k_\nu \in \mathbf{Z}_{>0}$.) Under the above hypotheses, Howard's construction of big Heegner points [How07] produces a class

$$\mathfrak{Z}_0 \in H^1(G_K, \mathbf{T}^\dagger),$$

where $\mathbf{T}$ is a self-dual twist of the big Galois representation $\rho_{\mathbf{f}}$ associated with $\mathbf{f}$. Under some additional hypothesis on the residual representation $\bar\rho_{\mathbf{f}} \simeq \bar\rho_f$ when $(D_K, N) > 1$, one can show that $\mathfrak{Z}_0$ lies in the so-called *strict Greenberg Selmer group* $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbf{T}^\dagger) \subseteq H^1(G_K, \mathbf{T}^\dagger)$, and so its image under the specialization map ν_f produces a second class $\nu_f(\mathfrak{Z}_0) \in \mathrm{Sel}(K, V_f(r))$.

Theorem A (Theorem 6.5)**.** *Assume in addition that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K, $\bar\rho_f|_{G_K}$ is irreducible, and $\bar\rho_f$ is ramified at every prime $q \mid N$ nonsplit in K. Then*

$$\nu_f(\mathfrak{Z}_0) = \left(1 - \frac{p^{r-1}}{\nu_f(\mathbf{a}_p)}\right)^2 \frac{\Phi^{\text{ét}}_{f,K}(\Delta_r^{\text{heeg}})}{u_K (2\sqrt{-D_K})^{r-1}},$$

where $u_K = |\mathcal{O}_K^\times|/2$.

This subsumes the main result of [Cas13], which only implies the equality in Theorem A under the assumption of Howard's "horizontal nonvanishing conjecture" [How07, Conj. 2.2.2] and the nondegeneracy of the cyclotomic p-adic height pairing. The class $\mathfrak{Z}_0$ is obtained from Howard's big Heegner point $\mathfrak{X}_1$ of conductor 1, and more generally Theorem 6.5 establishes a comparison between the Selmer classes constructed from Heegner cycles of conductor $c \geqslant 1$ prime to Np and the corresponding higher weight specializations of the big Heegner point $\mathfrak{X}_c$. In particular, Theorem 6.5 answers a question raised by Howard (see [How07, p. 93]).

Similarly as in [Cas13], the proof of Theorem A follows from relating the cohomology classes involved to special values of L-functions. More precisely, extending work of Bertolini–Darmon–Prasanna [BDP13] and Brakočević [Bra11], in [CH17] we constructed an anticyclotomic p-adic L-function $\mathscr{L}_{\mathfrak{p},\psi}(f)$ interpolation central critical values for the Rankin–Selberg convolution of f with certain theta series. Moreover, also in [CH17] we constructed a compatible system of cohomology classes $\mathbf{z}_f$ interpolating the p-adic étale Abel–Jacobi images of (generalized) Heegner cycles of p-power conductor, and extending the p-adic Gross–Zagier formula of [BDP13] we obtained an "explicit reciprocity law"

$$\text{(1.1)} \qquad \langle \mathcal{L}_{\mathfrak{p},\psi}(\mathbf{z}_f), \omega_f \otimes t^{1-2r} \rangle = -\mathscr{L}_{\mathfrak{p},\psi}(f)$$

relating $\mathscr{L}_{\mathfrak{p},\psi}(f)$ to the image of $\mathbf{z}_f$ under a Perrin-Riou "big logarithm map". In Section 2 of this paper, we construct a two-variable p-adic L-function $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ interpolating the p-adic L-functions of [CH17] attached to the different specializations f_ν of $\mathbf{f}$; in particular,

$$\nu_f(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})) = \mathscr{L}_{\mathfrak{p},\psi}(f). \tag{1.2}$$

The key new ingredient in our proof of Theorem A is then the connection that we find between $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ and the system $\mathfrak{Z}_\infty$ of Howard's big Heegner points of p-power conductor.

Theorem B (Theorem 5.3)**.** *There is a Perrin-Riou big logarithm map $\mathcal{L}_{\mathfrak{p},\boldsymbol{\xi}}$ sending Howard's system of big Heegner points $\mathfrak{Z}_\infty$ attached to $\mathbf{f}$ over the anticyclotomic $\mathbf{Z}_p$-extension of K to the two-variable p-adic L-function $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$.*

The construction of the Perrin-Riou map $\mathcal{L}_{\mathfrak{p},\boldsymbol{\xi}}$ is given in Section 3, building upon work of Ochiai [Och03] and Loeffler–Zerbes [LZ14], and the proof of the "explicit reciprocity law" of Theorem B is obtained in Section 5 after a suitable extension of the calculations in [Cas13]. With this result at hand, the proof of Theorem A follows easily by specializing the equality in Theorem 5.3 at ν_f, using (1.2) and the interpolation property of the map $\mathcal{L}_{\mathfrak{p},\boldsymbol{\xi}}$, and comparing it with the equality in (1.1).

The second purpose of this paper is to exploit the p-adic variation of Heegner points in Hida families to establish certain new cases of the equivariant Birch–Swinnerton-Dyer conjecture for rational elliptic curves with complex multiplication. More precisely, let $A/\mathbf{Q}$ be an elliptic curve with CM, and let

$$\varrho : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_E(V_\varrho) \simeq \mathrm{GL}_2(E)$$

be a 2-dimensional odd and irreducible Artin representation factoring through a finite quotient $\mathrm{Gal}(F/\mathbf{Q})$ and with values in a finite extension $E \subseteq \mathbf{C}$ of $\mathbf{Q}$. Let $T_p(A)$ be the p-adic Tate module of A, and set $V_p(A) := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p(A)$. Associated to the compatible system $V_p(A) \otimes \imath_p V_\varrho$ of p-adic representations of $G_{\mathbf{Q}}$ is a Artin–Hasse–Weil L-function $L(A/\mathbf{Q}, \varrho, s)$. This is defined for $\mathrm{Re}(s) > 3/2$ by an absolutely convergent Euler product of degree 4, and by [Hec27] and [KW09] it is known to admit an analytic continuation to the entire complex plane, with a functional equation relating its values at s and $2-s$. The equivariant Birch–Swinnerton-Dyer conjecture then predicts that

$$\mathrm{ord}_{s=1} L(A/\mathbf{Q}, \varrho, s) \overset{?}{=} \dim_L \mathrm{Hom}_{G_{\mathbf{Q}}}(V_\varrho, A(F)_L), \tag{1.3}$$

and that

$$\mathrm{Hom}_{G_{\mathbf{Q}}}(V_\varrho, Ш_{p^\infty}(A/F)_L) \overset{?}{=} \{0\} \tag{1.4}$$

for all primes p, where $Ш_{p^\infty}(A/F)$ is the p-primary component of the Tate–Shafarevich group of A/F, and for any abelian group M let set $M_L := M \otimes_{\mathbf{Z}} L$. Let N_A and N_ϱ be the conductor of A and ϱ, respectively, and denote by $\mathrm{Sel}(F, V_p A) \subseteq H^1(G_F, V_p(A))$ the Bloch–Kato Selmer group for $V_p(A)|_{\mathrm{Gal}(\overline{\mathbf{Q}}/F)}$.

Theorem C. *Let $A/\mathbf{Q}$ be an elliptic curve of conductor N_A and with complex multiplication by an imaginary quadratic field K, let $p \nmid 6N_\varrho N_A$ be a prime, and let $\mathfrak{P}$ be a prime of E above p. Assume that:*

- *N_ϱ and N_A are coprime;*
- *$p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K;*
- *K satisfies hypothesis* (heeg) *relative to N_ϱ;*
- *$\varrho(\mathrm{Frob}_p)$ has distinct eigenvalues modulo $\mathfrak{P}$.*

If $L(A/\mathbf{Q}, \varrho, 1) \neq 0$, then

$$\mathrm{Hom}_{G_{\mathbf{Q}}}(V_\varrho, \mathrm{Sel}(F, V_p(A))_L) = \{0\}.$$

In particular, (1.3) *and* (1.4) *hold.*

The conclusion that (1.3) holds under the nonvanishing of $L(A/\mathbf{Q},\varrho,s)$ at $s=1$ was already contained in earlier work of Bertolini–Darmon–Rotger [BDR15, Thm. A], while recent work of Kings–Loeffler–Zerbes [KLZ17, Thm. 11.7.4] establishes an analog of Theorem C for rational elliptic curves *without* complex multiplication (the CM case is excluded in [KLZ17] by the "big image hypothesis" of [*loc.cit.*, §11.1]). Thus the new content of Theorem C is the vanishing of the ϱ-isotypical component of $Ш_{p^\infty}(A/F)_L$ for "half" of the primes p under the nonvanishing of $L(A/\mathbf{Q},\varrho,1)$.

Let us conclude this Introduction with a few words about the proof of Theorem C. Denote by $L(f/K,\chi,s)$ the Rankin–Selberg L-function for the convolution of a cusp form $f\in S_k(\Gamma_1(N))$ with a Hecke character χ of K. From the explicit reciprocity law of Theorem B, we deduce a proof of the implication

$$L(f_\nu/K,\chi\mathbf{N}^{k_\nu/2},0)\neq 0 \implies \nu(\mathfrak{Z}_\infty)^{\chi^{-1}}\neq 0,$$

for $\nu:\mathbb{I}\to\overline{\mathbf{Q}}_p$ of weight $k_\nu>0$ and certain anticyclotomic Hecke characters χ. Since Howard's big Heegner points satisfies the axioms of an Euler system, one can deduce from Kolyvagin's methods (as extended in [CH17, §7.2] to the anticyclotomic setting) a proof of the implication

$$L(f_\nu/K,\chi\mathbf{N}^{k_\nu/2},0)\neq 0 \implies \mathrm{Sel}(K,V_{\nu,\chi})=\{0\},$$

where $\mathrm{Sel}(K,V_{\nu,\chi})$ is the Bloch–Kato Selmer group for the representation $V_{f_\nu}|_{G_K}\otimes\chi$. Since by [KW09] any Artin representation ϱ as in Theorem C is attached to some $g\in S_1(\Gamma_1(N_\varrho))$, taking χ so that $\chi\mathbf{N}^{1/2}$ corresponds to the grossencharacter of A, $\mathbf{f}$ to be a Hida family passing through g, and specializing the resulting $\mathfrak{Z}_\infty$ to weight one, the proof of Theorem C follows.

Some notations and definitions. For any place v of number field E, we let $\mathrm{rec}_v:E_v^\times\to G_{E_v}^{\mathrm{ab}}$ and $\mathrm{rec}_E:E^\times\backslash\mathbb{A}_E^\times\to G_E^{\mathrm{ab}}$ be the local and global reciprocity map, respectively, with geometric normalizations. If $\phi:\mathbf{Z}_p^\times\to\mathbf{C}^\times$ is a continuous character of conductor p^n, the Gauss sum of ϕ is defined by

$$\mathfrak{g}(\phi)=\sum_{u\in(\mathbf{Z}/p^n\mathbf{Z})^\times}\phi(u)\mathbf{e}(u/p^n),$$

where $\mathbf{e}(z)=\exp(2\pi iz)$, and if $\chi:\mathbf{Q}_p\to\mathbf{C}^\times$ is a continuous character of conductor p^n, we define the ε-factor of χ by $\varepsilon(\chi)=p^n\chi^{-1}(p^n)\mathfrak{g}(\chi^{-1})^{-1}$.

2. p-adic Rankin L-series

In this section, we give the construction of a two-variable anticyclotomic p-adic L-function $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ attached to a Hida family $\mathbf{f}$ and an imaginary quadratic field K in which $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits. Such construction closely parallels the one-variable construction by Brakočević [Bra11], and was essentially contained in [Bra12].

2.1. Geometric and p-adic modular forms.

Fix a prime p, and let $N\geqslant 3$ be an integer prime to p.

Definition 2.1. Let k be an integer and let B be a $\mathbf{Z}_{(p)}$-algebra. A *geometric modular form* f of weight k on $\Gamma_1(Np^\infty)$ defined over B is a rule which assigns to every triple $(A,\eta,\omega)_{/C}$, over an arbitrary B-algebra C, consisting of:

- an elliptic curve A/C;
- a $\Gamma_1(Np^\infty)$-level structure η on A, i.e., an immersion
$$\eta=(\eta^{(p)},\eta_p):\boldsymbol{\mu}_N\oplus\boldsymbol{\mu}_{p^\infty}\hookrightarrow A[N]\oplus A[p^\infty]$$
as group schemes over S;
- a C-basis ω of $H^0(A,\Omega^1_{A/C})$,

a value $f(A,\eta,\omega) \in C$ depending only on the isomorphism class of (A,η,ω) over C and such that:

(1) For any B-algebra homomorphism $\varphi : C \to C'$, we have
$$f((A,\eta,\omega)\otimes_C C') = \varphi(f(A,\eta,\omega));$$
(2) For all $\lambda \in C^\times$, we have
$$f(A,\eta,\lambda\omega) = \lambda^{-k} f(A,\eta,\omega);$$
(3) Letting $(\mathrm{Tate}(q),\eta_{\mathrm{can}},\omega_{\mathrm{can}})_{/B((q))}$ be the Tate elliptic curve $\mathbf{G}_m/q^{\mathbf{Z}}$ equipped with its canonical level structure η_{can} and differential ω_{can}, we have
$$f(\mathrm{Tate}(q),\eta_{\mathrm{can}},\omega_{\mathrm{can}}) \in B[[q]].$$

Let $\mathrm{Ig}(N)_{/\mathbf{Z}_{(p)}}$ be the Igusa scheme parameterizing pairs $(A,\eta)_{/S}$ of elliptic curves equipped with $\Gamma_1(Np^\infty)$-level structure over arbitrary locally Noetherian $\mathbf{Z}_{(p)}$-schemes S. The generic fiber $\mathrm{Ig}(N)_{/\mathbf{Q}}$ is given by

$$\mathrm{Ig}(N)_{/\mathbf{Q}} = \varprojlim_s Y_1(Np^s)_{/\mathbf{Q}}, \tag{2.1}$$

where $Y_1(Np^s)_{/\mathbf{Q}}$ is the usual open modular curve of level $\Gamma_1(Np^s)$, and a geometric modular form f as in Definition 2.1 can be viewed as a section of a certain sheaf on $\mathrm{Ig}(N)_{/\mathbf{Z}_{(p)}}$.

Definition 2.2. Denote by Γ^{wt} the group $1+p\mathbf{Z}_p \subseteq \mathbf{Z}_p^\times$, let $k \in \mathbf{Z}_p$ and let $\varepsilon : \Gamma^{\mathrm{wt}} \to \boldsymbol{\mu}_{p^\infty}$ be a finite order character. A *p-adic modular form* of tame level N and weight (k,ε), defined over a p-adic ring R (i.e., $R \simeq \varprojlim_m R/p^mR$) is a function f on the formal completion $\widehat{\mathrm{Ig}}(N)_{/R}$ of $\mathrm{Ig}(N)_{/R}$ satisfying
$$f|\langle u\rangle(A,\eta) := f(A,\eta^{(p)},\eta_p u) = \varepsilon(u)u^k f(A,\eta),$$
for all $u \in \Gamma^{\mathrm{wt}}$ and $[(A,\eta)] = [(A,\eta^{(p)},\eta_p)] \in \widehat{\mathrm{Ig}}(N)_{/R}$.

Denote by $V_p(N;R)$ the space of p-adic modular forms of tame level N defined over a p-adic ring R. Associated with every geometric modular form f on $\Gamma_1(Np^\infty)$ over R there is a p-adic modular form $\widehat{f} \in V_p(N;R)$ defined by the rule
$$\widehat{f}(A,\eta) = f(A,\eta,\widehat{\omega}(\eta_p)), \quad \text{for all } [(A,\eta)] \in \widehat{\mathrm{Ig}}(N)_{/R},$$
where $\widehat{\omega}(\eta_p)$ is the canonical differential on A arising from the isomorphism of formal groups $\widehat{\eta}_p : \widehat{\mathbf{G}}_m \simeq \widehat{A}$ induced by $\eta_p : \boldsymbol{\mu}_{p^\infty} \hookrightarrow A[p^\infty]$.

