

HEEGNER CYCLES AND p -ADIC L -FUNCTIONS

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ABSTRACT. In this paper, we deduce the vanishing of Selmer groups for the Rankin–Selberg convolution of a cusp form with a theta series of higher weight from the nonvanishing of the associated L -value, thus establishing the rank 0 case of the Bloch–Kato conjecture in these cases. Our methods are based on the connection between Heegner cycles and p -adic L -functions, building upon recent work of Bertolini, Darmon and Prasanna, and on an extension of Kolyvagin’s method of Euler systems to the anticyclotomic setting. In the course of the proof, we also obtain a higher weight analogue of Mazur’s conjecture (as proven in weight 2 by Cornut–Vatsal), and as a consequence of our results, we deduce from Nekovář’s work a proof of the parity conjecture in this setting.

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1. INTRODUCTION

Let $f \in S_{2r}^{\text{new}}(\Gamma_0(N))$ be a newform of weight $2r \geq 2$ and level N . Fix an odd prime $p \nmid N$. Let F/\mathbf{Q}_p be a finite extension containing the image of the Fourier coefficients of f under a fixed embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$, and denote by

$$\rho_f : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \text{Aut}_F(V_f(r)) \simeq \text{GL}_2(F)$$

the self-dual Tate twist of the p -adic Galois representation associated to f . Let K/\mathbf{Q} be an imaginary quadratic field of odd discriminant $-D_K < -3$ and let $\chi : G_K := \text{Gal}(\overline{\mathbf{Q}}/K) \rightarrow F^\times$ be a locally algebraic anticyclotomic character. The G_K -representation

$$V_{f,\chi} := V_f(r) \otimes \chi$$

is then conjugate self-dual, and the associated Rankin L -series $L(f, \chi, s)$ satisfies a functional equation relating its values at s and $2r - s$. The Bloch–Kato conjectures (see [BK90], [FPR94]), which provide a vast generalization of the Birch–Swinnerton–Dyer conjecture and Dirichlet’s class number formula, predict in this context the equality

$$\text{(BK)} \quad \text{ord}_{s=r} L(f, \chi, s) \stackrel{?}{=} \dim_F \text{Sel}(K, V_{f,\chi})$$

between the order of vanishing at the central point of the Rankin L -series $L(f, \chi, s)$ and the size of the Bloch–Kato Selmer group $\text{Sel}(K, V_{f,\chi})$ for the representation $V_{f,\chi}$.

Hypothesis (H). The following hypotheses are assumed throughout.

- (a) $p \nmid 2(2r - 1)!N\varphi(N)$;

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- (b) the conductor of χ is prime to N ;
- (c) N is a product of primes split in K ;
- (d) $p = \mathfrak{p}\bar{\mathfrak{p}}$ is split in K , where \mathfrak{p} is induced by ι_p .

Our first arithmetic application is the proof of new “rank zero” cases of conjecture (BK).

Theorem A. *Assume further that the newform f is ordinary at p . If $L(f, \chi, r) \neq 0$, then*

$$\dim_F \text{Sel}(K, V_{f, \chi}) = 0.$$

Remark. Let $\epsilon(V_{f, \chi}) = \pm 1$ be the sign of the functional equation of $L(f, \chi, s)$. The non-vanishing of $L(f, \chi, r)$ implies $\epsilon(V_{f, \chi}) = +1$. On the other hand, under our Hypothesis (H), the global sign $\epsilon(V_{f, \chi})$ is completely determined by the local sign at the archimedean place, which in turn depends on the infinity type of χ . More precisely, let $c\mathcal{O}_K$ be the conductor of χ and let $(j, -j)$ be its infinity type, so that for every $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{c\mathcal{O}_K}$ we have

$$\chi(\text{rec}_p(\alpha)) = (\alpha/\bar{\alpha})^j,$$

where $\text{rec}_p : (K \otimes \mathbf{Q}_p)^\times \rightarrow G_K^{ab}$ is the geometrically normalized local reciprocity law map at p . Then one can show that

$$\epsilon(V_{f, \chi}) = +1 \iff j \geq r \text{ or } j \leq -r.$$

In particular, the characters χ for which Theorem A applies are all of infinite order.

Let $\Gamma_K^- := \text{Gal}(K_\infty/K)$ be the Galois group of the anticyclotomic \mathbf{Z}_p -extension of K . Write $c = c_o p^s$ with $p \nmid c_o$. Suppose that $\chi = \psi\phi_0$, where ψ is an anticyclotomic character of infinity type $(r, -r)$ and conductor $c_o\mathcal{O}_K$ and ϕ_0 is a p -adic character of Γ_K^- . The proof of Theorem A rests on the study of a p -adic L -function $\mathcal{L}_{\mathfrak{p}, \psi}(f) \in \bar{\mathbf{Z}}_p[[\Gamma_K^-]]$ defined by the interpolation of the central critical values $L(f, \psi\phi, r)$, as ϕ runs over a Zariski-dense subset of p -adic characters of Γ_K^- . In a slightly different form, this p -adic L -function was introduced in the earlier work of Bertolini, Darmon and Prasanna [BDP13], where they proved a remarkable formula relating the values of $\mathcal{L}_{\mathfrak{p}, \psi}(f)$ at unramified characters *outside* the range of interpolation to the p -adic Abel–Jacobi images of generalized Heegner cycles.

Let $T_f(r)$ be a $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ -stable \mathcal{O}_F -lattice in $V_f(r)$. As a key step toward the proof of Theorem A, we produce Iwasawa cohomology classes

$$\mathbf{z}_f \in H_{\text{Iw}}^1(K_\infty, V_f(r)) := \varprojlim_{K \subset K' \subset K_\infty} H^1(K', T_f(r)) \otimes_{\mathcal{O}_F} F$$

interpolating generalized Heegner cycles over the anticyclotomic tower. Moreover, based on an extension of the calculations of [BDP13] we prove an “explicit reciprocity law”:

$$\langle \mathcal{L}_{\mathfrak{p}, \psi}(\mathbf{z}_f), \omega_f \otimes t^{1-2r} \rangle = (-c_o^{r-1}) \cdot \mathcal{L}_{\mathfrak{p}, \psi}(f)$$

(*cf.* Theorem 5.7) relating the p -adic L -function $\mathcal{L}_{\mathfrak{p}, \psi}(f)$ to the image of the classes \mathbf{z}_f under a variant of Perrin-Riou’s *big logarithm map* $\mathcal{L}_{\mathfrak{p}, \psi}$. The assumption that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K and the p -ordinarity of f are crucially used at this point. The non-ordinary case will be treated in a forthcoming work of S. Kobayashi.