2.2. **$\mathbb{I}$-adic modular forms.** Let $\mathfrak{O}$ be the ring of integers of a finite extension of $L/\mathbf{Q}_p$, and set $\Lambda_{\mathfrak{O}}^{\mathrm{wt}} = \mathfrak{O}[[\Gamma^{\mathrm{wt}}]]$. For any $k \in \mathbf{Z}$ and $\varepsilon : \Gamma^{\mathrm{wt}} \to \boldsymbol{\mu}_{p^\infty}$ let
$$\nu_{k,\varepsilon} : \Lambda_{\mathfrak{O}}^{\mathrm{wt}} \longrightarrow \overline{\mathbf{Q}}_p$$
be the $\mathfrak{O}$-algebra homomorphism defined by $u \mapsto \varepsilon(u)u^{k-2}$ for $u \in \Gamma^{\mathrm{wt}}$.

Definition 2.3. Let $\mathbb{I}$ be a $\Lambda_{\mathfrak{O}}^{\mathrm{wt}}$-algebra. We say that a $\mathfrak{O}$-algebra homomorphism $\nu : \mathbb{I} \to \overline{\mathbf{Q}}_p$ is an *arithmetic prime* if the composition
$$\Lambda_{\mathfrak{O}}^{\mathrm{wt}} \longrightarrow \mathbb{I} \xrightarrow{\nu} \overline{\mathbf{Q}}_p$$
is of the form $\nu_{k,\varepsilon}$ for some $k \in \mathbf{Z}_{\geqslant 2}$ and $\varepsilon : \Gamma^{\mathrm{wt}} \to \boldsymbol{\mu}_{p^\infty}$. We then say that ν has weight (k,ε), omitting ε from the notation if $\varepsilon = \mathbb{1}$.

We denote by $\mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$ the set of arithmetic primes of $\mathbb{I}$, which we may also view as a subset of $\mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p)$, and for each $\nu \in \mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$ let F_ν be the residue field of $\ker(\nu)$, and $\mathcal{O}_\nu \subseteq F_\nu$ be the valuation ring.

Definition 2.4. Let $\psi_0 : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathfrak{O}^\times$ be a Dirichlet character modulo Np, and let $\mathbb{I}$ be a finite flat $\Lambda_{\mathfrak{O}}^{\mathrm{wt}}$-algebra.

(1) An $\mathbb{I}$-*adic modular form* of tame level N and character ψ_0 is a formal q-expansion
$$\mathbf{f} = \sum_{n=0}^{\infty} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$$
such that for all $\nu \in \mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$ of weight (k,ε), the q-series $\sum_n \nu(\mathbf{a}_n)q^n$ is the q-expansion of a p-adic modular form $\mathbf{f}_\nu \in V_p(N;\mathcal{O}_\nu)$ of weight (k,ε). Denote by $G(N,\psi_0;\mathbb{I})$ the $\mathbb{I}$-module of such formal q-expansions.

(2) We say that $\mathbf{f} \in G(N,\psi_0;\mathbb{I})$ is *arithmetic* if, for all $\nu \in \mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$, $\mathbf{f}_\nu = \widehat{f_\nu}$ is the p-adic avatar of a classical modular form
$$f_\nu \in M_k(\Gamma_0(Np^s), \psi_0\varepsilon\omega^{2-k}),$$
where $s > 0$ is the power of p in the conductor of ε, and $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p^\times$ is the Teichmüller character; and say that $\mathbf{f}$ is *cuspidal* if f_ν is a cusp form for all $\nu \in \mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$. Denote by $S^a(N,\psi_0;\mathbb{I}) \subseteq G(N,\psi_0;\mathbb{I})$ the submodule consisting of cuspidal arithmetic $\mathbb{I}$-adic modular forms.

(3) We say that $\mathbf{f} \in S^a(N,\psi_0;\mathbb{I})$ is *ordinary* if the U_p-operator acts invertibly on $\mathbf{f}_\nu$ for all $\nu \in \mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$, and let $S^{\mathrm{ord}}(N,\psi_0;\mathbb{I}) \subseteq S^a(N,\psi_0;\mathbb{I})$ be the corresponding submodule.

Define
$$V_p(N,\psi_0;\mathbb{I}) := V_p(N,\psi_0;\mathfrak{O})\widehat{\otimes}_{\mathfrak{O}}\mathbb{I},$$
and let $[z] : \mathbf{Z}_p^\times \to \mathfrak{O}[[\mathbf{Z}_p^\times]]^\times$ be the character given by inclusion of group-like elements. The space $V_p(N,\psi_0;\mathbb{I})$ is thus equipped with two different actions of $z \in \Gamma^{\mathrm{wt}}$: one via the diamond operators $\langle z\rangle$ acting on the first factor of the above completed tensor product, and the other via multiplication by $[z]$ on the second factor.

Proposition 2.5. *There is a canonical* $\mathbb{I}$*-module isomorphism*
$$G(N,\psi_0;\mathbb{I}) = \{\mathbf{f} \in V_p(N,\psi_0;\mathbb{I}) : \mathbf{f}|\langle z\rangle = [z]\mathbf{f}, \quad \forall z \in \mathbf{Z}_p^\times\}.$$

Proof. See [Hid00, Thm. 3.2.16]. □

Thus by extension of scalars we may evaluate any $\mathbf{f} \in G(N,\psi_0;\mathbb{I})$ at a point $x \in \widehat{\mathrm{Ig}}(N)(\mathbb{I})$, producing an element $\mathbf{f}(x) \in \mathbb{I}$. This will be used in §2.3 to define measures associated with $\mathbf{f}$ which, for appropriate choices of x (defined in §2.4), interpolate special values of L-functions.

2.3. **Modular measures.** Let W be a finite extension of $\widehat{\mathbf{Z}}_p^{\mathrm{ur}}$, the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$, and denote by $\mathrm{Cont}(\mathbf{Z}_p, W)$ the space of continuous W-valued functions on $\mathbf{Z}_p$. Let
$$\mathrm{Meas}(\mathbf{Z}_p, W) := \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cont}(\mathbf{Z}_p, W), W)$$
be the space of W-valued measures of $\mathbf{Z}_p$. As usual, if $\mu \in \mathrm{Meas}(\mathbf{Z}_p, W)$ and $\phi \in \mathrm{Cont}(\mathbf{Z}_p, W)$, we use the notation $\int_{\mathbf{Z}_p}\phi(z)\mathrm{d}\mu(z) := \mu(\phi)$.

The *Amice transform* $\mathscr{A}_\mu(T) \in W[[T]]$ of a measure μ is the power series
$$\mathscr{A}_\mu(T) := \sum_{m=0}^{\infty} c_m(\mu)T^m,$$
where $c_m(\mu) = \int_{\mathbf{Z}_p}\binom{z}{m}\mathrm{d}\mu(z)$. One easily checks that
$$\int_{\mathbf{Z}_p} z^n \mathrm{d}\mu(z) = \left(T\frac{d}{dT}\right)^n \mathscr{A}_\mu(T)\big|_{T=0}$$

for all $n \geqslant 0$, and by Mahler's theorem the rule $\mu \mapsto \mathscr{A}_\mu$ defines an isomorphism $\mathrm{Meas}(\mathbf{Z}_p, W) \simeq W[[T]]$ of p-adic Banach algebras.

For $\mathbb{I} = \Lambda_{\mathfrak{O}}^{\mathrm{wt}}$, any $\mathbb{I}$-adic modular form $\mathbf{f} = \sum_{n=0}^{\infty} \mathbf{a}_n q^n \in G(N, \psi_0; \mathbb{I})$ defines a measure $\mu_{\mathbf{f}}$ on Γ^{wt} with values in $V_p(N, \psi_0; \mathfrak{O})$ by

$$\int_{\Gamma^{\mathrm{wt}}} z^k \mathrm{d}\mu_{\mathbf{f}}(z) = \sum_{n=0}^{\infty} \nu_k(\mathbf{a}_n) q^n,$$

for all $k \in \mathbf{Z}$; by linearity, the same is true for modular forms over any finite $\Lambda_{\mathfrak{O}}^{\mathrm{wt}}$-algebra $\mathbb{I}$.

Let d be the operator on $G(N, \psi_0; \mathbb{I})$ given by

$$d : \sum_{n=0}^{\infty} \mathbf{a}_n q^n \mapsto \sum_{n=0}^{\infty} n \mathbf{a}_n q^n,$$

and for each $m \in \mathbf{Z}_{\geqslant 0}$ let $\binom{d}{m}$ denote the operator given by $\sum_n \mathbf{a}_n q^n \mapsto \sum_n \binom{n}{m} \mathbf{a}_n q^n$.

Definition 2.6. For any $\mathbf{f} \in G(N, \psi_0; \mathbb{I})$ and $x \in \widehat{\mathrm{Ig}}(N)(\mathbb{I})$, let $\mu_{\mathbf{f},x}$ be the $\mathbb{I}$-valued measure on $\mathbf{Z}_p$ determined by

$$\int_{\mathbf{Z}_p} \binom{z}{m} \mathrm{d}\mu_{\mathbf{f},x}(z) = \binom{d}{m} \mathbf{f}(x),$$

for all $m \geqslant 0$.

Setting $\mathbf{f}^\flat := \sum_{(n,p)=1} \mathbf{a}_n q^n \in G(N, \psi_0; \mathbb{I})$, it is easy to see that the associated measure $\mu_{\mathbf{f}^\flat,x}$ is then supported on $\mathbf{Z}_p^\times$.

2.4. **CM points.** Let K be an imaginary quadratic field of odd discriminant $-D_K < -3$, let $p > 2$ be a prime split in K, and write

$$p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}},$$

where $\mathfrak{p}$ is the prime of K above p induced by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$. We shall assume throughout that K satisfies the following *Heegner hypothesis* relative to a fixed integer $N > 0$ prime to p:

(heeg) there is an ideal $\mathfrak{N} \subseteq \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z}$.

The existence of such $\mathfrak{N}$, which will be fixed from now on, amounts to the requirement that every prime $q \mid N$ is either split in K or it is ramified in K with $q^2 \nmid N$.

For each positive integer c let $\mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_K$ be the order of K of that conductor, and let H_c be the corresponding ring class field, so that $\mathrm{Gal}(H_c/K) \simeq \mathrm{Pic}(\mathcal{O}_c)$ by the Artin reciprocity map. For each $\mathcal{O}_c$-ideal $\mathfrak{a}$ prime to $\mathfrak{N}\mathfrak{p}$, let $A_{\mathfrak{a}}/H_c$ be the CM elliptic curve with the complex uniformization $A_{\mathfrak{a}}(\mathbf{C}) = \mathbf{C}/\mathfrak{a}^{-1}$, and equip $A_{\mathfrak{a}}$ with the $\Gamma_1(Np^\infty)$-level structure

$$\eta_a : \boldsymbol{\mu}_N \oplus \boldsymbol{\mu}_{p^\infty} \hookrightarrow A_{\mathfrak{a}}[N] \oplus A_{\mathfrak{a}}[p^\infty]$$

defined in [CH17, p.6], where $a \in \widehat{K}^\times$ is chosen so that $a\widehat{K} \cap \mathcal{O}_c = \mathfrak{a}$. The pair $(A_{\mathfrak{a}}, \eta_a)$ defines a point $x_{\mathfrak{a}} \in \mathrm{Ig}(N)$ defined over a discrete valuation ring inside $\mathcal{V} := \imath_p^{-1}(\mathcal{O}_{\mathbf{C}_p}) \cap K^{\mathrm{ab}}$, where K^{ab} is the maximal abelian extension of K in $\overline{\mathbf{Q}}$. We let x_c denote the point $x_{\mathfrak{a}}$ for $\mathfrak{a} = \mathcal{O}_c$.

Write $c = c_o p^n$ with $p \nmid c_o$ and $n \geqslant 0$, and decompose $c_o = c_o^+ c_o^-$ with c_o^+ (resp. c_o^-) only divisible by primes which are split (resp. nonsplit) in K. We similarly decompose $N = N^+ N^-$, and set $\mathfrak{C}^+ := c_o^+ \mathcal{O}_K$ and $\mathfrak{N}^+ := N^+ \mathcal{O}_K$. Fix a square-root $\sqrt{-D_K} \in K$, and set

$$\vartheta := (D_K + \sqrt{-D_K})/2.$$

Following [CH17, §2.4], we define the matrix $\varsigma^{(\infty)} = (\varsigma_q) \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$ by

- $\varsigma_q = 1$, if $q \nmid c_o^+ N^+ p$,
- $\varsigma_q = (\overline{\vartheta} - \vartheta)^{-1} \begin{pmatrix} \overline{\vartheta} & \vartheta \\ 1 & 1 \end{pmatrix}$, if $q = \mathfrak{q}\overline{\mathfrak{q}}$ with $\mathfrak{q} \mid \mathfrak{C}^+\mathfrak{N}^+\mathfrak{p}$,

and the matrix $\gamma_c = (\gamma_{c,q}) \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$ by

- $\gamma_{c,q} = 1$, if $q \nmid cNp$,
- $\gamma_{c,q} = \begin{pmatrix} q^{\mathrm{ord}_q(c)} & 1 \\ 0 & 1 \end{pmatrix}$, if $q = \mathfrak{q}\overline{\mathfrak{q}}$ with $\mathfrak{q} \mid \mathfrak{C}^+\mathfrak{N}^+\mathfrak{p}$,
- $\gamma_{c,q} = \begin{pmatrix} 1 & 0 \\ 0 & q^{\mathrm{ord}_q(c) - \mathrm{ord}_q(N)} \end{pmatrix}$, if $q \mid c_o^- N^-$,

and set $\xi_c := \varsigma^{(\infty)}\gamma_c$. Under the complex uniformization

$$[\cdot] : \mathfrak{H} \times \mathrm{GL}_2(\widehat{\mathbf{Q}}) \longrightarrow \mathrm{Ig}(N)(\mathbf{C})$$

deduced from (2.1) and the standard uniformization of $Y_1(Np^s)$, the pair (ϑ, ξ_c) is sent to x_c. Moreover, by Shimura's reciprocity law, if $a \in \widehat{K}^{(p)\times}$ and $\mathfrak{a} = a\widehat{\mathcal{O}}_c \cap K$ is the associated fractional ideal of $\mathcal{O}_c$, then

$$[(\vartheta, \overline{a}^{-1}\xi_c)] = x_{\mathfrak{a}} = x_c^{\sigma_{\mathfrak{a}}} \in \mathrm{Ig}(N)(H_c(\mathfrak{p}^\infty)),$$

where $\sigma_{\mathfrak{a}} = \mathrm{rec}_K(a^{-1})|_{H_c(\mathfrak{p}^\infty)} \in \mathrm{Gal}(H_c(\mathfrak{p}^\infty)/K)$ is the Artin symbol of $\mathfrak{a}$ over the compositum of H_c with the ray class field of K of conductor $\mathfrak{p}^\infty$, and $a \mapsto \overline{a}$ denoted the action of the nontrivial automorphism $\tau \in \mathrm{Gal}(K/\mathbf{Q})$ on $\mathbb{A}_K$.

2.5. **Anticyclotomic Hecke characters.** We say that a Hecke character $\psi : K^\times \backslash \mathbb{A}_K^\times \to \mathbf{C}^\times$ has infinity type (ℓ_1, ℓ_2), with $\ell_1, \ell_2 \in \frac{1}{2}\mathbf{Z}$ such that $\ell_1 - \ell_2 \in \mathbf{Z}$, if

$$\psi_\infty(z) = z^{\ell_1 - \ell_2}(z\overline{z})^{\ell_2},$$

where for each place v of K, we let $\psi_v : K_v^\times \to \mathbf{C}^\times$ be the component of ψ at v. The conductor of ψ is the largest ideal $\mathfrak{c} \subseteq \mathcal{O}_K$ such that $\psi_{\mathfrak{q}}(u) = 1$ for all $u \in (1 + \mathfrak{c}\mathcal{O}_{K,\mathfrak{q}})^\times \subseteq K_{\mathfrak{q}}^\times$. If ψ has conductor $\mathfrak{c}_\psi$ and $\mathfrak{a}$ is any fractional ideal of K prime to $\mathfrak{c}_\psi$, we write $\psi(\mathfrak{a})$ for $\psi(a)$, where $a \in \widehat{K}^\times$ is such that $a\widehat{\mathcal{O}}_K \cap K = \mathfrak{a}$ and $a_{\mathfrak{q}} = 1$ for all $\mathfrak{q}$ dividing $\mathfrak{c}_\psi$. As a function on fractional ideals, then ψ satisfies $\psi((\alpha)) = \alpha^{\ell_2 - \ell_1}(\alpha\overline{\alpha})^{-\ell_2}$ for all $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{\mathfrak{c}_\psi}$.

Definition 2.7. If $\psi = \psi_{\mathrm{fin}}\psi_\infty$ is a Hecke character of K with has infinity type (ℓ_1, ℓ_2), the *p-adic avatar* of ψ is the character $\widehat{\psi} : K^\times \backslash \widehat{K}^\times \to \mathbf{C}_p^\times$ defined by

$$\widehat{\psi}(z) = \imath_p \imath_\infty^{-1}(\psi_{\mathrm{fin}}(z)) z_{\mathfrak{p}}^{\ell_1} z_{\overline{\mathfrak{p}}}^{\ell_2}.$$

Via the reciprocity map rec_K, we shall often regard $\widehat{\psi}$ as a Galois character $\widehat{\psi} : G_K \to \mathbf{C}_p^\times$.

Let $H_{p^\infty} = \bigcup_n H_{p^n}$ be the ring class field of K of conductor p^∞, and set $\widetilde{\Gamma} := \mathrm{Gal}(H_{p^\infty}/K)$. We say that a Hecke character $\psi : K^\times \backslash \mathbb{A}_K^\times \to \mathbf{C}^\times$ is *anticyclotomic* if $\psi|_{\mathbb{A}^\times} = \mathbb{1}$. The infinity type of such ψ is of the form $(\ell, -\ell)$, and the correspondence $\psi \mapsto \widehat{\psi}$ establishes a bijection between the set of anticyclotomic Hecke characters of K of conductor dividing p^∞ and the set of locally algebraic $\mathbf{C}_p$-valued characters of $\widetilde{\Gamma}$.

2.6. **A two-variable anticyclotomic p-adic L-function.** Throughout this section, we let $\mathbf{f} \in G(N, \psi_0; \mathbb{I})$ be an $\mathbb{I}$-adic modular form as in Definition 2.4. Also, let $\lambda : K^\times \backslash \mathbb{A}_K^\times \to \mathfrak{O}^\times$ be a fixed Hecke character of infinity type $(1, 0)$ and conductor prime to Np.

Definition 2.8. Let $\varepsilon_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ be the p-adic cyclotomic character, and let $i \in \mathbf{Z}/(p-1)\mathbf{Z}$ be such that $\psi_0|_{(\mathbf{Z}/p\mathbf{Z})^\times} = \omega^i$.

(1) Define the *critical character* $\Theta : G_{\mathbf{Q}} \to \mathbb{I}^\times$ by

$$\Theta(\sigma) := \omega^{i/2}(\sigma) \cdot [\langle \varepsilon_{\rm cyc}(\sigma)\rangle^{1/2}],$$

where $\langle\cdot\rangle^{1/2} : \mathbf{Z}_p^\times \to \Gamma^{\rm wt}$ is the composition of the projection $\mathbf{Z}_p^\times \twoheadrightarrow \Gamma^{\rm wt}$ with the map $x \mapsto x^{1/2}$.

(2) Define the $\mathbb{I}$-adic character $\boldsymbol{\chi} : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{I}^\times$ by

$$\boldsymbol{\chi}(x) := \Theta(\mathrm{rec}_{\mathbf{Q}}(\mathrm{N}_{K/\mathbf{Q}}(x))).$$

(3) Denote by $\langle\lambda\rangle$ the composition of λ with the projection onto the $\mathbf{Z}_p$-free quotient of $\mathfrak{O}^\times$, which then takes values in $\Gamma^{\rm wt}$, and define $\boldsymbol{\xi} : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{I}^\times$ by

$$\boldsymbol{\xi}(x) = \lambda^{1-\tau}(x)\cdot[\langle\lambda^{1-\tau}(x)\rangle^{1/2}],$$

where $\lambda^{1-\tau}(x) := \lambda(x)/\lambda(\overline{x})$.

Remark 2.9. Recall that we assume $p > 2$ and note that implicit in Definition 2.8 is a choice of a lift of i to $\mathbf{Z}/2(p-1)\mathbf{Z}$; we fix either one of the two possible choices, *cf.* [How07, Rem. 2.1.3].

Assume that $c_o\mathcal{O}_K$ is the conductor of $\lambda^{1-\tau}$, and for any $\mathcal{O}_{c_o}$-ideal $\mathfrak{a}$ prime to $\mathfrak{N}\mathfrak{p}$, let $\mu_{\mathbf{f}^\flat,\mathfrak{a}}$ be the measure $\mu_{\mathbf{f}^\flat,x}$ of Definition 2.6 associated to the CM point $x_{\mathfrak{a}} \in \mathrm{Ig}(N)$ of §2.4. Setting $T = t-1$, we shall denote by $\mu_{\mathbf{f}^\flat_{\mathfrak{a}}}$ the measure on $\mathbf{Z}_p^\times$ characterized by

$$\mathscr{A}_{\mu_{\mathbf{f}^\flat_{\mathfrak{a}}}}(T) = \mathscr{A}_{\mu_{\mathbf{f}^\flat,\mathfrak{a}}}((1+T)^{\mathbf{N}(\mathfrak{a})^{-1}\sqrt{-D_K}^{-1}} - 1),$$

and if $\phi : \mathbf{Z}_p^\times \to \mathfrak{O}^\times$ is any continuous character, define $\mathbf{f}^\flat_{\mathfrak{a}} \otimes \phi(t) \in \mathbb{I}[[t-1]]$ by

$$\begin{aligned}\mathbf{f}^\flat_{\mathfrak{a}} \otimes \phi(t) &= \int_{\mathbf{Z}_p} \phi(x)t^x \mathrm{d}\mu_{\mathbf{f}^\flat_{\mathfrak{a}}}(x)\\ &= \sum_{m\geqslant 0}\left[\int_{\mathbf{Z}_p}\phi(x)\binom{x}{m}\mathrm{d}\mu_{\mathbf{f}^\flat_{\mathfrak{a}}}(x)\right](t-1)^m.\end{aligned}$$

Definition 2.10. The *two-variable anticyclotomic p-adic L-function* attached to $\mathbf{f}$ and $\boldsymbol{\xi}$ is the $\mathbb{I}$-valued measure $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ on $\widetilde{\Gamma}$ given by

$$\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})(\phi) = \sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_{c_o})} \boldsymbol{\xi}\boldsymbol{\chi}^{-1}(\mathfrak{a})\mathbf{N}(\mathfrak{a})^{-1}\cdot\left(\mathbf{f}^\flat_{\mathfrak{a}}\otimes\phi|[\mathfrak{a}]\right)(A_{\mathfrak{a}},\eta_{\mathfrak{a}}),$$

for all $\phi : \widetilde{\Gamma} \to \mathbf{C}_p^\times$, where $\phi|[\mathfrak{a}]$ is the character on $\mathbf{Z}_p^\times$ defined by $\phi|[\mathfrak{a}](z) := \phi(\mathrm{rec}_K(a)\mathrm{rec}_{\mathfrak{p}}(z))$.

Now we describe the interpolation property satisfied by $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$. For the statement, recall that if $f = \sum_{n=1}^\infty a_n(f)q^n$ is a normalized newform of weight $k > 0$ and ψ is an anticyclotomic Hecke character of conductor $c\mathcal{O}_K$, the Rankin L-series $L(f/K,\psi,s)$ is defined by the analytic continuation of the Dirichlet L-series defined by

$$L(f/K,\psi,s) = \zeta(2s+1-k)\sum_{\mathfrak{a}}\frac{a_{\mathrm{N}(\mathfrak{a})}(f)\psi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s},$$

for $s \in \mathbf{C}$ with $\mathrm{Re}(s) > \frac{k+1}{2}$, where the sum is over the integral ideals $\mathfrak{a}$ of K with $(\mathfrak{a}, c\mathcal{O}_K) = 1$. In terms of automorphic L-functions, we have

$$L(f/K,\psi,s) = L\Big(s - \frac{k-1}{2}, \pi_K\otimes\psi\Big), \tag{2.2}$$

where π_K is the base change to K of the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f. Thus since $\pi_K\otimes\psi$ is self-dual, $L(f/K,\psi,s)$ satisfies a functional equation relating is values at s and $k-s$.

For any $\mathfrak{O}$-algebra homomorphism $\nu_k : \mathbb{I} \to \overline{\mathbf{Q}}_p$ with $k > 0$ and ψ an anticyclotomic Hecke character of K of conductor $c_o p^n \mathcal{O}_K$ with $p \nmid c_o$, define the p-adic multiplier $\mathscr{E}_{\mathfrak{p}}(f_\nu, \psi)$ by

$$\mathscr{E}_{\mathfrak{p}}(f_\nu, \psi) = \begin{cases} \bigl(1 - \frac{\nu(\mathbf{a}_p)\psi_{\overline{\mathfrak{p}}}(p)}{p^{k/2}}\bigr)\bigl(1 - \frac{\psi_{\overline{\mathfrak{p}}}(p)\varepsilon_\nu(p)p^{k/2-1}}{\nu(\mathbf{a}_p)}\bigr) & \text{if } n = 0; \\ \frac{\varepsilon(\psi_{\mathfrak{p}}^{-1})}{p^n} & \text{if } n \geqslant 1, \end{cases}$$

where ε_ν is the nebentypus of f_ν, and set

$$L^{\mathrm{alg}}(f_\nu/K, \psi, k/2) := \frac{\Gamma(k+\ell)\Gamma(\ell+1)}{(2\pi)^{k+2\ell+1}(\operatorname{Im}\vartheta)^{k+2\ell}} \cdot \frac{L(f_\nu/K, \psi, k/2)}{\Omega_K^{2k+4\ell}},$$

where $\Omega_K \in \mathbf{C}^\times$ is a complex period attached to K as in [CH17, §2.5].

Theorem 2.11. *Let $\nu = \nu_k$ for some $k > 0$ and let $\widehat{\phi}$ be the p-adic avatar of an anticyclotomic Hecke character ϕ of K of infinity type $(\ell, -\ell)$ with $\ell \geqslant 0$ and conductor $c_o p^n \mathcal{O}_K$ with $p \nmid c_o$. Then:*

$$\frac{\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))(\widehat{\phi})^2}{\Omega_p^{2k+4\ell}} = L^{\mathrm{alg}}(f_\nu/K, \xi_\nu\phi, k/2) \cdot \mathscr{E}_{\mathfrak{p}}(f_\nu, \xi_\nu\phi)^2 \cdot \phi(\mathfrak{N}^{-1}) \cdot 2^3 \cdot c_o\varepsilon(f_\nu) \cdot w_K^2\sqrt{D_K},$$

where $\varepsilon(f_\nu)$ is the global root number of f_ν, $w_K := |\mathcal{O}_K^\times|$, and $\Omega_p \in R_0^\times$ is a p-adic period as in [CH17, §2.5].

Proof. Let ν be as in the statement and set $f = f_\nu$. Then

$$\Theta_\nu(z) = z^{k/2-1}$$

for all $z \in \mathbf{Z}_p^\times$, and hence $\chi_\nu(\mathfrak{a}) = \mathbf{N}(\mathfrak{a})^{k/2-1}$. Specializing $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ at ν we thus see that

$$\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))(\widehat{\phi}) = \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_{c_o})} \xi_\nu(\mathfrak{a})\mathbf{N}(\mathfrak{a})^{-r} \cdot \Bigl(\widehat{f}_{\mathfrak{a}}^{\flat} \otimes \phi|[\mathfrak{a}]\Bigr)(A_{\mathfrak{a}}, \eta_{\mathfrak{a}}).$$

Since ξ_ν is the p-adic avatar of an anticyclotomic Hecke character of infinity type $(k/2, -k/2)$, the above shows that $\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))$ agrees with the R_0-valued measure $\mathscr{L}_{\mathfrak{p},\xi_\nu}(f)$ on $\widetilde{\Gamma}$ constructed in [CH17, §3.3], so the result follows from [*loc.cit.*, Prop. 3.8]. (Note that in [CH17] only cusp form of even weights $k \geqslant 2$ are considered, but the construction of $\mathscr{L}_{\mathfrak{p},\xi_\nu}(f)$ readily extends to any $k \in \mathbf{Z}_{\geqslant 1}$, and the results quoted from [Hsi14] are available in this level of generality.) ☐

Corollary 2.12. *For every $\nu = \nu_k$ with $k > 0$, the function $\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))$ is not identically zero.*

Proof. As shown in the proof of Theorem 2.11, the specialization $\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))$ agrees with the p-adic L-function $\mathscr{L}_{\mathfrak{p},\xi_\nu}(f)$ constructed in [CH17, §3.3] with $f = f_\nu$, and so the result similarly follows from [*loc.cit.*, Thm. 3.9]. ☐

3. Big logarithm maps

In this section we construct a Perrin-Riou big logarithm map adapted to our global anticyclotomic setting. Starting with [PR94], the cyclotomic theory of these maps has been widely studied in the literature; see e.g. [Ber03] and the references therein. The construction we give here combines work of Ochiai [Och03] and Loeffler–Zerbes [LZ14].

3.1. **Ochiai's map for nearly ordinary deformations.**

We keep the notations introduced in §2.2 and §2.6; in particular, $\mathfrak{O}$ denotes the ring of integers of finite extension of $L/\mathbf{Q}_p$ and $\mathbb{I}$ is a finite flat extension of $\Lambda_{\mathfrak{O}}^{\mathrm{wt}} = \mathfrak{O}[[\Gamma^{\mathrm{wt}}]]$. We also identify $G_{\mathbf{Q}_p} := \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ with the decomposition group $D_p \subseteq G_{\mathbf{Q}}$ determined by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$.

Definition 3.1. Let $\mathbb{T}$ be a free $\mathbb{I}$-module of rank 2 equipped with a continuous linear action of $G_{\mathbf{Q}}$. We say that $\mathbb{T}$ is a *p-ordinary deformation* if:

(i) the action of $G_{\mathbf{Q}}$ on $\det(\mathbb{T})$ is given by $\Theta^{-2}\varepsilon_{\rm cyc}^{-1}$;

(ii) there exists a filtration as $G_{\mathbf{Q}_p}$-modules

$$0 \longrightarrow \mathscr{F}^+\mathbb{T} \longrightarrow \mathbb{T} \longrightarrow \mathscr{F}^-\mathbb{T} \longrightarrow 0 \tag{3.1}$$

with $\mathscr{F}^{\pm}\mathbb{T}$ free of rank 1 over $\mathbb{I}$, and with the action on $\mathscr{F}^+\mathbb{T}$ being unramified.

Fix a p-ordinary deformation $\mathbb{T}$ as in Definition 3.1, and let $\Psi : G_{\mathbf{Q}_p} \to \mathbb{I}^\times$ be the unramified character giving the action of $G_{\mathbf{Q}_p}$ on $\mathscr{F}^+\mathbb{T}$. Let $\Gamma_{\rm cyc}$ be the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, and let $\Lambda_{\rm cyc}$ be the free $\mathbf{Z}_p[[\Gamma_{\rm cyc}]]$-module of rank one where $G_{\mathbf{Q}_p}$ acts via the inverse of the canonical character $G_{\mathbf{Q}_p} \twoheadrightarrow \Gamma_{\rm cyc} \hookrightarrow \mathbf{Z}_p[[\Gamma_{\rm cyc}]]^\times$.

Definition 3.2. Set $\mathcal{I} := \mathbb{I}\widehat{\otimes}\mathbf{Z}_p[[\Gamma_{\rm cyc}]]$. The *nearly p-ordinary deformation* associated to $\mathbb{T}$ is the $\mathcal{I}$-module

$$\mathcal{T} := \mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\rm cyc}$$

equipped with the diagonal $G_{\mathbf{Q}_p}$-action. From (3.1), $\mathcal{T}$ fits in an exact sequence of $\mathbb{I}[[G_{\mathbf{Q}_p}]]$-modules

$$0 \longrightarrow \mathscr{F}^+\mathcal{T} \longrightarrow \mathcal{T} \longrightarrow \mathscr{F}^-\mathcal{T} \longrightarrow 0$$

with $\mathscr{F}^\pm\mathcal{T} := \mathscr{F}^\pm\mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\rm cyc}$.