With the result at hand, the proof of Theorem A follows easily. Indeed, by the interpolation property of $\mathcal{L}_{\mathfrak{p}, \psi}(f)$, the nonvanishing of the L -value $L(f, \chi, r)$ in the statement implies the nonvanishing of the value of $\mathcal{L}_{\mathfrak{p}, \psi}(f)$ at $\phi_0 = \psi^{-1}\chi$; by our explicit reciprocity law, this translates into the nonvanishing of the natural image of \mathbf{z}_f in $H^1(K_{\mathfrak{p}}, V_f(r) \otimes \chi^{-1})$. Combined with a suitable extension of Kolyvagin’s method of Euler systems with local conditions at p (see §7), we then use this to establish the vanishing of $\text{Sel}(K, V_{f, \chi})$.

Remark. Under more stringent hypotheses, a version of Theorem A was proven in [Cas14]. The strategy followed in *loc.cit.* is the same as in this paper, but with our classes \mathbf{z}_f replaced by the specializations $\nu_f(\mathfrak{Z}_\infty)$ of Howard’s system of big Heegner points [How07] attached to the Hida family passing through f . In particular, a key ingredient in [Cas14] is the proof of a certain “two-variable” explicit reciprocity law, which specializes to a relation between $\mathcal{L}_{\mathfrak{p}, \psi}(f)$ and the image of $\nu_f(\mathfrak{Z}_\infty)$ under $\mathcal{L}_{\mathfrak{p}, \psi}$. Comparing the resulting two formulas for $\mathcal{L}_{\mathfrak{p}, \psi}(f)$, the equality

$$\nu_f(\mathfrak{Z}_\infty) = \mathbf{z}_f$$

follows easily, yielding an important refinement of the main result of [Cas13].

Next we consider the case

$$\epsilon(V_{f,\chi}) = -1 \iff -r < j < r,$$

so the central L -value $L(f, \chi, r)$ vanishes, and we expect the nonvanishing of Selmer groups. In §4.4, we construct the classes $z_{f,\chi,n} \in H^1(K_n, V_{f,\chi})$ over ring class fields K_n of K . These classes are obtained by taking the χ -component of the p -adic Abel–Jacobi image of generalized Heegner cycles, and they enjoy the properties of an *anticyclotomic Euler system*. The aforementioned extension of Kolyvagin’s methods to the anticyclotomic setting, which follows from a combination of arguments developed by Nekovář [Nek95] and Bertolini–Darmon [BD90], also applies to Hecke characters χ with infinity types $(j, -j)$ with $-r < j < r$, and by these methods we obtain a proof of the following result without the p -ordinary hypothesis on f . Put $z_{f,\chi} := \text{cor}_{K_c/K}(z_{f,\chi,c})$.

Theorem B. *Assume that $\epsilon(V_{f,\chi}) = -1$. If $z_{f,\chi} \neq 0$, then*

$$\text{Sel}(K, V_{f,\chi}) = F \cdot z_{f,\chi}.$$

Remark. The expected extension of the Gross–Zagier formula of [Zha97] to generalized Heegner cycles, together with the conjectural injectivity of the p -adic Abel–Jacobi map [Nek00, Conj. (2.1.2)], would yield a proof of the implication $L'(f, \chi, r) \neq 0 \implies z_{f,\chi} \neq 0$, for any χ as above with $\epsilon(V_{f,\chi}) = -1$. In these favorable circumstances, our Theorem B would imply conjecture (BK) in the “rank one” case.

Appealing to the nonvanishing results of [Hsi14], in Theorem 3.7 we show that the p -adic L -function $\mathcal{L}_{p,\psi}(f) \in \overline{\mathbf{Z}}_p[[\Gamma_K^-]]$ is nonzero, and hence, as χ varies, all but finitely many of the values $L(f, \chi, r)$ appearing in Theorem A are nonzero; our result thus covers most cases of conjecture (BK) for those χ . Moreover, combined with [Nek07, Corollary (5.3.2)], the above generic nonvanishing and our Theorems A and B yield a proof of the “parity conjecture” for $V_{f,\chi}$.

Theorem C. *Suppose that f is ordinary at p . Then*

$$\text{ord}_{s=r} L(f, \chi, s) \equiv \dim_F \text{Sel}(K, V_{f,\chi}) \pmod{2}.$$

That is, the equality predicted by conjecture (BK) holds modulo 2.

Finally, we note that the nontriviality of $\mathcal{L}_{p,\psi}(f)$, combined with our extension of the p -adic Gross–Zagier formula of [BDP13], immediately yields an analogue of Mazur’s nonvanishing conjecture [Maz84] for generalized Heegner cycles and ranks of Selmer groups (see Theorem 6.3).

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Notation and definitions. We let p be a prime and fix embeddings $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ and $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ throughout. Let $\mathbf{A} = \mathbf{A}_{\mathbf{Q}}$ be the adèle ring of \mathbf{Q} . Let $\psi = \prod_q \psi_q : \mathbf{Q} \backslash \mathbf{A} \rightarrow \mathbf{C}^\times$ be the standard additive character with $\psi_\infty(x) = \exp(2\pi i x)$. For each finite prime q , denote by $\text{ord}_q : \mathbf{Q}^\times \rightarrow \mathbf{Z}$ the normalized valuation with $\text{ord}_q(q) = 1$. If N is a positive integer, denote by μ_N the group scheme of N -th roots of unity. We set $\mu_N = \mu_N(\overline{\mathbf{Q}})$ and $\zeta_N := \exp(\frac{2\pi i}{N})$.

If $\phi : \mathbf{Z}_q^\times \rightarrow \mathbf{C}^\times$ is a continuous character of conductor q^n , define the Gauss sum

$$\mathfrak{g}(\phi) = \sum_{u \in (\mathbf{Z}/q^n \mathbf{Z})^\times} \phi(u) \zeta_p^u.$$

By definition, $\mathfrak{g}(\mathbf{1}) = 1$ for the trivial character $\mathbf{1}$. If F is a finite extension of \mathbf{Q}_q and π is an irreducible representation of $\text{GL}_n(F)$ ($n = 1, 2$), we let

$$\varepsilon(s, \pi) := \varepsilon(s, \pi, \psi_q \circ \text{Tr}_{F/\mathbf{Q}_q})$$

be the local ε -factor attached to the additive character $\psi_q \circ \text{Tr}_{F/\mathbf{Q}_q}$ (see [Sch02, Section 1.1] for the definition and basic properties). If $\chi : \mathbf{Q}_q^\times \rightarrow \mathbf{C}^\times$ is a character of conductor q^n , then we have

$$(1.1) \quad \varepsilon(s, \chi) = \mathfrak{g}(\chi^{-1}) \cdot \chi(-q^n) q^{-ns}, \quad \varepsilon(s, \chi) \varepsilon(1-s, \chi^{-1}) = \chi(-1).$$

If L is a number field or a local field, we denote by G_L the absolute Galois group of L and by \mathcal{O}_L the ring of integers of L .

2. MODULAR CURVES AND CM POINTS

2.1. Igusa schemes and modular curves. Let $N \geq 3$ be an integer prime to p , and let $\mathrm{Ig}(N)_{/\mathbf{Z}_{(p)}}$ be the Igusa scheme over $\mathbf{Z}_{(p)}$, which is the moduli space parameterizing elliptic curves with $\Gamma_1(Np^\infty)$ -level structure. More precisely, for each locally noetherian scheme S over $\mathbf{Z}_{(p)}$, $\mathrm{Ig}(N)(S)$ is the set of isomorphism classes of pairs (A, η) consisting of an elliptic curve A over S and a $\Gamma_1(Np^\infty)$ -level structure $\eta = (\eta^{(p)}, \eta_p) : \mu_N \oplus \mu_{p^\infty} \hookrightarrow A[N] \oplus A[p^\infty]$, an immersion as group schemes over S . For a non-negative integer n , let $Y_1(Np^n)_{/\mathbf{Q}}$ be the usual open modular curve of level $\Gamma_1(Np^n)$. Put

$$U_1(Np^n) = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) \mid g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{Np^n} \right\},$$

$$U_0(Np^n) = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{Np^n} \right\}.$$

Letting \mathbb{H} be the complex upper half-plane, the curve $Y_1(Np^n)$ admits the complex uniformization

$$Y_1(Np^n)(\mathbf{C}) = \mathrm{GL}_2(\mathbf{Q})^+ \backslash \mathbb{H} \times \mathrm{GL}_2(\widehat{\mathbf{Q}}) / U_1(Np^n),$$

where $\mathrm{GL}_2(\mathbf{Q})^+$ is the subgroup of $\mathrm{GL}_2(\mathbf{Q})$ with positive determinants. Since the generic fiber $\mathrm{Ig}(N)_{/\mathbf{Q}}$ is given by

$$\mathrm{Ig}(N)_{/\mathbf{Q}} = \varprojlim_n Y_1(Np^n)_{/\mathbf{Q}},$$

this yields a map

$$\mathbb{H} \times \mathrm{GL}_2(\widehat{\mathbf{Q}}) \rightarrow \mathrm{Ig}(N)(\mathbf{C}), \quad x = (\tau_x, g_x) \mapsto [(A_x, \eta_x)].$$

We now give an explicit construction of pairs (A_x, η_x) of complex elliptic curves with $\Gamma_1(Np^\infty)$ -level structure. Let $V = \mathbf{Q}e_1 \oplus \mathbf{Q}e_2$ be the two-dimensional \mathbf{Q} -vector space equipped with the symplectic pairing

$$\langle ae_1 + be_2, ce_1 + de_2 \rangle = ad - bc,$$

and let $\mathrm{GL}_2(\mathbf{Q})$ act on V from the right via

$$(xe_1 + ye_2) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (xa + cy)e_1 + (xb + yd)e_2.$$

For $\tau \in \mathbb{H}$, define the map $p_\tau : V \rightarrow \mathbf{C}$ by

$$p_\tau(ae_1 + be_2) = a\tau + b.$$

Then p_τ induces an isomorphism $V_{\mathbf{R}} := \mathbf{R} \otimes_{\mathbf{Q}} V \simeq \mathbf{C}$. Let \mathbf{L} be the standard lattice $\mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, and for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$ define the \mathbf{Z} -lattice $\mathbf{L}_g \subset V$ by

$$\mathbf{L}_g := (\widehat{\mathbf{Z}}e_1 \oplus \widehat{\mathbf{Z}}e_2)g' \cap V,$$

where g' is the main involution defined by

$$g' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = g^{-1} \det g.$$

The \mathbf{C} -pair (A_x, η_x) attached to $x = (\tau_x, g_x) \in \mathbb{H} \times \mathrm{GL}_2(\widehat{\mathbf{Q}})$ is then given by

$$A_x = \mathbf{C}/L_x, \quad L_x := p_{\tau_x}(\mathbf{L}_{g_x}),$$

and the $\Gamma_1(Np^\infty)$ -level structure $\eta_x = (\eta_x^{(p)}, \eta_{x,p})$ is given by the immersions

$$\eta_x^{(p)} : \mu_N \hookrightarrow N^{-1}\mathbf{Z}/\mathbf{Z} \otimes L_x, \quad \zeta_N^j \mapsto p_{\tau_x}(j/N \otimes e_2 g'_x),$$

$$\eta_{x,p} : \mu_{p^\infty} \hookrightarrow \mathbf{Q}_p/\mathbf{Z}_p \otimes L_x, \quad \zeta_{p^n}^j \mapsto p_{\tau_x}(j/p^n \otimes e_2 g'_x).$$

Here we have used the identification $\mathbf{Q}/\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{L}_{g'_x} = \mathbf{Q}/\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{L}_{g_x}$. The lattice $L_x \subset \mathbf{C}$ is called the *period lattice* of A_x attached to the standard differential form dw , with w the standard complex coordinate of \mathbf{C} .

2.2. Modular forms. We briefly recall the definitions and standard facts about geometric and p -adic modular forms. The basic references are [Kat73], [Kat78] and [Hid04].

Geometric modular forms.

Definition 2.1. Let k be an integer, and let B be a $\mathbf{Z}(p)$ -algebra. A geometric modular form f of weight k on $\mathrm{Ig}(N)$ defined over B is a rule assigning to every triple $(A, \eta, \omega)_{/C}$ over a B -algebra C , consisting of a point $[(A, \eta)] \in \mathrm{Ig}(N)(C)$ and a basis ω of $H^0(A, \omega_{A/C})$ over C , a value $f(A, \eta, \omega) \in C$ such that the following conditions are satisfied:

- (G1) $f(A, \eta, \omega) = f(A', \eta', \omega') \in C$ if $(A, \eta, \omega) \simeq (A', \eta', \omega')$ over C .
- (G2) If $\varphi : C \rightarrow C'$ is any B -algebra homomorphism, then

$$f((A, \eta, \omega) \otimes_C C') = \varphi(f(A, \eta, \omega)).$$

- (G3) $f(A, \eta, t\omega) = t^{-k} f(A, \eta, \omega)$ for all $t \in C^\times$.
- (G4) Letting $(\mathrm{Tate}(q), \eta_{\mathrm{can}}, \omega_{\mathrm{can}})$ be the Tate elliptic curve $\mathbf{G}_m/q^{\mathbf{Z}}$ with the canonical level structure η_{can} and the canonical differential ω_{can} over $\mathbf{Z}((q))$, the value $f(\mathrm{Tate}(q), \eta_{\mathrm{can}}, \omega_{\mathrm{can}})$ lies in $B[[q]]$. We call

$$f(\mathrm{Tate}(q), \eta_{\mathrm{can}}, \omega_{\mathrm{can}}) \in B[[q]]$$

the algebraic Fourier expansion of f .