Let $\epsilon : \Gamma_{\rm cyc} \simeq 1 + p\mathbf{Z}_p$ be the isomorphism induced by the p-adic cyclotomic character. We denote by $\mathcal{X}^a_{\mathfrak{O}}(\Gamma_{\rm cyc})$ the set of continuous characters $\sigma : \Gamma_{\rm cyc} \to \overline{\mathbf{Q}}_p^\times$ of the form $\sigma = \epsilon^{w_\sigma}\sigma_o$ for some integer $w_\sigma \geqslant 0$, called the *weight* of σ, and some finite order character σ_o. We then say that σ has *conductor* p^r if so does σ_o seen as a character on $\mathbf{Z}_p^\times$.

Recall the set $\mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ from Definition 2.3, and for every pair $(\nu, \sigma) \in \mathcal{X}^a_{\mathfrak{O}}(\mathbb{I}) \times \mathcal{X}^a_{\mathfrak{O}}(\Gamma_{\rm cyc})$ let $\mathcal{O}_{\nu,\sigma}$ be the extension of $\mathcal{O}_\nu$ generated by the values of σ. With a slight abuse, we shall also denote by $\mathcal{O}_{\nu,\sigma}$ the free $\mathcal{O}_{\nu,\sigma}$-module of rank one where $G_{\mathbf{Q}_p}$ acts via the character σ. Define

$$T_{\nu,\sigma} := \mathcal{T} \otimes_{\mathcal{I},\nu} \mathcal{O}_{\nu,\sigma}, \qquad V_{\nu,\sigma} := T_{\nu,\sigma} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$
$$\mathscr{F}^\pm T_{\nu,\sigma} := \mathscr{F}^\pm\mathcal{T} \otimes_{\mathcal{I},\nu} \mathcal{O}_{\nu,\sigma}, \qquad \mathscr{F}^\pm V_{\nu,\sigma} := \mathscr{F}^\pm T_{\nu,\sigma} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

and for every finite extension $F/\mathbf{Q}_p$, let

$$\mathrm{Sp}_{\nu,\sigma} : H^1(F, \mathscr{F}^+\mathcal{T}) \longrightarrow H^1(F, \mathscr{F}^+T_{\nu,\sigma}) \longrightarrow H^1(F, \mathscr{F}^+V_{\nu,\sigma})$$

be the induced maps on cohomology.

For V a finite-dimensional L-vector space with a continuous linear action of G_F, we denote by $\mathbf{D}_{{\rm dR},F}(V)$ the filtered $(L \otimes_{\mathbf{Q}_p} F)$-module

$$\mathbf{D}_{{\rm dR},F}(V) := (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\rm dR})^{G_F},$$

where $\mathbf{B}_{\rm dR}$ is Fontaine's ring of p-adic de Rham periods. If V is a de Rham G_F-representation (i.e., $\dim_F\mathbf{D}_{{\rm dR},F}(V) = \dim_L V$), then for any finite extension E/F there is a canonical isomorphism $D_{{\rm dR},E}(V) = E \otimes_F D_{{\rm dR},F}(V)$. Denote by $\langle\ ,\ \rangle$ the de Rham pairing

$$\langle\ ,\ \rangle : \mathbf{D}_{{\rm dR},F}(V) \times \mathbf{D}_{{\rm dR},F}(V^*(1)) \longrightarrow L \otimes_{\mathbf{Q}_p} F \longrightarrow \mathbf{C}_p,$$

where $V^* = \mathrm{Hom}_L(V, L)$. Let $\mathbf{B}_{\rm cris} \subseteq \mathbf{B}_{\rm dR}$ be the crystalline period ring and define

$$\mathbf{D}_{{\rm cris},F}(V) := (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\rm cris})^{G_F};$$

this is an $(L \otimes_{\mathbf{Q}_p} F_0)$-module equipped with the action of crystalline Frobenius Φ, where F_0 is the maximal unramified subfield of F. When $F = \mathbf{Q}_p$, we write $\mathbf{D}_{\rm dR}(V) = \mathbf{D}_{{\rm dR},\mathbf{Q}_p}(V)$ and $\mathbf{D}_{\rm cris}(V) = \mathbf{D}_{{\rm cris},\mathbf{Q}_p}(V)$. If V is a crystalline representation (i.e., $\dim_{F_0}\mathbf{D}_{{\rm cris},F}(V) = \dim_L V$), then we have a canonical isomorphism $F \otimes_{F_0} \mathbf{D}_{{\rm cris},F}(V) = \mathbf{D}_{{\rm dR},F}(V)$. Suppose further that

$$\mathbf{D}_{{\rm cris},F}(V)^{\Phi=1} = \{0\}.$$

Then we denote by log the Bloch–Kato logarithm map

$$\log := \log_{F,V} : H^1_{\rm f}(F,V) \longrightarrow \frac{\mathbf{D}_{\rm dR,\mathit{F}}(V)}{{\rm Fil}^0\mathbf{D}_{\rm dR,\mathit{F}}(V)} = {\rm Fil}^0\mathbf{D}_{\rm dR,\mathit{F}}(V^*(1))^\vee,$$

where $H^1_{\rm f}(F,V)\subseteq H^1(F,V)$ is the Bloch–Kato subspace [BK90, (3.7.2)], and denote by $\exp^*$ the dual exponential map

$$\exp^* := \exp^*_{F,V} : H^1(F,V^*(1)) \longrightarrow {\rm Fil}^0\mathbf{D}_{\rm dR,\mathit{F}}(V^*(1))$$

obtained by dualizing the Bloch–Kato exponential map

$$\exp := \exp_{F,V} : \frac{\mathbf{D}_{\rm dR,\mathit{F}}(V)}{{\rm Fil}^0\mathbf{D}_{\rm dR,\mathit{F}}(V)} \longrightarrow H^1(F,V)$$

with respect to the de Rham and local Tate pairings (*cf.* [LZ14, §2.4]).

Definition 3.3. Let $\mathbb{T}$ be a p-ordinary deformation, and set

$$\mathbb{D} := (\mathscr{F}^+\mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathbf{Z}}_p^{\rm nr})^{G_{\mathbf{Q}_p}}, \tag{3.2}$$

where the $G_{\mathbf{Q}_p}$-action on $\mathscr{F}^+\mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathbf{Z}}_p^{\rm nr}$ is the diagonal one. Also set

$$\mathcal{D} := \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[[\Gamma_{\rm cyc}]].$$

Fix a compatible system $(\zeta_{p^r})_r$ of p-power roots of unity. Then as in [Och03, Def. 3.12], for every $(\nu,\sigma)\in\mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})\times\mathcal{X}^a_{\mathfrak{O}}(\Gamma_{\rm cyc})$ there are specialization maps

$$\mathrm{Sp}_{\nu,\sigma} : \mathcal{D} \longrightarrow \mathbf{D}_{\rm dR}(\mathscr{F}^+V_{\nu,\sigma}). \tag{3.3}$$

Theorem 3.4. *Let $\gamma_o\in\Gamma_{\rm cyc}$ be a topological generator, and define*

$$\mathcal{J} := (\Psi(\mathrm{Fr}_p)-1, \gamma_o-1) \subseteq \mathcal{I}.$$

For any finite unramified extension $F/\mathbf{Q}_p$ with ring of integers $\mathcal{O}_F$ there exists an injective $\mathcal{I}$-linear map

$$\mathcal{E}_F^{\Gamma_{\rm cyc}} : \mathcal{J}(\mathcal{D}\otimes_{\mathbf{Z}_p}\mathcal{O}_F) \longrightarrow H^1(F,\mathscr{F}^+\mathcal{T})$$

such that for every $\nu\in\mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ of weight $k\geqslant 2$ and $\sigma\in\mathcal{X}^a_{\mathfrak{O}}(\Gamma_{\rm cyc})$ of weight w with $1\leqslant w\leqslant k-1$ and conductor p^n, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{J}(\mathcal{D}\otimes_{\mathbf{Z}_p}\mathcal{O}_F) & \xrightarrow{\ \mathcal{E}_F^{\Gamma_{\rm cyc}}\ } & H^1(F,\mathscr{F}^+\mathcal{T}) \\ \big\downarrow{\scriptstyle \mathrm{Sp}_{\nu,\sigma}} & & \big\downarrow{\scriptstyle \mathrm{Sp}_{\nu,\sigma}} \\ \mathbf{D}_{\rm dR,\mathit{F}}(\mathscr{F}^+V_{\nu,\sigma}) & \longrightarrow & H^1(F,\mathscr{F}^+V_{\nu,\sigma}), \end{array}$$

where the bottom horizontal map is given by

$$(-1)^{w-1}(w-1)!\cdot\exp\times\begin{cases} \big(1-\frac{p^{w-1}}{\Psi_\nu(\mathrm{Fr}_p)}\big)\big(1-\frac{\Psi_\nu(\mathrm{Fr}_p)}{p^w}\big)^{-1} & \textit{if } n=0; \\ \mathfrak{g}(\sigma_o^{-1})\big(\frac{p^{w-1}}{\Psi_\nu(\mathrm{Fr}_p)}\big)^r & \textit{if } n\geqslant 1. \end{cases}$$

Proof. See [Och03, Prop. 5.3]. □

3.2. **Going up the unramified $\mathbf{Z}_p$-extension.** Let $U := \mathrm{Gal}(F_\infty/\mathbf{Q}_p)$ be the Galois group of the unramified $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, let F_n be the subfield of F_∞ with $\mathrm{Gal}(F_n/\mathbf{Q}_p) \simeq \mathbf{Z}/p^n\mathbf{Z}$, and set $U_n := \mathrm{Gal}(F_\infty/F_n)$. Let $y_n : \mathcal{O}_{F_n} \to \mathcal{O}_{F_n}[U/U_n]$ be the $\mathbf{Z}_p$-linear map defined by

$$y_n(x) = \sum_{\sigma \in U/U_n} x^\sigma[\sigma^{-1}],$$

and let $\mathcal{S}_n$ be the image of y_n.

For any $x \in \mathcal{O}_{F_{n+1}}$, it is readily seen that the image of $y_{n+1}(x)$ in $\mathcal{O}_{F_{n+1}}[U/U_n]$ agrees with $y_n(\mathrm{Tr}_{F_{n+1}/F_n}(x))$, and hence passing to the inverse limits with respect to the trace maps, we obtain an isomorphism

$$\varprojlim_n y_n : \varprojlim_n \mathcal{O}_{F_n} \xrightarrow{\simeq} \mathcal{S}_\infty := \varprojlim_n \mathcal{S}_n. \tag{3.4}$$

Proposition 3.5. *The module $\mathcal{S}_\infty$ is free of rank 1 over $\mathbf{Z}_p[[U]]$, and it is identified with*

$$\{g \in \widehat{\mathcal{O}}_{F_\infty}[[U]] : \ g^u = [u]g \ \text{ for all } u \in U\},$$

where $\widehat{\mathcal{O}}_{F_\infty}$ is the completion of the ring of integers of F_∞.

Proof. See [LZ14, Prop. 3.2, Prop. 3.6]. □

3.3. **A two-variable regulator map for ordinary deformations.** Let $G = \mathrm{Gal}(L_\infty/\mathbf{Q}_p)$ be the Galois group of the unique $\mathbf{Z}_p^2$-extension of $\mathbf{Q}_p$, and note that $L_\infty = \mathbf{Q}_{p,\infty}F_\infty$. As in §3.1, we let $\mathbb{T}$ be a p-ordinary deformation in the sense of Definition 2.4, and let $\Psi : G_{\mathbf{Q}_p} \to \mathbb{I}^\times$ be the unramified character giving the action on the unramified $\mathbb{I}$-line $\mathscr{F}^+\mathbb{T} \subseteq \mathbb{T}$.

Definition 3.6. An arithmetic prime $\nu \in \mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$ is *exceptional* for $\mathbb{T}$ if $\nu = \nu_2$ and $\Psi_\nu(\mathrm{Fr}_p) = 1$.

For any subquotient $\mathbb{M}$ of $\mathbb{T}$ define

$$H^1_{\mathrm{Iw}}(L_\infty, \mathbb{M}) := \varprojlim_L H^1(L, \mathbb{M}),$$

where the limit is over the finite extensions $L/\mathbf{Q}_p$ contained in L_∞ with respect to the corestriction maps.

Theorem 3.7. *Let $\lambda := \Psi(\mathrm{Fr}_p) - 1$. There is an injective $\mathbb{I}[[G]]$-linear map*

$$\mathcal{L}^G : H^1_{\mathrm{Iw}}(L_\infty, \mathscr{F}^+\mathbb{T}) \longrightarrow \lambda^{-1} \cdot \mathcal{J}(\mathbb{D} \widehat{\otimes}_{\mathbf{Z}_p} \widehat{\mathcal{O}}_{F_\infty}[[G]])$$

such that for every $\mathfrak{Y}_\infty \in H^1_{\mathrm{Iw}}(L_\infty, \mathscr{F}^+\mathbb{T})$, and for every non-exceptional $\nu \in \mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$ of weight $k \geqslant 2$ and Hodge–Tate character $\phi : G \to L^\times$ of conductor p^n and Hodge–Tate weight[1] *w, with $1 \leqslant w \leqslant k-1$, we have*

$$\mathcal{L}^G(\mathfrak{Y}_\infty)(\nu, \phi) = \frac{(-1)^{w-1}}{(w-1)!} \cdot \log(\nu(\mathfrak{Y}_\infty)^{\phi^{-1}}) \times \begin{cases} \left(1 - \frac{\Psi_\nu(\mathrm{Fr}_p)}{p^w}\right)\left(1 - \frac{p^{w-1}}{\Psi_\nu(\mathrm{Fr}_p)}\right)^{-1} & \text{if } n = 0; \\ \varepsilon(\phi)^{-1}\Psi_\nu(\mathrm{Fr}_p^n) & \text{if } n \geqslant 1, \end{cases}$$

where $\varepsilon(\phi)$ is the ε-factor of ϕ.

Proof. For each $n \geqslant 0$, let

$$\mathcal{E}_{F_n}^{\Gamma_{\mathrm{cyc}}} : \mathcal{J}(\mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_{F_n}) \longrightarrow H^1(F_n, \mathscr{F}^+\mathcal{T})$$

be the big exponential map of Theorem 3.4 for the unramified extension $F_n/\mathbf{Q}_p$, and using (3.4) define

$$\mathcal{E}^G := \varprojlim_n \mathcal{E}_{F_n}^{\Gamma_{\mathrm{cyc}}} : \mathcal{J}(\mathcal{D} \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{S}_\infty) \longrightarrow H^1_{\mathrm{Iw}}(F_\infty, \mathscr{F}^+\mathcal{T}).$$

[1]In this paper, we adopt the convention that the Hodge–Tate weight of $\varepsilon_{\mathrm{cyc}}$ is $+1$. Thus the Hodge–Tate weights of a p-adic de Rham representation V are the integers w such that $\mathrm{Fil}^{-w}\mathbf{D}_{\mathrm{dR}}(V) \supsetneq \mathrm{Fil}^{-w+1}\mathbf{D}_{\mathrm{dR}}(V)$.

By Shapiro's lemma, we view $\mathcal{E}^G$ as taking values in $H^1_{\rm Iw}(L_\infty, \mathscr{F}^+\mathbb{T})$. Since each $\mathcal{E}^{\Gamma_{\rm cyc}}_{F_n}$ has cokernel killed by λ, it is readily seen that $\mathcal{E}^G$ is an injective $\mathbb{I}[[G]]$-linear map with cokernel killed by λ, and hence given any $\mathfrak{Y}_\infty \in H^1_{\rm Iw}(L_\infty, \mathscr{F}^+\mathbb{T})$, the assignment

$$\mathcal{L}^G(\mathfrak{Y}_\infty) := \lambda^{-1} \cdot (\mathcal{E}^G)^{-1}(\lambda \cdot \mathfrak{Y}_\infty)$$

is a well-defined element in

$$\lambda^{-1} \cdot \mathcal{J}(\mathcal{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{S}_\infty) \hookrightarrow \lambda^{-1} \cdot \mathcal{J}(\mathcal{D}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathcal{O}}_{F_\infty}[[U]]) \simeq \lambda^{-1} \cdot \mathcal{J}(\mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathcal{O}}_{F_\infty}[[G]]).$$

Thus constructed, the interpolation properties of $\mathcal{L}^G$ for each non-exceptional $\nu \in \mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ then follow as in [LZ14, Thm. 4.15]. □

Definition 3.8. Let $\mathbf{f} \in \mathbb{I}[[q]]$ be an ordinary $\mathbb{I}$-adic newform of tame level N (prime to p). We say that an arithmetic prime $\nu \in \mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ is *p-old* if $\mathbf{f}_\nu$ is the p-stabilization of an ordinary newform of level N.