If f is a geometric modular form of weight k defined over a subring $\mathcal{O} \subset \mathbf{C}$, then f gives rise to a holomorphic function $\mathbf{f} : \mathbb{H} \times \mathrm{GL}_2(\widehat{\mathbf{Q}}) \rightarrow \mathbf{C}$ by the rule

$$\mathbf{f}(x) = f(A_x, \eta_x, 2\pi i dw), \quad x \in \mathbb{H} \times \mathrm{GL}_2(\widehat{\mathbf{Q}}),$$

where w is the standard complex coordinate of $A_x = \mathbf{C}/L_x$. This function \mathbf{f} satisfies the transformation rule:

$$\mathbf{f}(\alpha\tau, \alpha g) = (\det \alpha)^{-\frac{k}{2}} J(\alpha, \tau)^k \cdot \mathbf{f}(\tau, g) \quad (\alpha \in \mathrm{GL}_2(\mathbf{Q})^+),$$

where $J : \mathrm{GL}_2(\mathbf{R})^+ \times \mathbb{H} \rightarrow \mathbf{C}$ is the automorphy factor defined by

$$J(g, \tau) = (\det g)^{-\frac{1}{2}} \cdot (c\tau + d) \quad (g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}).$$

Moreover, the function $\mathbf{f}(-, 1) : \mathbb{H} \rightarrow \mathbf{C}$ is a classical elliptic modular form of weight k with analytic Fourier expansion

$$\mathbf{f}(\tau, 1) = \sum_{n \geq 0} \mathbf{a}_n(f) e^{2\pi i n \tau},$$

and we have the equality between algebraic and analytic Fourier expansions (*cf.* [Kat78, §1.7])

$$f(\mathrm{Tate}(q), \eta_{\mathrm{can}}, \omega_{\mathrm{can}}) = \sum_{n \geq 0} \mathbf{a}_n(f) q^n \in \mathcal{O}[[q]].$$

We say that f is of level $\Gamma_0(Np^n)$ if $\mathbf{f}(\tau, gu) = \mathbf{f}(\tau, g)$ for all $u \in U_0(Np^n)$.

p -adic modular forms. Let R be a p -adic ring, and let $\widehat{\mathrm{Ig}}(N)_{/R} := \varprojlim_m \mathrm{Ig}(N)_{/R/p^m R}$ be the formal completion of $\mathrm{Ig}(N)_{/R}$. Define the space $V_p(N, R)$ of p -adic modular forms of level N by

$$\begin{aligned} V_p(N, R) &:= H^0(\widehat{\mathrm{Ig}}(N)_{/R}, \mathcal{O}_{\widehat{\mathrm{Ig}}(N)_{/R}}) \\ &= \varprojlim_m H^0(\mathrm{Ig}(N), \mathcal{O}_{\mathrm{Ig}(N)} \otimes R/p^m R). \end{aligned}$$

Thus elements in $V_p(N, R)$ are formal functions on the Igusa tower $\mathrm{Ig}(N)$. We say that a p -adic modular form f is of weight $k \in \mathbf{Z}_p$ if for every $u \in \mathbf{Z}_p^\times$, we have

$$(2.1) \quad f(A, \eta) = u^{-k} f(A, \eta^{(p)}, \eta_p u), \quad [(A, \eta)] = [(A, \eta^{(p)}, \eta_p)] \in \widehat{\mathrm{Ig}}(N)_{/R}.$$

If f is a geometric modular form defined over R , then we can associate to f a p -adic modular form \widehat{f} , called the *p -adic avatar* of f , as follows. Let C be a complete local R -algebra, and let (A, η) be an elliptic curve with $\Gamma_1(Np^\infty)$ -level structure. The p^∞ -level structure $\eta_p : \mu_{p^\infty} \hookrightarrow A[p^\infty]$ induces an

isomorphism $\widehat{\eta}_p : \widehat{\mathbf{G}}_m \simeq \widehat{A}$ (here \widehat{A} is the formal group of A), which in turn gives rise to a differential $\widehat{\omega}(\eta_p) \in \text{Lie}(A) = \text{Lie}(\widehat{A}) \simeq \text{Lie}(\widehat{\mathbf{G}}_m) = C$. Then \widehat{f} is the p -adic modular form defined by the rule

$$\widehat{f}(A, \eta) = f(A, \eta, \widehat{\omega}(\eta_p)), \quad [(A, \eta)] \in \widehat{\text{Ig}}(N)/R$$

(cf. [Kat78, (1.10.15)]). It follows from the definition that if f is a geometric modular form of weight k and level $\Gamma_0(Np^n)$, then \widehat{f} is a p -adic modular form of weight k .

2.3. CM points (I). Let K be an imaginary quadratic field of discriminant $-D_K < 0$, and denote by $z \mapsto \bar{z}$ the complex conjugation on \mathbf{C} , which gives the non-trivial automorphism of K . In this section, we assume that $p > 2$ is a prime split in \mathcal{O}_K and write

$$p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}},$$

where \mathfrak{p} is the prime ideal above p determined by the embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$.

Define $\vartheta \in K$ by

$$\vartheta = \frac{D' + \sqrt{-D_K}}{2}, \quad D' = \begin{cases} D_K & \text{if } 2 \nmid D_K, \\ D_K/2 & \text{if } 2 \mid D_K. \end{cases}$$

Then $\mathcal{O}_K = \mathbf{Z} + \mathbf{Z}\vartheta$ and $\vartheta\bar{\vartheta}$ is a local uniformizer of \mathbf{Q}_q for q ramified in K . If M is a positive integer, we decompose $M = M^+M^-$, with the prime factors of M^+ (resp. M^-) split (resp. inert or ramified) in K . For each prime $q = \mathfrak{q}\bar{\mathfrak{q}}$ split in K , we write

$$\mathbf{Z}_q \otimes_{\mathbf{Z}} \mathcal{O}_K = \mathbf{Z}_q e_{\bar{\mathfrak{q}}} \oplus \mathbf{Z}_q e_{\mathfrak{q}},$$

where $e_{\mathfrak{q}}$ and $e_{\bar{\mathfrak{q}}}$ are the idempotents in $\mathbf{Z}_q \otimes_{\mathbf{Z}} \mathcal{O}_K$ corresponding to \mathfrak{q} and $\bar{\mathfrak{q}}$, respectively.

We assume that $N\mathcal{O}_K = \mathfrak{N}\bar{\mathfrak{N}}$ for some ideal \mathfrak{N} of \mathcal{O}_K . Let c be a positive integer, let $\mathcal{O}_c := \mathbf{Z} + c\mathcal{O}_K$ be the order of conductor c , and let K_c be the ring class field of K of conductor c . Let \mathfrak{a} be a fractional ideal of \mathcal{O}_c , and let $a \in \widehat{K}^\times$ with $a\widehat{K} \cap \mathcal{O}_c = \mathfrak{a}$. To the ideal \mathfrak{a} and the finite idele a , we associate a \mathbf{C} -pair $(A_{\mathfrak{a}}, \eta_a)$ of complex CM elliptic curves with $\Gamma_1(Np^\infty)$ -level structure as follows. Define $A_{\mathfrak{a}}$ to be the complex elliptic curve $\mathbf{C}/\mathfrak{a}^{-1}$. For each prime $q \mid pN$, let \mathfrak{q} be the prime of \mathcal{O}_K above q with $\mathfrak{q} \mid \mathfrak{N}\bar{\mathfrak{p}}$, and let $a_{\mathfrak{q}} \in \mathbf{Q}_q$ be the \mathfrak{q} -component of a . We then have $(\mathbf{Z}_q \otimes_{\mathbf{Z}} \mathfrak{a}^{-1}) \cap \mathbf{Q}_q e_{\mathfrak{q}} = \mathbf{Z}_q a_{\mathfrak{q}}^{-1} c e_{\mathfrak{q}}$ and the exact sequence

$$A_{\mathfrak{a}}[q^\infty] = \mu_{q^\infty} \otimes a_{\mathfrak{q}}^{-1} c e_{\mathfrak{q}} \hookrightarrow A_{\mathfrak{a}}[q^\infty] \twoheadrightarrow \mathbf{Q}_q/\mathbf{Z}_q \otimes a_{\bar{\mathfrak{q}}}^{-1} e_{\bar{\mathfrak{q}}},$$

and we define $\eta_a = (\eta_a^{(p)}, \eta_{a,p}) : \mu_N \oplus \mu_{p^\infty} \xrightarrow{\sim} A_{\mathfrak{a}}[\mathfrak{N}] \oplus A_{\mathfrak{a}}[p^\infty] \hookrightarrow A_{\mathfrak{a}}[N] \oplus A_{\mathfrak{a}}[p^\infty]$ to be the embedding determined by the isomorphism $\mu_{q^n} \xrightarrow{\sim} A_{\mathfrak{a}}[q^n]$ sending

$$\zeta_{q^n}^j \mapsto \begin{cases} j/q^n \otimes a_{\bar{\mathfrak{q}}}^{-1} q^{\text{ord}_{\mathfrak{q}}(c)} e_{\mathfrak{q}} & \text{if } q \mid N^+p, \\ j/q^n \otimes a_{\bar{\mathfrak{q}}}^{-1} & \text{if } q \mid N^-. \end{cases}$$

Denote by \mathcal{V} the valuation ring $\iota_p^{-1}(\mathcal{O}_{\mathbf{C}_p}) \cap K^{\text{ab}}$. It follows from the theory of complex multiplication [Shi98, 18.6, 21.1] combined with the criterion of Serre–Tate [ST68] that $(A_{\mathfrak{a}}, \eta_a)$ actually descends to a discrete valuation ring \mathcal{V}_0 inside \mathcal{V} . Thus $[(A_{\mathfrak{a}}, \eta_a)]$ is defined over \mathcal{V}_0 and belongs to $\text{Ig}(N)(\mathcal{V}_0)$. We call $[(A_{\mathfrak{a}}, \eta_a)] \in \text{Ig}(N)(\mathcal{V})$ the CM point attached to (\mathfrak{a}, a) .

If \mathfrak{a} is a prime-to- $\mathfrak{N}\bar{\mathfrak{p}}$ integral ideal of \mathcal{O}_c , we write $(A_{\mathfrak{a}}, \eta_a)$ for the triple $(A_{\mathfrak{a}}, \eta_a)$ with \mathfrak{q} -component $a_{\mathfrak{q}} = 1$ for every $\mathfrak{q} \mid \mathfrak{N}\bar{\mathfrak{p}}$. If $\mathfrak{a} = \mathcal{O}_c$, we write (A_c, η_c) for $(A_{\mathcal{O}_c}, \eta_{\mathcal{O}_c})$. In this case, we see immediately from the construction that $A_{\mathfrak{a}} = A_c/A_c[\mathfrak{a}]$ and the isogeny $\lambda_{\mathfrak{a}} : A_c \rightarrow A_{\mathfrak{a}}$ induced by the quotient map $\mathbf{C}/\mathcal{O}_c \rightarrow \mathbf{C}/\mathfrak{a}^{-1}$ yields $\eta_{\mathfrak{a}} = \lambda_{\mathfrak{a}} \circ \eta_c$.

2.4. CM points (II). We give an explicit complex uniformization of the CM points introduced above. Consider the embedding $K \hookrightarrow M_2(\mathbf{Q})$ given by

$$a\vartheta + b \mapsto \begin{pmatrix} a(\vartheta + \bar{\vartheta}) + b & -a\vartheta\bar{\vartheta} \\ a & b \end{pmatrix}.$$

For each $g \in \text{GL}_2(\widehat{\mathbf{Q}})$, denote by $[(\vartheta, g)]$ the image of (ϑ, g) in $\varprojlim_n Y_1(Np^n)(\mathbf{C}) = \text{Ig}(N)(\mathbf{C})$. Shimura's reciprocity law for CM points (cf. [Hid04, Cor. 4.20]) implies that $[(\vartheta, g)] \in \text{Ig}(N)(K^{\text{ab}})$ and

$$(2.2) \quad \text{rec}_K(a)[(\vartheta, g)] = [(\vartheta, \bar{a}g)] \quad (a \in \widehat{K}^\times),$$

where $\text{rec}_K : K^\times \backslash \widehat{K}^\times \rightarrow \text{Gal}(K^{\text{ab}}/K)$ is the *geometrically normalized* reciprocity law map.