Note that if $\nu \in \mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ has weight $k > 2$ and trivial nebentypus, then ν is p-old (see [How07, Lemma 2.1.5]), and that any p-old arithmetic prime is also non-exceptional.

Corollary 3.9. *Let $\nu \in \mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ be a p-old arithmetic prime. Then for every Hodge–Tate character $\phi : G \to L^\times$ of Hodge–Tate weight $w \leqslant 0$ and conductor p^n we have*

$$\mathcal{L}^G(\mathfrak{Y}_\infty)(\nu, \phi) = (-w)! \cdot \exp^*(\nu(\mathfrak{Y}_\infty)^{\phi^{-1}}) \times \begin{cases} \big(1 - \frac{\Psi_\nu({\rm Fr}_p)}{p^w}\big)\big(1 - \frac{p^{w-1}}{\Psi_\nu({\rm Fr}_p)}\big)^{-1} & \text{if } n = 0; \\ \varepsilon(\phi)\Psi_\nu({\rm Fr}_p^n) & \text{if } n \geqslant 1. \end{cases}$$

Proof. The specialization of $\mathcal{L}^G$ at ν gives rise to an $\mathcal{O}_\nu[[G]]$-linear map

$$\mathcal{L}^G_\nu : H^1_{\rm Iw}(L_\infty, \mathscr{F}^+V_\nu) \longrightarrow \mathbf{D}_{\rm dR}(\mathscr{F}^+V_\nu) \otimes_{\mathbf{Z}_p} \widehat{\mathcal{O}}_{F_\infty}[[G]]$$

which by Theorem 3.7 enjoys the same interpolation properties at a dense set characters of G as the map $\mathcal{L}^G_V$ constructed in [LZ14, Thm. 4.7] for $V = \mathscr{F}^+V_\nu$. (Note that since ν is p-old, $\mathscr{F}^+V_\nu$ is indeed a crystalline $G_{\mathbf{Q}_p}$-representation.) Since $\mathcal{L}^G_{\mathscr{F}^+V_\nu}$ is uniquely determined by its values at such characters (for every given class in $H^1_{\rm Iw}(L_\infty, \mathscr{F}^+V_\nu)$), the result follows from [LZ14, Thm. 4.15]. □

4. Big Heegner points

Let $f \in S_k(\Gamma_0(N))$ be a p-ordinary newform of level N prime to $p > 3$, and let $K/\mathbf{Q}$ be an imaginary quadratic field as in §2.4; in particular, K satisfies condition (heeg) relative to N. Let $L/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathfrak{O}$ containing the Fourier coefficients of f. In this section, we briefly recall Howard's construction of big Heegner points associated to the ordinary $\mathbb{I}$-adic newform passing through f.

4.1. Galois representations associated to Hida families.

Let $X_{s/\mathbf{Q}}$ be the compactified modular curve whose non-cuspidal points classify isomorphism classes of triples (E, C, π) with:

- E is an elliptic curve over an arbitrary $\mathbf{Q}$-scheme S;
- C is a cyclic subgroup of E of order N;
- π is a point of E of exact order p^s.

Let $J_s := {\rm Pic}^0(X_s) \otimes_{\mathbf{Z}} \mathfrak{O}$ be the Jacobian of X_s, denote by $\mathfrak{h}_s$ the $\mathfrak{O}$-algebra generated by the Hecke operators T_ℓ ($\ell \nmid Np$), U_ℓ ($\ell \mid Np$), and $\langle a \rangle$ ($a \in (\mathbf{Z}/N\mathbf{Z})^\times$) acting of J_s by Albanese functoriality, and let

$$e^{\rm ord} := \lim_{m\to\infty} U_p^{m!}$$

be Hida's ordinary projector. By [Hid86, Thm. 1.1], the algebra $\mathfrak{h}^{\rm ord} := \varprojlim_s e^{\rm ord}\mathfrak{h}_s$ is finite flat over $\Lambda^{\rm wt}_{\mathfrak{O}}$. Let $\mathfrak{h}^{\rm ord}_{\mathfrak{m}}$ be the local summand of $\mathfrak{h}^{\rm ord}$ through which the algebra homomorphism

$\lambda_f : \mathfrak{h}^{\mathrm{ord}} \to \mathfrak{O}$ defined by f factors, let $\mathfrak{a} \subseteq \mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}$ be the unique minimal prime containing the kernel of λ_f, and set

$$\mathbb{I} := \mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}/\mathfrak{a}.$$

Letting $\mathbf{a}_n \in \mathbb{I}$ be the image of $T_n \in \mathfrak{h}^{\mathrm{ord}}$, the formal q-expansion $\mathbf{f} = \sum_{n\geqslant 1} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$ is an ordinary $\mathbb{I}$-adic newform of tame level N and character ω^{k-2} in the sense of Definition 2.4.

Let κ_L be the residue field of L and denote by $\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_L)$ the semisimple residual representation attached to f.

Theorem 4.1. *Assume that $\bar{\rho}_f$ is irreducible and p-distinguished. Then the following hold:*

- *The module*
$$\mathbf{T} := \left(\varprojlim_s e^{\mathrm{ord}}(\mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathfrak{O})\right) \otimes_{\mathfrak{h}^{\mathrm{ord}}} \mathbb{I}$$
is free of rank 2 over $\mathbb{I}$.
- *The Galois representation*
$$\rho_{\mathbf{f}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbf{T}) \simeq \mathrm{GL}_2(\mathbb{I})$$
is unramified outside Np with
$$\mathrm{trace}(\rho_{\mathbf{f}})(\mathrm{Fr}_\ell^{-1}) = \mathbf{a}_\ell, \qquad \det(\rho_{\mathbf{f}})(\mathrm{Fr}_\ell^{-1}) = \ell[\ell],$$
for all $\ell \nmid Np$, where Fr_ℓ^{-1} is an arithmetic Frobenius.
- *As a representation of $G_{\mathbf{Q}_p}$, there is an exact sequence*
$$0 \longrightarrow \mathscr{F}^+\mathbf{T} \longrightarrow \mathbf{T} \longrightarrow \mathscr{F}^-\mathbf{T} \longrightarrow 0 \tag{4.1}$$
with $\mathscr{F}^{\pm}\mathbf{T} \simeq \mathbb{I}$, and with the action of $G_{\mathbf{Q}_p}$ on $\mathscr{F}^-\mathbf{T}$ given by the unramified character $\alpha : G_{\mathbf{Q}_p} \to \mathbb{I}^\times$ sending Fr_p^{-1} to $\mathbf{a}_p$.

Proof. This follows from [MT90, Thm. 7] and [Wil88, Thm. 2.2.2]. □

4.2. **Howard's big Heegner points.** Fix a positive integer c prime to Np. The CM points $x_{cp^n} \in \mathrm{Ig}(N)(\mathbf{C})$ constructed in §2.4 descend to points $P_{cp^n,s} \in X_s(H_{cp^n}(\boldsymbol{\mu}_{p^s}))$, for all $n \geqslant s$.

Lemma 4.2.

(1) *For all $\sigma \in \mathrm{Gal}(H_{cp^n}(\boldsymbol{\mu}_{p^s})/H_{cp^n})$, we have*
$$P^{\sigma}_{cp^n,s} = \langle \vartheta(\sigma) \rangle \cdot P_{cp^n,s},$$
where $\vartheta : \mathrm{Gal}(H_{cp^n}(\boldsymbol{\mu}_{p^s})/H_{cp^n}) \to \mathbf{Z}_p^\times/\{\pm 1\}$ is such that $\vartheta^2 = \varepsilon_{\mathrm{cyc}}$.

(2) *If $n \geqslant s > 1$, then*
$$\sum_{\sigma \in \mathrm{Gal}(H_{cp^n}(\boldsymbol{\mu}_{p^s})/H_{cp^{n-1}}(\boldsymbol{\mu}_{p^s}))} \alpha_s(P^{\sigma}_{cp^n,s}) = U_p \cdot P_{cp^n,s-1},$$
where $\alpha_s : X_s \to X_{s-1}$ is the map given by $(E, C, \pi) \mapsto (E, C, p \cdot \pi)$ on non-cuspidal moduli.

(3) *If $n \geqslant s \geqslant 1$, then*
$$\sum_{\sigma \in \mathrm{Gal}(H_{cp^n}(\boldsymbol{\mu}_{p^s})/H_{cp^{n-1}}(\boldsymbol{\mu}_{p^s}))} P^{\sigma}_{cp^n,s} = U_p \cdot P_{cp^{n-1},s}.$$

Proof. From the construction of x_{cp^n}, it is immediate to see that the point $P_{cp^n,s}$ for $n \geqslant s$ agrees with the point $h_{cp^{n-s},s} \in X_s(\mathbf{C})$ corresponding to the triple $(A_{cp^{n-s},s}, \mathfrak{n}_{cp^{n-s},s}, \pi_{cp^{n-s},s})$ with:

- $A_{cp^{n-s},s}(\mathbf{C}) = \mathbf{C}/\mathcal{O}_{cp^n}$;
- $\mathfrak{n}_{cp^{n-s},s} = A_{cp^{n-s},s}[\mathfrak{N} \cap \mathcal{O}_{cp^n}]$;
- $\pi_{cp^{n-s},s}$ a generator of the kernel of the cyclic p^s-isogeny $\mathbf{C}/\mathcal{O}_{cp^n} \to \mathbf{C}/\mathcal{O}_{cp^{n-s}}$,

as constructed in [How07, §2.2]. The proof of properties (1), (2) and (3) thus follows from (the proof of) Corollary 2.2.2, Lemma 2.2.4 and Proposition 2.3.1 of [How07], respectively. □

Set $L_{c,s} := H_{cp^s}(\boldsymbol{\mu}_{p^s})$, and let e_i denote the idempotent of $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ projecting to the ω^i-th isotypical component. As in [How07, Cor. 2.2.2], it follows easily from Lemma 4.2 that the points $e_{k-2}e^{\mathrm{ord}}P_{cp^{t+1+s},s}$ define classes

$$y_{cp^{t+1},s} \in e^{\mathrm{ord}}J_s(L_{cp^{t+1},s})$$

which satisfy

$$y^\sigma_{cp^{t+1},s} = \Theta(\sigma)\cdot y_{cp^{t+1},s}, \quad \text{for all } \sigma\in \mathrm{Gal}(L_{cp^{t+1},s}/H_{cp^{t+1+s}}), \tag{4.2}$$

where Θ is the character defined in (2.2), viewed as acting on J_s via the diamond operators.

Definition 4.3. For any $\Lambda^{\mathrm{wt}}_{\mathfrak{O}}$-module M equipped with a linear $G_{\mathbf{Q}}$-action, we let $M^\dagger$ denote its twist by the character Θ^{-1}.

Thus by (4.2) we have

$$y_{cp^{t+1},s} \in H^0(H_{cp^{t+1+s}}, e^{\mathrm{ord}}J_s(L_{cp^{t+1},s})^\dagger).$$

For any $m>0$, let $\mathfrak{G}_{H_m}$ be the Galois group of the maximal extension of H_m unramified outside the primes above Np. By Lemma 4.2, the image of $y_{cp^{t+1},s}$ under the composite map

$$H^0(H_{cp^{t+1+s}}, e^{\mathrm{ord}}J_s^{\mathrm{ord}}(L_{cp^{t+1},s})^\dagger) \xrightarrow{\mathrm{Cor}} H^0(H_{cp^{t+1}}, e^{\mathrm{ord}}J_s(L_{cp^{t+1},s})^\dagger)$$
$$\xrightarrow{\mathrm{Kum}} H^1(\mathfrak{G}_{H_{cp^{t+1}}}, e^{\mathrm{ord}}\mathrm{Ta}_p(J_s)^\dagger)$$

defines a class $\mathfrak{X}_{cp^{t+1},s}$ satisfying

$$\alpha_{s*}\mathfrak{X}_{cp^{t+1},s} = U_p\cdot \mathfrak{X}_{cp^{t+1},s-1}$$

under the map

$$\alpha_{s*}: H^1(\mathfrak{G}_{H_{cp^{t+1}}}, e^{\mathrm{ord}}\mathrm{Ta}_p(J_s)^\dagger) \longrightarrow H^1(\mathfrak{G}_{H_{cp^{t+1}}}, e^{\mathrm{ord}}\mathrm{Ta}_p(J_{s-1})^\dagger)$$

induced by $\alpha_s : X_s\to X_{s-1}$ by Albanese functoriality.

Definition 4.4. The *big Heegner point of conductor cp^{t+1}* is the class

$$\mathfrak{X}_{cp^{t+1}} \in H^1(\mathfrak{G}_{H_{cp^{t+1}}}, \mathbf{T}^\dagger)$$

defined as the image of $\varprojlim_s U_p^{-s}\cdot\mathfrak{X}_{cp^{t+1},s}$ under the natural map

$$\varprojlim_s H^1(\mathfrak{G}_{H_{cp^{t+1}}}, e^{\mathrm{ord}}\mathrm{Ta}_p(J_s)^\dagger) \longrightarrow H^1(\mathfrak{G}_{H_{cp^{t+1}}}, \mathbf{T}^\dagger).$$

By inflation, we shall view $\mathfrak{X}_{cp^{t+1}}$ as a class in $H^1(H_{cp^{t+1}}, \mathbf{T}^\dagger)$.

By [How07, Prop. 2.3.1], the classes

$$\mathfrak{Z}_{c,t} := U_p^{-t}\cdot \mathfrak{X}_{cp^{t+1}} \in H^1(H_{cp^t}, \mathbf{T}^\dagger) \tag{4.3}$$

are compatible under the corestriction maps, thus defining a class

$$\mathfrak{Z}_{c,\infty} := \varprojlim_t \mathfrak{Z}_{c,t} \in H^1_{\mathrm{Iw}}(H_{cp^\infty}, \mathbf{T}^\dagger).$$

The maximal free quotient of $\mathrm{Gal}(H_{cp^\infty}/K)$ is the Galois group Γ of the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K, and so for every character χ of $\Delta_c := \ker(\mathrm{Gal}(H_{cp^\infty}/K)\twoheadrightarrow\Gamma)$, we obtain a class

$$\mathfrak{Z}^\chi_{c,\infty} \in H^1_{\mathrm{Iw}}(K_\infty, \mathbf{T}^\dagger\otimes\chi).$$

For $c=1$ and $\chi=\mathbb{1}$ this is the Iwasawa cohomology class $\mathfrak{Z}_\infty$ considered in [How04, §3.3].

5. Explicit reciprocity law

As in previous sections, let $\Gamma = \mathrm{Gal}(K_\infty/K)$ (resp. $G = \mathrm{Gal}(L_\infty/\mathbf{Q}_p)$) be the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of K (resp. the unique $\mathbf{Z}_p^2$-extension of $\mathbf{Q}_p$). We assume that $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K and for each $v \mid p$ in K with a slight abuse of notation we let $K_{\infty,v}$ be the completion of K_∞ at a fixed prime above v.

5.1. **Regulator map for the anticyclotomic $\mathbf{Z}_p$-extension of K.** Recall the $\mathbb{I}$-adic Hecke character $\boldsymbol{\xi} : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{I}^\times$ in (2.2), based on the choice of an $\mathfrak{O}$-valued Hecke character λ of K of conductor prime to Np and infinity type $(1, 0)$. With a slight abuse of notation, we also let $\boldsymbol{\xi} : G_K \to \mathbb{I}^\times$ be the character defined by

$$\boldsymbol{\xi}(\sigma) := [\langle\widehat{\lambda}(\sigma)\widehat{\lambda}^{-1}(\tau\sigma\tau)\rangle^{1/2}]$$

where $\tau \in G_K$ is the nontrivial automorphism of $\mathrm{Gal}(K/\mathbf{Q})$, and set

$$\mathbb{T} := \mathbf{T}|_{G_K} \otimes \Theta^{-1}\boldsymbol{\xi}^{-1}\varepsilon_{\mathrm{cyc}}^{-1}. \tag{5.1}$$

By Theorem 4.1, if v is a place of K above p, the restriction of $\mathbb{T}$ to a decomposition group at v takes the form

$$\mathbb{T}_{|G_{K_v} = G_{\mathbf{Q}_p}} \ : \ \begin{pmatrix} \alpha^{-1}\Theta\boldsymbol{\xi}^{-1} & * \\ & \alpha\Theta^{-1}\boldsymbol{\xi}^{-1}\varepsilon_{\mathrm{cyc}}^{-1} \end{pmatrix} \tag{5.2}$$

on a suitable $\mathbb{I}$-basis. Since $\alpha^{-1}\Theta\boldsymbol{\xi}^{-1}$ is an unramified character of $G_{K_\mathfrak{p}}$, the representation (5.2) for $v = \mathfrak{p}$ is an ordinary deformation in the sense of Definition 3.1, and hence associated with it we may consider the regulator map $\mathcal{L}^G$ of Theorem 3.7.