Let $c_o = c_o^+ c_o^-$ be a positive integer prime to p and fix a decomposition $c_o^+ \mathcal{O}_K = \mathfrak{C}\bar{\mathfrak{C}}$. Define $\varsigma^{(\infty)} = (\varsigma_q) \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$ by $\varsigma_q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $q \nmid c_o^+ N^+ p$, and

$$\varsigma_q = (\bar{\vartheta} - \vartheta)^{-1} \begin{pmatrix} \bar{\vartheta} & \vartheta \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(K_q) = \mathrm{GL}_2(\mathbf{Q}_q) \text{ if } q = \mathfrak{q}\bar{\mathfrak{q}} \text{ with } \mathfrak{q} \mid \mathfrak{C}\mathfrak{N}\mathfrak{p}.$$

Let $c = c_o p^n$ with $n \geq 0$. We define $\gamma_c = \prod_q \gamma_{c,q} \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$ by $\gamma_{c,q} = 1$ if $q \nmid c N p$,

$$\gamma_{c,q} = \begin{pmatrix} q^{\mathrm{ord}_q(c)} & 1 \\ 0 & 1 \end{pmatrix} \text{ if } q = \mathfrak{q}\bar{\mathfrak{q}} \text{ with } \mathfrak{q} \mid \mathfrak{C}\mathfrak{N}\mathfrak{p},$$

$$\gamma_{c,q} = \begin{pmatrix} 1 & 0 \\ 0 & q^{\mathrm{ord}_q(c) - \mathrm{ord}_q(N)} \end{pmatrix} \text{ if } q \mid c^- N^-.$$

Let $\xi_c := \varsigma^{(\infty)} \gamma_c \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$ be the product. An elementary computation shows that $\mathcal{O}_c = \mathfrak{p}_\vartheta(\mathbf{L}_{\xi_c})$ and that for $q = \mathfrak{q}\bar{\mathfrak{q}}$ with $\mathfrak{q} \mid \mathfrak{C}\mathfrak{N}\mathfrak{p}$, we have

$$(2.3) \quad \varsigma_q \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = (ae_{\bar{\mathfrak{q}}} + be_{\mathfrak{q}}) \varsigma_q \quad (a, b \in \mathbf{Q}_q^\times),$$

and

$$(2.4) \quad \xi'_{c,q} : \mathbf{Z}_q e_1 \oplus \mathbf{Z}_q e_2 \simeq \mathbf{Z}_q \otimes_{\mathbf{Z}} \mathcal{O}_c, \quad \mathfrak{p}_\vartheta(e_2 \xi'_{c,q}) = q^{\mathrm{ord}_q(c)} e_{\mathfrak{q}},$$

so we have $[(\vartheta, \xi_c)] = [(A_c, \eta_c)]$. Define

$$x_c := [(A_c, \eta_c)] = [(\vartheta, \xi_c)] \in \mathrm{Ig}(N)(\mathbf{C}).$$

In general, if $a \in \widehat{K}^{(cp)\times}$ and $\mathfrak{a} = a\widehat{\mathcal{O}}_c \cap K$ is a fractional ideal of \mathcal{O}_c , we let

$$\sigma_{\mathfrak{a}} := \mathrm{rec}_K(a^{-1})|_{K_c(\mathfrak{p}^\infty)} \in \mathrm{Gal}(K_c(\mathfrak{p}^\infty)/K),$$

where $K_c(\mathfrak{p}^\infty)$ is the compositum of K_c and the ray class field of K of conductor \mathfrak{p}^∞ . Thus $\sigma_{\mathfrak{a}}$ is the image of \mathfrak{a} under the classical Artin map. We have

$$(2.5) \quad x_{\mathfrak{a}} := [(A_{\mathfrak{a}}, \eta_{\mathfrak{a}})] = [(\vartheta, \bar{a}^{-1} \xi_c)] = x_c^{\sigma_{\mathfrak{a}}} \in \mathrm{Ig}(N)(K_c(\mathfrak{p}^\infty)).$$

Here the first equality can be verified by noting that the main involution induces the complex conjugation on \mathbf{A}_K^\times and using (2.3), and the second equality follows from Shimura's reciprocity law for CM points (2.2).

2.5. CM periods. Let $\widehat{\mathbf{Q}}_p^{\mathrm{ur}}$ be the p -adic completion of the maximal unramified extension $\mathbf{Q}_p^{\mathrm{ur}}$ of \mathbf{Q}_p , and let \mathcal{W} be the ring of integers of $\widehat{\mathbf{Q}}_p^{\mathrm{ur}}$. If \mathfrak{a} is a prime-to- $\mathfrak{p}\mathfrak{N}$ fractional ideal of \mathcal{O}_c with $p \nmid c$, then $(A_{\mathfrak{a}}, \eta_{\mathfrak{a}})$ has a model defined over $\mathcal{V}^{\mathrm{ur}} := \mathcal{W} \cap K^{\mathrm{ab}}$. In the sequel, we shall still denote this model by $(A_{\mathfrak{a}}, \eta_{\mathfrak{a}})$ and simply write A for $A_{\mathcal{O}_K}$.

Fix a Néron differential ω_A of A over $\mathcal{V}^{\mathrm{ur}}$. There exists a unique prime-to- p isogeny $\lambda_{\mathfrak{a}} : A_{\mathfrak{a}} \rightarrow A$ inducing the identity map on both the complex Lie algebras $\mathbf{C} = \mathrm{Lie} A_{\mathfrak{a}}(\mathbf{C}) \rightarrow \mathbf{C} = \mathrm{Lie} A(\mathbf{C})$ via the complex uniformizations and on the \mathfrak{p} -divisible groups $\boldsymbol{\mu}_{p^\infty} = A_{\mathfrak{a}}[\mathfrak{p}^\infty] \rightarrow \boldsymbol{\mu}_{p^\infty} = A[\mathfrak{p}^\infty]$ via the level structures at p . Letting $\omega_{A_{\mathfrak{a}}} := \lambda_{\mathfrak{a}}^* \omega_A$ be the pull-back of ω_A , we see that there exists a pair $(\Omega_K, \Omega_p) \in \mathbf{C}^\times \times \mathcal{W}^\times$ such that

$$\Omega_K \cdot 2\pi i dw = \Omega_p \cdot \widehat{\omega}(\eta_{\mathfrak{a},p}) = \omega_{A_{\mathfrak{a}}},$$

where w is the standard complex coordinate of $\mathbf{C}/\mathfrak{a}^{-1} = A_{\mathfrak{a}}(\mathbf{C})$. The pair (Ω_K, Ω_p) are called the complex and p -adic periods of K . Note that the ratio Ω_K/Ω_p does not depend on the choice of Néron differential ω_A .

3. ANTICYCLOTOMIC p -ADIC L -FUNCTIONS

In this section, we review the anticyclotomic p -adic L -functions that were originally constructed in [Bra11], [BDP13] and [Hsi14] from various points of view. Our purpose is to extend their interpolation formulae to include p -ramified characters and to prove the nonvanishing of these p -adic L -functions, so we find it more convenient to adopt the approach of [Bra11], based on the use of Serre–Tate coordinates.

3.1. t -expansion of p -adic modular forms. Let $\mathbf{x} = [(A_0, \eta)]$ be a point in the Igusa tower $\mathrm{Ig}(N)(\overline{\mathbb{F}}_p)$ and let $\widehat{S}_{\mathbf{x}} \hookrightarrow \mathrm{Ig}(N)_{/\mathcal{W}}$ be the local deformation space of \mathbf{x} over \mathcal{W} . The p^∞ -level structure η_p determines a point $P_{\mathbf{x}} \in T_p(A_0^\dagger)$, where A_0^\dagger is the dual abelian variety of A_0 and $T_p(A_0^\dagger) = \varprojlim_n A_0^\dagger[p^n](\overline{\mathbb{F}}_p)$ is the p -adic Tate module of A_0^\dagger . Let $\lambda_{\mathrm{can}}: A_0 \simeq A_0^\dagger$ be the canonical principal polarization.

For each deformation \mathcal{A}/R over a local Artinian ring (R, \mathfrak{m}_R) , let $q_{\mathcal{A}}: T_p(A_0) \times T_p(A_0^\dagger) \rightarrow 1 + \mathfrak{m}_R$ be the Serre–Tate bilinear form attached to \mathcal{A}/R (see [Kat81]). The *canonical Serre–Tate coordinate* $t: \widehat{S}_{\mathbf{x}} \rightarrow \widehat{\mathbf{G}}_m$ is defined by

$$t(\mathcal{A}) := q_{\mathcal{A}}(\lambda_{\mathrm{can}}^{-1}(P_{\mathbf{x}}), P_{\mathbf{x}})$$

and yields an identification $\mathcal{O}_{\widehat{S}_{\mathbf{x}}} = \mathcal{W}[[t-1]]$.

Let $f \in V_p(N, \mathcal{W})$ be a p -adic modular form over \mathcal{W} . The t -expansion $f(t)$ of f around \mathbf{x} is defined by

$$f(t) = f|_{\widehat{S}_{\mathbf{x}}} \in \mathcal{W}[[t-1]],$$

and we let df be the p -adic measure on \mathbf{Z}_p such that

$$\int_{\mathbf{Z}_p} t^x df(x) = f(t).$$

Moreover, if $\phi: \mathbf{Z}_p \rightarrow \mathcal{O}_{\mathbf{C}_p}$ is any continuous function, we define $f \otimes \phi(t) \in \mathcal{O}_{\mathbf{C}_p}[[t-1]]$ by

$$f \otimes \phi(t) = \int_{\mathbf{Z}_p} \phi(x) t^x df = \sum_{n \geq 0} \int_{\mathbf{Z}_p} \phi(x) \binom{x}{n} df(x) \cdot (t-1)^n.$$

Lemma 3.1. *If $\phi: \mathbf{Z}_p/p^n \mathbf{Z}_p \rightarrow \mathcal{O}_{\mathbf{C}_p}$, then*

$$f \otimes \phi(t) = p^{-n} \sum_{u \in \mathbf{Z}/p^n \mathbf{Z}} \sum_{\zeta \in \mu_{p^n}} \zeta^{-u} \phi(u) f(t\zeta).$$

If $\phi: \mathbf{Z}_p \rightarrow \mathbf{Z}_p$ is $z \mapsto z^k$, then

$$f \otimes \phi(t) = \left[t \frac{d}{dt} \right]^k (f).$$

PROOF. This is well-known. For example, see [Hid93, §3.5 (5)]. \square

3.2. Serre–Tate coordinates of CM points. Suppose that c is a positive integer with $p \nmid c$. Let \mathfrak{a} be a prime-to- $c\mathfrak{N}p$ integral ideal of \mathcal{O}_c , and let $a \in \widehat{K}^{(c\mathfrak{N}p)^\times}$ be such that $\mathfrak{a} = a\widehat{\mathcal{O}}_c \cap K$. Define $N(\mathfrak{a})$ by

$$\begin{aligned} N(\mathfrak{a}) &:= \text{degree of the } \mathbf{Q}\text{-isogeny } \mathbf{C}/\mathcal{O}_K \rightarrow \mathbf{C}/\mathfrak{a}^{-1} \\ &= c^{-1} \#(\mathcal{O}_c/\mathfrak{a}) = c^{-1} |a|_{\mathbf{A}_K}^{-1}. \end{aligned}$$

Let $x_{\mathfrak{a}} = [(A_{\mathfrak{a}}, \eta_{\mathfrak{a}})] \in \mathrm{Ig}(N)(\mathcal{V})$ be the CM point attached to \mathfrak{a} and let t be the canonical Serre–Tate coordinate of $\mathbf{x}_{\mathfrak{a}} := x_{\mathfrak{a}} \otimes_{\mathcal{V}} \overline{\mathbb{F}}_p$. We will use the following notation: for each $z \in \mathbf{Q}_p$, set

$$(3.1) \quad \mathbf{n}(z) := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_p) \subset \mathrm{GL}_2(\widehat{\mathbf{Q}}).$$

Put

$$x_{\mathfrak{a}} * \mathbf{n}(z) := [(\vartheta, \bar{a}^{-1} \xi_c \mathbf{n}(z))] \in \mathrm{Ig}(N)(\mathcal{V}).$$

Lemma 3.2. *Let $u \in \mathbf{Z}_p$. We have $(x_{\mathfrak{a}} * \mathbf{n}(up^{-n})) \otimes \overline{\mathbb{F}}_p = \mathbf{x}_{\mathfrak{a}}$, and*

$$t(x_{\mathfrak{a}} * \mathbf{n}(up^{-n})) = \zeta_{p^n}^{-uN(\mathfrak{a})^{-1} \sqrt{-D_K}^{-1}}.$$

PROOF. Let $(\mathcal{A}, \eta_{\mathcal{A}})_{/\mathcal{V}_0}$ be a model of the CM elliptic curve $(A_{\mathfrak{a}}, \eta_{\mathfrak{a}})$ over a discrete valuation ring $\mathcal{V}_0 \subset \mathcal{V}$. Let $\overline{\mathcal{A}} = \mathcal{A} \otimes \overline{\mathbb{F}}_p$. Recall that $e_{\mathfrak{p}}$ and $e_{\overline{\mathfrak{p}}}$ are idempotents in $K \otimes_{\mathbf{Q}} K \hookrightarrow \mathbf{Q}_p \otimes_{\mathbf{Q}} K$ corresponding to \mathfrak{p} and $\overline{\mathfrak{p}}$ respectively. In fact

$$e_{\mathfrak{p}} = \frac{\vartheta \otimes 1 - 1 \otimes \overline{\vartheta}}{\vartheta - \overline{\vartheta}}, \quad e_{\overline{\mathfrak{p}}} = \frac{\vartheta \otimes 1 - 1 \otimes \vartheta}{\vartheta - \overline{\vartheta}},$$

so we have

$$(3.2) \quad \mathrm{p}_{\vartheta}(e_{1\zeta'_p}) = e_{\overline{\mathfrak{p}}}, \quad \mathrm{p}_{\vartheta}(e_{2\zeta'_p}) = e_{\mathfrak{p}}.$$