In the following, we let $\mathbb{T}$ be the representation (5.2) for $v = \mathfrak{p}$ and identify $\mathrm{Gal}(K_{\infty,\mathfrak{p}}/K_\mathfrak{p})$ with Γ via $G_{\mathbf{Q}_p} = G_{K_\mathfrak{p}} \hookrightarrow G_K$. Recall the module $\mathbb{D}$ of Definition 3.3.

Lemma 5.1. *There exists an element $\omega_\mathbf{f}^\vee \in \mathbb{D}$ such that*

$$\langle\nu(\omega_\mathbf{f}^\vee), \omega_{f_\nu}\rangle = 1$$

for all $\nu \in \mathcal{X}_\mathfrak{O}^a(\mathbb{I})$, where $\omega_{f_\nu} \in \mathbf{D}_{\mathrm{dR}}(\mathscr{F}^+V_\nu)$ is the differential associated to the p-stabilized newform f_ν.

Proof. This is [KLZ17, Prop. 10.1.1(1)]. □

Proposition 5.2. *Let $\lambda := \mathbf{a}_p \cdot \Theta\boldsymbol{\xi}^{-1}(\mathrm{Fr}_p) - 1 \in \mathbb{I}$ and set $\widetilde{\mathbb{I}} := \mathbb{I}[\lambda^{-1}] \otimes_{\mathbf{Z}_p} \widehat{\mathcal{O}}_{F_\infty}$. There exists an $\widetilde{\mathbb{I}}[[\Gamma]]$-linear map*

$$\mathrm{tw}_{-1}\mathcal{L}_{\omega_\mathbf{f}}^\Gamma : H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, \mathscr{F}^+\mathbb{T}(1)) \longrightarrow \widetilde{\mathbb{I}}[[\Gamma]]$$

with the following interpolation property. Let $\mathfrak{Y}_\infty \in H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, \mathscr{F}^+\mathbb{T}(1))$ and let $\widehat{\phi} : \Gamma \to L^\times$ be the p-adic avatar of an anticyclotomic Hecke character of K of conductor p^n and infinity type $(\ell, -\ell)$.

(i) *If $\nu \in \mathcal{X}_\mathfrak{O}^a(\mathbb{I})$ is non-exceptional and $\ell \leqslant 0$, then:*

$$\mathrm{tw}_{-1}\mathcal{L}_{\omega_\mathbf{f}}^\Gamma(\mathfrak{Y}_\infty)(\nu, \widehat{\phi}) = \frac{\varepsilon(\widehat{\phi}^{-1})}{\nu(\mathbf{a}_p^n)\chi_\nu\xi_\nu^{-1}(p_\mathfrak{p}^n)} \cdot \frac{\mathscr{P}^*(\nu, \phi^{-1})}{\mathscr{P}(\nu, \phi)} \cdot \frac{(-1)^\ell}{(-\ell)!} \cdot \log_{/\omega_{f_\nu}}(\nu(\mathfrak{Y}_\infty)^{\widehat{\phi}^{-1}});$$

(ii) *If $\nu \in \mathcal{X}_\mathfrak{O}^a(\mathbb{I})$ is p-old and $\ell > 0$, then:*

$$\mathrm{tw}_{-1}\mathcal{L}_{\omega_\mathbf{f}}^\Gamma(\mathfrak{Y}_\infty)(\nu, \widehat{\phi}) = \frac{\varepsilon(\widehat{\phi}^{-1})}{\nu(\mathbf{a}_p^n)\chi_\nu\xi_\nu^{-1}(p_\mathfrak{p}^n)} \cdot \frac{\mathscr{P}^*(\nu, \phi^{-1})}{\mathscr{P}(\nu, \phi)} \cdot (\ell - 1)! \cdot \exp^*_{/\omega_{f_\nu}}(\nu(\mathfrak{Y}_\infty)^{\widehat{\phi}^{-1}}),$$

where

$$\frac{\mathscr{P}^*(\nu,\phi^{-1})}{\mathscr{P}(\nu,\phi)} = \begin{cases} \frac{\left(1-\nu(\mathbf{a}_p)\chi_\nu\xi_\nu^{-1}\phi^{-1}(\mathfrak{p})\right)}{\left(1-p\nu(\mathbf{a}_p)^{-1}\chi_\nu\xi_\nu^{-1}\phi(\mathfrak{p})\right)} & \text{if } n=0; \\ 1 & \text{if } n \geqslant 1, \end{cases}$$

and $\log_{/\omega_{f_\nu}}$ *and* $\exp^*_{/\omega_{f_\nu}}$ *denote the Bloch–Kato logarithm and dual exponential maps paired against* ω_{f_ν}.

Proof. In view of (5.2), the action of $G_{\mathbf{Q}_p}$ on $\mathscr{F}^+\mathbb{T}$ is given by the unramified character sending Fr_p to $\mathbf{a}_p\cdot\Theta\boldsymbol{\xi}^{-1}(\mathrm{Fr}_p) = \mathbf{a}_p\cdot\boldsymbol{\chi}\boldsymbol{\xi}^{-1}(p_\mathfrak{p})$, and so by Theorem 3.7 and Lemma 5.1 we may consider the map

$$\mathcal{L}^G_{\omega_\mathbf{f}} : H^1_{\mathrm{Iw}}(L_\infty,\mathscr{F}^+\mathbb{T}) \longrightarrow \widetilde{\mathbb{I}}[[G]]$$

defined by the rule

$$\mathcal{L}^G(-) = \mathcal{L}^G_{\omega_\mathbf{f}}(-)\cdot(\omega_\mathbf{f}^\vee\otimes 1).$$

Let $\mathrm{tw}_{-1}\mathcal{L}^G_{\omega_\mathbf{f}}$ be the composite map

$$\mathrm{tw}_{-1}\mathcal{L}^G_{\omega_\mathbf{f}} : H^1_{\mathrm{Iw}}(L_\infty,\mathscr{F}^+\mathbb{T}(1)) \xrightarrow{\otimes(\zeta_{p^r})_r^{-1}} H^1_{\mathrm{Iw}}(L_\infty,\mathscr{F}^+\mathbb{T}) \xrightarrow{\mathcal{L}^G_{\omega_\mathbf{f}}} \widetilde{\mathbb{I}}[[G]] \xrightarrow{\mathrm{Tw}_{\varepsilon_{\mathrm{cyc}}^{-1}}} \widetilde{\mathbb{I}}[[G]],$$

where

- (ζ_{p^r}) is the $\mathbb{I}$-linear map induced by $(x_r \bmod p^r) \mapsto (x_r\otimes\zeta_{p^r} \bmod p^r)$;
- $\mathrm{Tw}_{\varepsilon_{\mathrm{cyc}}^{-1}}$ is the $\mathbb{I}$-linear isomorphism given by $g\mapsto\varepsilon_{\mathrm{cyc}}^{-1}(g)g$ for $g\in G$,

and let $\mathbb{J}$ be the kernel of the natural projection $\mathbb{I}[[G]]\twoheadrightarrow\mathbb{I}[[\Gamma]]$. The corestriction map

$$H^1_{\mathrm{Iw}}(L_\infty,\mathscr{F}^+\mathbb{T}(1))/\mathbb{J} \longrightarrow H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}^+\mathbb{T}(1))$$

is injective, and its cokernel is $H^2(L_\infty,\mathscr{F}^+\mathbb{T}(1))[\mathbb{J}]$, which vanishes since $H^0(K_{\infty,\mathfrak{p}},\mathscr{F}^+\mathbb{T}(1)) = \{0\}$ (as one can see e.g. by the argument right before [CH17, Lem. 5.5]). Quotienting $\mathrm{tw}_{-1}\mathcal{L}^G_{\omega_\mathbf{f}}$ by $\mathbb{J}$ we thus obtain a map

$$\mathrm{tw}_{-1}\mathcal{L}^\Gamma_{\omega_\mathbf{f}} : H^1(K_{\infty,\mathfrak{p}},\mathscr{F}^+\mathbb{T}(1)) \simeq H^1(L_\infty,\mathscr{F}^+\mathbb{T}(1))/\mathbb{J} \longrightarrow \widetilde{\mathbb{I}}[[\Gamma]]$$

with the desired properties following from Theorem 3.7 and Corollary 3.9. □

5.2. **Explicit reciprocity law for big Heegner points.** Let $\mathscr{L}_{\mathfrak{p},\xi}(\mathbf{f})$ be the two-variable p-adic L-function constructed in §2.6, and let $c\mathcal{O}_K$ be the conductor of $\lambda^{1-\tau}$, where λ is as used in the construction of the $\mathbb{I}$-adic character $\boldsymbol{\xi}$. On the other hand, let $\mathfrak{Z}_{c,\infty}\in H^1_{\mathrm{Iw}}(H_{cp^\infty},\mathbf{T}^\dagger)$ be Howard's systems of big Heegner points as recalled in §4.2. As shown in the proof of [How07, Prop. 2.4.5], for each place $v\mid p$ the restriction $\mathrm{res}_v(\mathfrak{Z}_{c,\infty})$ lies in the kernel of the natural map

$$H^1_{\mathrm{Iw}}(H_{cp^\infty,v},\mathbf{T}^\dagger) \longrightarrow H^1_{\mathrm{Iw}}(H_{cp^\infty,v},\mathscr{F}^-\mathbf{T}^\dagger)$$

induced by (4.1). In particular, by [How07, Lem. 2.4.4] the class $\mathrm{res}_\mathfrak{p}(\mathfrak{Z}_\infty)$ is the image of a unique class in $H^1_{\mathrm{Iw}}(H_{cp^\infty,\mathfrak{p}},\mathscr{F}^+\mathbf{T}^\dagger)$ which we shall still denote in the same manner. Moreover, since $\mathbf{T}^\dagger\otimes\boldsymbol{\xi}^{-1} = \mathbb{T}(1)$ by (5.1), the twist $\mathfrak{Z}_{c,\infty}^{\boldsymbol{\xi}^{-1}}$ lies in $H^1_{\mathrm{Iw}}(H_{cp^\infty},\mathbb{T}(1))$; in the following, we let $\mathfrak{Z}_{c,\infty}^{\boldsymbol{\xi}^{-1}}$ be the image of this class in $H^1_{\mathrm{Iw}}(K_\infty,\mathbb{T}(1))$ under corestriction, so that in particular we have $\mathrm{res}_\mathfrak{p}(\mathfrak{Z}_{c,\infty}^{\boldsymbol{\xi}^{-1}})\in H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}^+\mathbb{T}(1))$.

Theorem 5.3. *The following equality holds in* $\widetilde{\mathbb{I}}[[\Gamma]]$*:*

$$\mathrm{tw}_{-1}\mathcal{L}^\Gamma_{\omega_\mathbf{f}}(\mathrm{res}_\mathfrak{p}(\mathfrak{Z}_{c,\infty}^{\boldsymbol{\xi}^{-1}})) = \mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})\cdot\sigma_{-1,\mathfrak{p}},$$

where $\sigma_{-1,\mathfrak{p}} := \mathrm{rec}_\mathfrak{p}(-1)|_{K_\infty}\in\Gamma$.

The proof of Theorem 5.3 will be an immediate consequence of the following result.

Proposition 5.4. *Let $\nu \in \mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ be an arithmetic prime of weight $(2,\varepsilon)$ with $\varepsilon : \Gamma^{\rm wt} \to \boldsymbol{\mu}_{p^\infty}$ of conductor p^s, and let $\widehat{\phi} : \Gamma \to L^\times$ be the p-adic avatar of an anticyclotomic Hecke character ϕ of K of infinity type $(1,-1)$ and conductor p^n. If $n \geqslant s$, then*

$$\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})(\nu,\widehat{\phi}^{-1}) = \frac{\phi_{\mathfrak{p}}(-1)\varepsilon(\phi_{\mathfrak{p}})}{\nu(\mathbf{a}_p^n)\chi_\nu\xi_\nu^{-1}(\mathrm{Fr}_p^n)} \cdot \log_{/\omega_{f_\nu}}(\mathrm{res}_{\mathfrak{p}}(\nu(\mathfrak{Z}_{c,\infty})^{\xi_\nu^{-1}\widehat{\phi}})).$$

Proof. Our hypotheses imply that the character $\xi_\nu\phi^{-1}$ has finite order and it factors through the $\mathrm{Gal}(H_{cp^{n+1}}/K)$. By the same calculation as in the proof of [CH17, Thm. 4.9] (see esp. [*loc.cit.*, (4.8)]) we obtain

$$\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})(\nu,\widehat{\phi}^{-1}) = \mathfrak{g}(\phi_{\mathfrak{p}}^{-1})p^{-n}\phi_{\mathfrak{p}}(p^n)\sum_{\sigma\in\mathrm{Gal}(H_{cp^{n+1}}/K)}\xi_\nu^{-1}\phi(\sigma)\chi_\nu^{-1}(\sigma)\cdot d^{-1}\widehat{f}_\nu^\flat(P^\sigma_{cp^{n+1},s}), \tag{5.3}$$

where $d^{-1}\widehat{f}_\nu$ is the p-adic modular form

$$d^{-1}\widehat{f}_\nu^\flat := \lim_{t\to -1} d^t\widehat{f}_\nu^\flat = \sum_{(n,p)=1}\nu(\mathbf{a}_n)n^{-1}q^n.$$

To proceed with the proof, we need to recall the definition of the Frobenius operator Frob on the space $V_p(N;R)$ of p-adic modular forms, where we take R to be a complete discrete valuation ring containing $\mathcal{O}_\nu$. If $x = [(A,\eta^{(p)},\eta_p)]$ is a point in $\widehat{\mathrm{Ig}}(N)_{/R}$ with

$$(\eta^{(p)},\eta_p) : \boldsymbol{\mu}_N \oplus \boldsymbol{\mu}_{p^\infty} \hookrightarrow A[N]\oplus A[p^\infty],$$

then η_p amounts to giving an isomorphism $\widehat{\eta}_p : \widehat{A} \simeq \widehat{\mathbf{G}}_m$ of formal groups over R, and we set

$$\mathrm{Frob}(x) := (A_0,\eta_0^{(p)},\eta_{0,p}),$$

where:

- $A_0 := A/\eta_p(\boldsymbol{\mu}_p)$ is the quotient of A by its canonical subgroup;
- $\eta_0^{(p)} := \lambda_0\circ\eta^{(p)} : \boldsymbol{\mu}_N \hookrightarrow A_0[N]$, where $\lambda_0 : A \to A_0$ is the natural projection;
- $\eta_{0,p} : \boldsymbol{\mu}_{p^\infty} \hookrightarrow A_0[p^\infty]$ induces $\widehat{\eta}_{0,p} := \widehat{\eta}_p\circ\widehat{\mu}_0$, where $\widehat{\mu}_0 : \widehat{A}_0 \simeq \widehat{A}$ is the isomorphism of formal groups induced by the dual isogeny $\mu_0 = \lambda_0^\vee$.

The action of Frob on $V_p(N;R)$ is then defined in the obvious manner, setting

$$\mathrm{Frob}(g)(x) := g(\mathrm{Frob}(x)),$$

for every $g \in V_p(N;R)$ and $x \in \widehat{\mathrm{Ig}}(N)_{/R}$.

Now let $F_{\omega_{f_\nu}}$ be the Coleman primitive of the differential ω_{f_ν} vanishing at the cusp ∞; this is a locally analytic p-adic modular form (as defined in [BDP13, p. 1083]) of weight 0 satisfying

$$dF_{\omega_{f_\nu}} = \omega_{f_\nu}$$

and characterized by the further requirement that

$$F_{\omega_{f_\nu}} - \frac{\nu(\mathbf{a}_p)}{p}\mathrm{Frob}(F_{\omega_{f_\nu}}) = d^{-1}\widehat{f}_\nu^\flat \tag{5.4}$$

(*cf.* [Cas13, Cor. 2.8]). In particular, note that $U_pF_{\omega_{f_\nu}} = \frac{\nu(\mathbf{a}_p)}{p}F_{\omega_{f_\nu}}$.

Let $F_{n,s}$ be a finite extension of $\imath_p(L_{cp^{n+1},s})$ in $\overline{\mathbf{Q}}_p$ such that the base-change $X_s\times_{\mathbf{Q}_p}F_{n,s}$ admits a stable model. The calculation in [Cas13, Prop. 2.9] applies to f and the classes

$$\Delta_{cp^{n+1},s} := (P_{cp^{n+1},s}) - (\infty), \quad \Delta_{cp^{n+1+s},s} := (P_{cp^{n+1+s},s}) - (\infty)$$

in $J_s(F_{n,s})$, yielding the formulae

$$\log_{\omega_{f_\nu}}(\Delta_{cp^{n+1},s}) = F_{\omega_{f_\nu}}(P_{cp^{n+1},s}), \quad \log_{\omega_{f_\nu}}(\Delta_{cp^{n+1+s},s}) = F_{\omega_{f_\nu}}(P_{cp^{n+1+s},s}), \tag{5.5}$$

where $\log_{\omega_{f_\nu}} : J_s(F_{n,s}) \to \mathbf{C}_p$ is the formal group logarithm associated with ω_{f_ν}.

Now define $Q_{cp^{n+1},s} \in J_s(L_{cp^{n+1},s}) \otimes_{\mathbf{Z}} F_\nu$ by

$$Q_{cp^{n+1},s} = \sum_{\sigma \in \mathrm{Gal}(H_{cp^{n+1+s}}/H_{cp^{n+1}})} \Delta^{\tilde{\sigma}}_{cp^{n+1+s},s} \otimes \chi_\nu^{-1}(\tilde{\sigma}), \tag{5.6}$$

where for each $\sigma \in \mathrm{Gal}(H_{cp^{n+1+s}}/H_{cp^{n+1}})$, $\tilde{\sigma}$ is an arbitrary lift of σ to $\mathrm{Gal}(L_{cp^{n+1},s}/H_{cp^{n+1}})$; by (4.2), the point $Q_{cp^{n+1},s}$ does not depend on the particular choice of lift. Taking lifts $\tilde{\sigma}$ in (5.6) which act trivially on $\boldsymbol{\mu}_{p^s}$ (as we may, since $H_{cp^{n+1+s}} \cap H_{cp^{n+1}}(\boldsymbol{\mu}_{p^s}) = H_{cp^{n+1}}$) and extending the map $\log_{\omega_f}$ by F_ν-linearity, we deduce from (5.5) that

$$\begin{aligned} \log_{\omega_{f_\nu}}(Q_{cp^{n+1},s}) &= \sum_{\tau \in \mathrm{Gal}(L_{cp^{n+1},s}/H_{cp^{n+1}}(\boldsymbol{\mu}_{p^s}))} F_{\omega_{f_\nu}}(P^\tau_{cp^{n+1+s},s}) \\ &= F_{\omega_{f_\nu}}(U_p^s \cdot P_{cp^{n+1+s},s}) \\ &= \left(\frac{\nu(\mathbf{a}_p)}{p}\right)^s \cdot F_{\omega_{f_\nu}}(P_{cp^{n+1},s}), \end{aligned} \tag{5.7}$$

using Lemma 4.2 for the second equality. Substituting (5.7) into (5.3) and using (5.4) we thus arrive at

$$\begin{aligned} \mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})(\nu, \widehat{\phi}^{-1}) &= \mathfrak{g}(\phi_{\mathfrak{p}}^{-1}) p^{-n} \phi_{\mathfrak{p}}(p^n) \\ &\quad \times \left(\frac{p}{\nu(\mathbf{a}_p)}\right)^s \sum_{\sigma \in \mathrm{Gal}(H_{cp^{n+1}}/K)} \xi_\nu^{-1}\phi\chi_\nu^{-1}(\sigma) \cdot \log_{\omega_f}(Q^\sigma_{cp^{n+1},s}). \end{aligned} \tag{5.8}$$

Recall that $\mathbf{T}^\dagger$ denotes the twist $\mathbf{T} \otimes \Theta^{-1}$, and note that $\mathbf{T}^\dagger \otimes_{\mathbb{I}} F_\nu \simeq \mathbf{T} \otimes_{\mathbb{I}} F_\nu$ as $G_{\mathbf{Q}(\boldsymbol{\mu}_{p^s})}$-representations. By Hida's control theorem (see e.g. [Hid86, Thm. 3.1(i)]), the natural map $\mathbf{T} \to \mathbf{T} \otimes_{\mathbb{I}} F_\nu$ factors as

$$\mathbf{T} \longrightarrow e^{\mathrm{ord}}\mathrm{Ta}_p(J_s) \longrightarrow \mathbf{T} \otimes_{\mathbb{I}} F_\nu,$$

and tracing through the definition of $\mathfrak{X}_{cp^{n+1}}$ in 4.2 we see that the image of $Q_{cp^{n+1},s}$ under the induced map

$$J_s(L_{cp^{n+1},s}) \otimes F_\nu \xrightarrow{e^{\mathrm{ord}} \circ \mathrm{Kum}} H^1(L_{cp^{n+1},s}, e^{\mathrm{ord}}\mathrm{Ta}_p(J_s) \otimes F_\nu) \longrightarrow H^1(L_{cp^{n+1},s}, \mathbf{T} \otimes_{\mathbb{I}} F_\nu)$$

agrees with the image of $U_p^s \cdot \nu(\mathfrak{X}_{cp^{n+1}})$ under the restriction

$$H^1(H_{cp^{n+1+s}}, \mathbf{T}^\dagger \otimes_{\mathbb{I}} F_\nu) \longrightarrow H^1(L_{cp^{n+1},s}, \mathbf{T}^\dagger \otimes_{\mathbb{I}} F_\nu) \simeq H^1(L_{cp^{n+1},s}, \mathbf{T} \otimes_{\mathbb{I}} F_\nu),$$

and hence

$$\log_{\omega_f}(Q_{cp^{n+1},s}) = \left(\frac{\nu(\mathbf{a}_p)}{p}\right)^s \cdot \log_{/\omega_f}(\mathrm{res}_{\mathfrak{p}}(\nu(\mathfrak{X}_{cp^{n+1}}))) \tag{5.9}$$

by the compatibility between the map $\log_{/\omega_f}$ in Proposition 5.2 and $\log_{\omega_f}$ (see [BK90, §3.10.1]).

Note that $\varepsilon(\phi_{\mathfrak{p}}) = \mathfrak{g}(\phi_{\mathfrak{p}}^{-1})\phi_{\mathfrak{p}}(-p^n)$. Thus substituting (5.9) into (5.8) and using (4.3) for the second equality, we conclude that

$$\begin{aligned} \mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})(\nu, \widehat{\phi}^{-1}) &= \frac{\phi_{\mathfrak{p}}(-1)\varepsilon(\phi_{\mathfrak{p}})}{\chi_\nu\xi_\nu^{-1}(p_{\mathfrak{p}}^n)} \sum_{\sigma \in \mathrm{Gal}(H_{cp^{n+1}}/K)} \xi_\nu^{-1}\phi(\sigma) \cdot \log_{/\omega_{f_\nu}}(\mathrm{res}_{\mathfrak{p}}(\nu(\mathfrak{X}_{cp^{n+1}})^\sigma)) \\ &= \frac{\phi_{\mathfrak{p}}(-1)\varepsilon(\phi_{\mathfrak{p}})}{\nu(\mathbf{a}_p^n)\chi_\nu\xi_\nu^{-1}(p_{\mathfrak{p}}^n)} \cdot \log_{/\omega_{f_\nu}}(\mathrm{res}_{\mathfrak{p}}(\nu(\mathfrak{Z}_{c,\infty})^{\xi_\nu^{-1}\widehat{\phi}})), \end{aligned}$$

as was to be shown. ∎

Proof of Theorem 5.3. In light of Proposition 5.2, the content of Proposition 5.4 amounts to the equality

$$\mathrm{tw}_{-1}\mathcal{L}^{\Gamma}_{\omega_{\mathbf{f}}}(\mathrm{res}_{\mathfrak{p}}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c,\infty}))(\nu,\widehat{\phi}^{-1}) = (\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})\cdot\sigma_{-1,\mathfrak{p}})(\nu,\widehat{\phi}^{-1}),$$

for all pairs (ν,ϕ) as in the statement of that result. Since an element in $\widetilde{\mathbb{I}}[[\Gamma]]$ is uniquely determined by its values at such set of pairs, the result follows. □

6. Applications

6.1. **Preparations.** Let $f\in S_k(\Gamma_1(N))$ be a p-ordinary newform of weight $k>0$ and level N prime to p defined over a finite extension L of $\mathbf{Q}_p$, and let K be an imaginary quadratic field satisfying hypothesis (heeg) relative to N and in which $p=\mathfrak{p}\overline{\mathfrak{p}}$. Let χ be the p-adic avatar of an anticyclotomic Hecke character of K of infinity type $(j,-j)$ with $j-k/2\in\mathbf{Z}$, and set

$$V_{f,\chi} := V_f(k/2)|_{G_K}\otimes\chi.$$

For S a finite set of places of K containing the primes above Np, and for every finite extension F/K inside $\overline{\mathbf{Q}}$, let $\mathfrak{G}_{F,S}$ be the Galois group of the maximal extension of F unramified outside the places above S. Recall that the *Bloch–Kato Selmer group* $\mathrm{Sel}(F,V_{f,\chi})$ is defined by

$$\mathrm{Sel}(F,V_{f,\chi}) = \ker\left(H^1(\mathfrak{G}_{F,S},V_{f,\chi}) \longrightarrow \prod_v \frac{H^1(F_v,V_{f,\chi})}{H^1_{\mathrm{f}}(F_v,V_{f,\chi})}\right), \tag{6.1}$$

where v runs over all places of F, and

$$H^1_{\mathrm{f}}(F_v,V_{f,\chi}) := \begin{cases} \ker\left(H^1(F_v,V_{f,\chi})\longrightarrow H^1(F_v^{\mathrm{ur}},V_{f,\chi})\right) & \text{if } v\nmid p;\\ \ker\left(H^1(F_v,V_{f,\chi})\longrightarrow H^1(F_v,V_{f,\chi}\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{cris}})\right) & \text{if } v\mid p.\end{cases}$$

For $T_{f,\chi} := T_f(k/2)|_{G_K}\otimes\chi$ with $T_f\subseteq V_f$ a $G_{\mathbf{Q}}$-stable lattice, we define $\mathrm{Sel}(F,T_{f,\chi})$ by the same recipe (6.1), replacing $H^1_{\mathrm{f}}(F_v,V_{f,\chi})$ by their natural preimages in $H^1(F_v,T_{f,\chi})$. By abuse of notation, we let $F_{\mathfrak{p}}$ denote the completion of F at any place above $\mathfrak{p}$, and similarly for $F_{\overline{\mathfrak{p}}}$.

Lemma 6.1. *If the infinity type of χ is $(j,-j)$ with $j-k/2\in\mathbf{Z}$ and $j\geqslant k/2$, then:*

$$H^1_{\mathrm{f}}(F_{\overline{\mathfrak{p}}},V_{f,\chi})=\{0\},\qquad H^1_{\mathrm{f}}(F_{\mathfrak{p}},V_{f,\chi})=H^1(F_{\mathfrak{p}},V_{f,\chi}).$$

In particular, the classes in the Bloch–Kato Selmer group $\mathrm{Sel}(F,V_{f,\chi})$ are trivial at all primes above $\overline{\mathfrak{p}}$ and satisfy no local condition at the primes above $\mathfrak{p}$.

Proof. Following our conventions (see the footnote in Theorem 3.7), we find that the Hodge–Tate weights of $V_{\overline{\mathfrak{p}}} := V_{f,\chi}|_{G_{F_{\overline{\mathfrak{p}}}}}$ are $k/2-j$ and $1-k/2-j$, which are non-positive integers under the above hypotheses, it follows that $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\overline{\mathfrak{p}}})=\mathbf{D}_{\mathrm{dR}}(V_{\overline{\mathfrak{p}}})$. Similarly, the Hodge–Tate weights of $V_{\mathfrak{p}} := V_{f,\chi}|_{G_{F_{\mathfrak{p}}}}$ are the strictly positive integers $k/2+j$ and $1-k/2+j$, which implies that $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{p}})=\{0\}$. The result thus follows from [BK90, Thm. 4.1(ii)]. □

We will also have use for the following generalized Selmer groups obtained by changing the local condition at the places above p in definition (6.1). For $v\mid p$ and $\mathcal{L}_v\in\{\emptyset,\mathrm{Gr},0\}$, set

$$H^1_{\mathcal{L}_v}(F_v,V_{f,\chi}) := \begin{cases} H^1(F_v,V_{f,\chi}) & \text{if } \mathcal{L}_v=\emptyset;\\ H^1(F_v,\mathscr{F}^+V_{f,\chi}) & \text{if } \mathcal{L}_v=\mathrm{Gr};\\ \{0\} & \text{if } \mathcal{L}_v=0,\end{cases}$$

and for $\mathcal{L}=\{\mathcal{L}_v\}_{v|p}$, define

$$H^1_{\mathcal{L}}(F,V_{f,\chi}) := \ker\left(H^1(\mathfrak{G}_{F,S},V_{f,\chi})\longrightarrow \prod_{v\nmid p}\frac{H^1(F_v,V_{f,\chi})}{H^1_{\mathrm{f}}(F_v,V_{f,\chi})}\times\prod_{v|p}\frac{H^1(F_v,V_{f,\chi})}{H^1_{\mathcal{L}_v}(F_v,V_{f,\chi})}\right).$$

In particular, by Lemma 6.1 we have

$$\mathrm{Sel}(F, V_{f,\chi}) = H^1_{\emptyset,0}(F, V_{f,\chi}). \tag{6.2}$$

As in §4.1, let $\mathbf{f} \in \mathbb{I}[[q]]$ be the $\mathbb{I}$-adic newform of tame level N attached to f, and let $\mathbf{T}$ be the associated big Galois representation.

Lemma 6.2. *Let F be a finite extension of K, and let v be a prime of F above a prime ℓ dividing (D_K, N). If $\bar{\rho}_f$ is ramified at ℓ, then $H^1(F_v^{\mathrm{ur}}, \mathbf{T}^\dagger)$ is $\mathbb{I}$-torsion free.*

Proof. This is well-known; see e.g. [Büy14, Lem. 3.12]. □

For F/K a finite extension, let $\mathrm{Sel}_{\mathrm{Gr}}(F, \mathbf{T}^\dagger) \subseteq H^1(\mathfrak{G}_{F,S}, \mathbf{T}^\dagger)$ be the strict Greenberg Selmer group of [How07, Def. 2.4.2].

Proposition 6.3. *If $\bar{\rho}_f$ is ramified at every prime $\ell \mid (D_K, N)$, then $\mathfrak{X}_c \in \mathrm{Sel}_{\mathrm{Gr}}(H_c, \mathbf{T}^\dagger)$ for all positive integers c prime to N.*

Proof. The proof of [How07, Prop. 2.4.5] shows that the localization $\mathrm{loc}_v(\mathfrak{X}_c)$ of $\mathfrak{X}_c$ at any place v of H_c lies in the local subspace $H^1_{\mathrm{Gr}}(H_{c,v}, \mathbf{T}^\dagger) \subseteq H^1(H_{c,v}, \mathbf{T}^\dagger)$ defining $\mathrm{Sel}_{\mathrm{Gr}}(H_c, \mathbf{T}^\dagger)$, except possibly at primes $v \mid \ell \mid N$ which are nonsplit in K, in which case it is shown that

$$\mathrm{loc}_v(\mathfrak{X}_c) \in \ker\left(H^1(H_{c,v}, \mathbf{T}^\dagger) \longrightarrow \frac{H^1(H^{\mathrm{ur}}_{c,v}, \mathbf{T}^\dagger)}{H^1(H^{\mathrm{ur}}_{c,v}, \mathbf{T}^\dagger)_{\mathrm{tors}}}\right),$$

where $H^1(H^{\mathrm{ur}}_{c,v}, \mathbf{T}^\dagger)_{\mathrm{tors}} \subseteq H^1(H^{\mathrm{ur}}_{c,v}, \mathbf{T}^\dagger)$ is the $\mathbb{I}$-torsion submodule. In light of Lemma 6.2, the result follows. □

6.2. **Higher weight specializations of big Heegner points.** In this section we show the connection between the higher weight specializations of big Heegner points and the étale Abel–Jacobi images of classical Heegner cycles [Nek95]. A first result along these lines was obtained in [Cas13] under a certain nonvanishing hypothesis (see [*loc.cit.*, Thm. 5.11]). In Theorem 6.5 below we remove that hypothesis, and find a relation between the global cohomology classes themselves, rather than just their cyclotomic p-adic heights.

Let $f \in S_{2r}(\Gamma_0(N))$ be a p-ordinary newform of even weight $2r \geqslant 2$, and let $T_f \subseteq V_f$ be a Galois stable lattice in the the p-adic Galois representations ρ_f attached to f. Fix an integer c prime to p, let Δ_c be the kernel of the projection $\mathrm{Gal}(H_{cp^\infty}/K) \twoheadrightarrow \Gamma$, and for every character χ of Δ_c, set

$$\mathrm{Sel}_{\mathrm{Gr}}(K_\infty, T_f(r) \otimes \chi) := \varprojlim_t \mathrm{Sel}(K_t, T_f(r) \otimes \chi),$$

where K_t is the subfield of K_∞ of degree p^t over K.

Lemma 6.4. *If $\bar{\rho}_f|_{G_K}$ is irreducible, then the restriction map*

$$\mathrm{res}_{\mathfrak{p}} : \mathrm{Sel}_{\mathrm{Gr}}(K_\infty, T_f(r) \otimes \chi) \longrightarrow H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, \mathscr{F}^+T_f(r) \otimes \chi)$$

is injective.

Proof. Since $\mathrm{Sel}_{\mathrm{Gr}}(K_\infty, T_f(r) \otimes \chi) \subseteq H^1_{\mathrm{Iw}}(K_\infty, T_f(r) \otimes \chi)$ is Λ-torsion-free by our hypothesis (see [How04, Lem. 2.2.9]), it suffices to show that $\ker(\mathrm{res}_{\mathfrak{p}})$ is Λ-torsion, or equivalently, that for infinitely many characters $\phi : \Gamma \to \boldsymbol{\mu}_{p^\infty}$ of p-power order, the ϕ-specialized map

$$\mathrm{Sel}_{\mathrm{Gr}}(K, V_f(r) \otimes \chi\phi) \longrightarrow H^1(K_{\mathfrak{p}}, \mathscr{F}^+V_f(r) \otimes \chi\phi) \tag{6.3}$$

is injective. Let $\mathbf{f} \in \mathbb{I}[[q]]$ be the $\mathbb{I}$-adic newform attached to f, and let $\nu \in \mathcal{X}^a_{\mathcal{O}}(\mathbb{I})$ be such that $\mathbf{f}_\nu$ is the ordinary p-stabilization of f. By Corollary 2.12, we have $\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))(\phi) \neq 0$ for all but finitely many ϕ, and by Theorem 5.3 this shows that $\mathrm{res}_{\mathfrak{p}}(\nu(\mathfrak{Z}_{c,\infty})^{\chi\phi}) \neq 0$ for all but finitely many ϕ, where $\nu(\mathfrak{Z}_{c,\infty})^{\chi\phi}$ is the image of $\nu(\mathfrak{Z}^\chi_{c,\infty})$ under the ϕ-specialization map

$$\mathrm{Sel}_{\mathrm{Gr}}(K_\infty, T_f(r) \otimes \chi) \longrightarrow \mathrm{Sel}_{\mathrm{Gr}}(K, V_f(r) \otimes \chi\phi).$$

Since by the results of [How07, §2.3] the class $\nu(\mathfrak{Z}_{c,\infty})^{\chi\phi}$ is the class over K of an Euler system for $T_f(r)\otimes\chi\phi$ in the sense of [CH17, Def. 7.2] with the Bloch–Kato local condition, by [CH17, Thm. 7.7] we have the implication

$$\nu(\mathfrak{Z}_{c,\infty})^{\chi\phi}\neq 0 \quad\Longrightarrow\quad \mathrm{Sel}_{\mathrm{Gr}}(K,V_f(r)\otimes\chi\phi)=L.\nu(\mathfrak{Z}_{c,\infty})^{\chi\phi}.$$

We thus conclude that (6.3) is injective, and so the result follows. □

We are now ready to prove a strong refinement of the main result of [Cas13], as advanced in [CH17, §1]. Let K be an imaginary quadratic field of odd discriminant $-D_K<0$ satisfying hypothesis (heeg) relative to N and in which the prime $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits (with $p\nmid 6N$, as usual) and fix a positive integer c prime to Np. Let α be the p-adic unit root of the p-th Hecke polynomial of f, and denote by

$$\mathbf{z}_{f,c,\alpha}\in H^1_{\mathrm{Iw}}(H_{cp^\infty},T_f(r))$$

the Λ-adic class constructed from generalized Heegner cycles in [CH17, §5.2] with tame conductor c, which defines a class in $\mathrm{Sel}_{\mathrm{Gr}}(H_{cp^\infty},T_f(r))$. We refer the reader to [Cas13, p. 1250] and [CH17, Eq. (4.6)] for the definition of the p-adic étale Abel–Jacobi images

$$\Phi^{\text{ét}}_{g,H_c}(\Delta^{\mathrm{heeg}}_{c,r}),\ \Phi^{\text{ét}}_{g,H_c}(\Delta^{\mathrm{bdp}}_{c,r})\ \in\mathrm{Sel}(H_c,T_g(r)).$$

of classical and generalized Heegner cycles, respectively, attached to an eigenform g of weight $2r\geqslant 2$. (For $r=1$, both reduce to Kummer images of classical Heegner points.) For the next theorem, write $k_0\geqslant 2$ for the weight of f, let $\mathbf{f}=\sum_{n=1}^\infty\mathbf{a}_nq^n\in\mathbb{I}[[q]]$ be the associated $\mathbb{I}$-adic newform of tame level N, and let $\mathfrak{Z}_{c,\infty}$ be Howard's system of big Heegner points attached to $\mathbf{f}$ and K, as recalled in §4.2.

Theorem 6.5. *Assume in addition that:*

- $k_0\equiv 2\pmod{p-1}$*;*
- $\bar\rho_f$ *is ramified at every prime* $q\mid(D_K,N)$*;*
- $\bar\rho_f$ *p-distinguished;*
- $\bar\rho_f|_{G_K}$ *is irreducible.*

Then for all $\nu\in\mathcal{X}^a_{\mathcal{O}}(\mathbb{I})$ *of weight* $2r>2$ *with* $2r\equiv k_0\pmod{2(p-1)}$ *and trivial nebentypus, we have*

$$\nu(\mathfrak{Z}_{c,\infty})\cdot c^{r-1}=\mathbf{z}_{f_\nu,c,\alpha}$$

as elements in $\mathrm{Sel}_{\mathrm{Gr}}(H_{cp^\infty},T_{f_\nu}(r))$*, where* $\alpha=\nu(\mathbf{a}_p)$*. In particular, for all such* ν *we have*

$$\nu(\mathfrak{Z}_{c,0})=\left(1-\frac{p^{r-1}}{\nu(\mathbf{a}_p)}\right)^2\cdot\frac{\Phi^{\text{ét}}_{f_\nu,H_c}(\Delta^{\mathrm{heeg}}_{r,c})}{u_c(2c\sqrt{-D_K})^{r-1}},\tag{6.4}$$

where $u_c=|\mathcal{O}_c^\times|/2$*.*

Proof. Letting χ be any fix character of Δ_c, it suffices to show the equality for the corresponding classes in $\mathrm{Sel}(K_\infty,V_f(r)\otimes\chi)$. Take $\boldsymbol{\xi}$ so that $\psi:=\xi_\nu$ restricts to the character χ on Δ_c, and let

$$\mathcal{L}^\psi_{\nu,\mathfrak{p}}:=\left\langle\mathcal{L}_{\mathfrak{p},\psi}(-),\omega_{f_\nu}\otimes t^{1-2r}\right\rangle: H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}^+V_{f_\nu}(r)\otimes\chi^{-1})\longrightarrow\Lambda_R(\Gamma)$$

be the map introduced in [CH17, §5.3] twisted by χ^{-1}. Then the map $\mathrm{tw}_{-1}\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}}$ of Proposition 5.2 twisted by $\boldsymbol{\xi}^{-1}$ specializes at ν to the map $\mathcal{L}_{\nu,\mathfrak{p}}$ twisted by ψ^{-1}, and so by Theorem 5.3 we have the relations

$$\mathcal{L}^\psi_{\nu,\mathfrak{p}}(\nu(\mathfrak{Z}_{c,\infty}))=\nu(\mathrm{tw}_{-1}\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}}(\mathrm{res}_{\mathfrak{p}}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c,\infty})))=\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})\cdot\sigma_{-1,\mathfrak{p}}).\tag{6.5}$$

On the other hand, as shown in the proof of Theorem 2.11, $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ specializes at ν to the p-adic L-function $\mathscr{L}_{\mathfrak{p},\psi}(f_\nu)$ of [CH17, §3.3], and so by the explicit reciprocity law of [*loc.cit.*, Thm. 5.7] we have the equalities

$$\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})\cdot\sigma_{-1,\mathfrak{p}}) = \mathscr{L}_{\mathfrak{p},\psi}(f_\nu)\cdot\sigma_{-1,\mathfrak{p}} = \mathcal{L}^{\psi}_{\nu,\mathfrak{p}}(\mathbf{z}_{f_\nu,c,\alpha})\cdot c^{1-r}, \tag{6.6}$$

where $\alpha=\nu(\mathbf{a}_p)$, since this is the U_p-eigenvalue of the p-stabilized newform f_ν.

Comparing (6.5) and (6.6), the proof of the first statement in Theorem 6.5 follows from Lemma 6.4 and the injectivity of $\mathcal{L}^{\psi}_{\nu,\mathfrak{p}}$. (The injectivity of this map is not explicitly stated in [CH17, §5.3], but it follows from the construction in [*loc.cit.*, Thm. 5.1] and [LZ14, Prop. 4.11].) In particular, by the construction of $\mathbf{z}_{f_\nu,c,\alpha}$ in [CH17, §5.2] (see [*loc.cit.*, Def. 5.2]), we obtain the relation

$$\nu(\mathfrak{Z}_{c,0}) = \frac{1}{u_c}\left(1-\frac{p^{r-1}}{\nu(\mathbf{a}_p)}\right)^2\cdot\Phi^{\text{ét}}_{f_\nu,H_c}(\Delta^{\text{bdp}}_{c,r}),$$

where $u_c=|\mathcal{O}_c^\times|/2$, and by [BDP17, Prop. 4.1.2] (with $r_1=2r-2$, $r_2=0$, and so $u=r-1$) relation (6.4) follows. □

6.3. **Proof of Theorem C.** With the same notations as in the Introduction, let $V_\varrho^\vee$ denote the contragredient of the representation V_ϱ, and let $g\in S_1(\Gamma_1(N_\varrho))$ be a cusp form whose associated Deligne–Serre representation V_g is isomorphic to $V_\varrho^\vee$ (the existence of g is a consequence of the proof [KW09] of Serre's modularity conjecture). If $\mathfrak{P}$ is prime of E above p as in the statement of Theorem C, we shall view g and V_g as the defined over the finite extension of $\mathbf{Q}_p$ given by the completion $L:=E_{\mathfrak{P}}$, and let $T_g\subseteq V_g$ be any $G_{\mathbf{Q}}$-stable $\mathcal{O}_L$-lattice.

Let $g_p\in S_1(\Gamma_0(p)\cap\Gamma_1(N_\varrho))$ be a p-stabilization of g. By [Wil88, Thm. 3], there exists an $\mathbb{I}$-adic newform $\mathbf{f}$ of tame conductor N_ϱ with $\nu_1(\mathbf{f})=g_p$, and our hypotheses on ϱ guarantee that the associated residual representation $\rho_{\mathbf{f}}$ is irreducible and p-distinguished. On the other hand, let λ be the grossencharacter associated to A by the theory of complex multiplication, and let $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ be the two-variable p-adic L-function of §2.6 constructed with the corresponding anticyclotomic $\mathbb{I}$-adic character $\boldsymbol{\xi}$. If $\mathfrak{Z}_{c,\infty}\in\mathrm{Sel}_{\mathrm{Gr}}(H_{cp^\infty},\mathbf{T}^\dagger)$ is Howard's system of big Heegner points attached to $\mathbf{f}$ and K, where as usual $c\mathcal{O}_K$ is the conductor of $\lambda^{1-\tau}$, setting $\chi:=\xi_{\nu_1}$ we see from Theorem 2.11 and Theorem 5.3 that

$$\begin{aligned} L(A/\mathbf{Q},\varrho,1)\neq 0 &\implies L(g/K,\chi\mathbf{N}^{-1/2},0)\neq 0\\ &\implies \nu_1(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))(\mathbb{1})\neq 0\\ &\implies \mathrm{res}_{\mathfrak{p}}(\nu_1(\mathfrak{Z}_{c,\infty})^{\chi^{-1}}))\neq 0,\end{aligned}\tag{6.7}$$

and so $\mathrm{res}_{\overline{\mathfrak{p}}}(\nu_1(\mathfrak{Z}_{c,\infty})^{\chi})\neq 0$ by the action of complex conjugation.

By the Euler system relations established in [How07, §2.3], $\nu_1(\mathfrak{Z}_{c,\infty})^{\chi^{-1}}$ is the class over K of an anticyclotomic Euler system for $T_g(1/2)\otimes\chi$ in the sense of [CH17, Def. 7.2] satisfying the local conditions defining $H^1_{\mathrm{Gr,Gr}}(K,V_{g,\chi})$, where $V_{g,\chi}:=V_g(1/2)|_{G_K}\otimes\chi$. Thus as in the proof of [CH17, Thm. 7.9] the last nonvanishing in (6.7) implies that

$$H^1_{\mathrm{Gr,Gr}}(K,V_{g,\chi}) = L\cdot(\nu_1(\mathfrak{Z}_{c,\infty})^{\chi^{-1}})^{\tau} = L\cdot\nu_1(\mathfrak{Z}_{c,\infty})^{\chi},$$

and since $\mathrm{res}_{\overline{\mathfrak{p}}}(\nu_1(\mathfrak{Z}_{c,\infty})^{\chi})\neq 0$, this implies that

$$H^1_{\mathrm{Gr},0}(K,V_{g,\chi}) = \{0\}. \tag{6.8}$$

From Poitou–Tate duality we obtain the exact sequence

$$\begin{aligned} 0\longrightarrow H^1_{0,\emptyset}(K,V_{g,\chi^{-1}})\longrightarrow H^1_{\mathrm{Gr},\emptyset}(K,V_{g,\chi^{-1}})&\xrightarrow{\mathrm{res}_{\mathfrak{p}}} H^1(K_{\mathfrak{p}},\mathscr{F}^+V_{g,\chi^{-1}})\\ &\longrightarrow H^1_{\emptyset,0}(K,V_{g,\chi})^\vee\longrightarrow H^1_{\mathrm{Gr},0}(K,V_{g,\chi})^\vee,\end{aligned}$$

and since $H^1(K_{\mathfrak{p}}, \mathscr{F}^+V_{g,\chi^{-1}})$ is one-dimensional, combining (6.7) and (6.8) we conclude that

$$H^1_{\emptyset,0}(K, V_{g,\chi}) = \{0\}, \tag{6.9}$$

and so $\mathrm{Sel}(K, V_p(A) \otimes V_\varrho^\vee)$ vanishes by Lemma 6.3.

Now let $H = \mathrm{Gal}(F/\mathbf{Q})$, where F is the splitting field of ϱ. Since $\mathrm{Hom}_{G_{\mathbf{Q}}}(V_\varrho, \mathrm{Sel}(F, V_p(A))_L)$ can be identified with the set of H-invariant classes in $\mathrm{Sel}(F, V_p(A)) \otimes V_\varrho^\vee = \mathrm{Sel}(F, V_p(A) \otimes V_\varrho^\vee)$, and the restriction map $\mathrm{Sel}(\mathbf{Q}, V_p(A) \otimes V_\varrho^\vee) \to \mathrm{Sel}(F, V_p(A) \otimes V_\varrho^\vee)^H$ is as isomorphism, the proof of Theorem C follows immediately from (6.9).

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Mathematics Department, Princeton University, Fine Hall, Princeton, NJ 08544-1000, USA
E-mail address: fcabello@math.princeton.edu