The complex uniformization $\alpha: (\mathbf{C}/\mathfrak{a}^{-1}, \eta_{\mathfrak{a}}) \simeq (\mathcal{A}, \eta_{\mathcal{A}})_{/\mathbf{C}}$ yields the identifications

$$\alpha: \mathbf{Z}_p e_{\bar{p}} \oplus \mathbf{Z}_p e_p = \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathfrak{a}^{-1} \simeq T_p(\mathcal{A})$$

and

$$\alpha: \boldsymbol{\mu}_{p^\infty} \otimes \mathbf{Z}_p e_p \simeq \widehat{\mathcal{A}}[p^\infty], \quad \alpha: \mathbf{Z}_p e_{\bar{p}} \simeq T_p(\overline{\mathcal{A}}).$$

Here $\widehat{\mathcal{A}}$ is the formal group attached to $\mathcal{A}/\mathcal{V}_0$. Let $\mathbf{e}_A: T_p(\mathcal{A}) \times T_p(\mathcal{A}^t) \rightarrow \mathbf{Z}_p$ be the Weil pairing. Let $\lambda_\vartheta: \mathcal{A} \rightarrow \mathcal{A}^t$ be the prime-to- p polarization induced by the Riemann form $\langle z, w \rangle_\vartheta = (\operatorname{Im} \vartheta)^{-1} \operatorname{Im}(z\bar{w})$ on $\mathbf{C}/\mathfrak{a}^{-1}$. The complex uniformization $\alpha^t = \alpha \circ \lambda_\vartheta: \mathbf{C} \rightarrow \mathcal{A}^t(\mathbf{C})$ induces $\alpha^t: \mathbf{C}/\mathbf{N}(\mathfrak{a})\mathfrak{a}^{-1} \simeq \mathcal{A}^t(\mathbf{C})$ and $\alpha^t: \mathbf{Z}_p e_{\bar{p}} \oplus \mathbf{Z}_p e_p \simeq T_p(\mathcal{A}^t)$ with $\alpha^t: \mathbf{Z}_p e_{\bar{p}} \simeq T_p(\overline{\mathcal{A}^t})$. By [Mum08, Theorem 1, page 237] and (3.2), we have

$$\mathbf{e}_A(a\alpha(e_{\bar{p}}) + b\alpha(e_p), c\alpha^t(e_{\bar{p}}) + d\alpha^t(e_p)) = -(ad - bc)(\vartheta - \bar{\vartheta})^{-1} = -(ad - bc)\sqrt{-D_K}^{-1}.$$

(Note the sign -1 .) The canonical polarization $\lambda_{\text{can}}: \mathcal{A} \simeq \mathcal{A}^t$ is given by $\alpha(z) \mapsto \alpha^t(\mathbf{N}(\mathfrak{a})z)$.

Let y be the complex point $(\vartheta, \bar{a}^{-1}\xi_{\mathbf{C}}\mathbf{n}(up^{-n}))$ and let $(\mathcal{B}, \eta_{\mathcal{B}})_{/\mathcal{V}_0}$ be a model of (A_y, η_y) over \mathcal{V}_0 (enlarging \mathcal{V}_0 if necessary), so $[(\mathcal{B}, \eta_{\mathcal{B}})] = [y]$. The period lattice L_y of $\mathcal{B}(\mathbf{C})$ is given by

$$L_y = \mathfrak{p}_\vartheta(L'), \quad L' = (\widehat{\mathbf{Z}}e_1 \oplus \widehat{\mathbf{Z}}e_2)\mathbf{n}(-up^{-n})\xi'_c a^{-1} \cap V.$$

By a direct computation and (3.2), we find that

$$\begin{aligned} \mathbf{Z}_p \otimes_{\mathbf{Z}} L_y &= \mathfrak{p}_\vartheta(\mathbf{Z}_p e_1 \oplus \mathbf{Z}_p e_2 \begin{pmatrix} 1 & -up^{-n} \\ 0 & 1 \end{pmatrix} \gamma'_{c,p} \zeta'_p) \\ &= \mathfrak{p}_\vartheta((\mathbf{Z}_p(e_1 - up^{-n}e_2) \oplus \mathbf{Z}_p e_2)\zeta'_p) \\ &= \mathbf{Z}_p(e_{\bar{p}} - up^{-n}e_p) \oplus \mathbf{Z}_p e_p, \end{aligned}$$

so the complex uniformization $\beta: (\mathbf{C}/L_y, \eta_y) \simeq (\mathcal{B}, \eta_{\mathcal{B}})_{/\mathbf{C}}$ induces the identification

$$\beta: \mathbf{Z}_p(e_{\bar{p}} - \frac{u}{p^n}e_p) \oplus \mathbf{Z}_p e_{\bar{p}} \simeq T_p(\mathcal{B}).$$

With the above preparations, we see that over \mathbf{C} there are natural isomorphisms

$$\mathcal{A}/\mathcal{A}[\mathfrak{p}] \simeq \mathcal{B}/\mathcal{B}[\mathfrak{p}] \simeq \mathbf{C}/\mathfrak{a}^{-1}\mathfrak{p}_c^{-1} \quad (\mathfrak{p}_c = \mathfrak{p} \cap \mathcal{O}_c)$$

induced by the inclusions of L_y and \mathfrak{a}^{-1} in $\mathfrak{a}^{-1}\mathfrak{p}_c^{-1}$, which extend uniquely to an isomorphism $\mathcal{A}/\mathcal{A}[\mathfrak{p}] \simeq \mathcal{B}/\mathcal{B}[\mathfrak{p}]$ over \mathcal{V}_0 ([FC90, Prop. 2.7]). By construction, $\mathcal{A}[\mathfrak{p}]$ and $\mathcal{B}[\mathfrak{p}]$ are connected components of $\mathcal{A}[p]$ and $\mathcal{B}[p]$, so we get the isomorphism $(\overline{\mathcal{A}}, \eta_{\overline{\mathcal{A}}})^{\sigma_p} \simeq (\overline{\mathcal{B}}, \eta_{\overline{\mathcal{B}}})^{\sigma_p}$, where $(-)^{\sigma_p}$ denotes the conjugate of the p -th power Frobenius σ_p , and hence $(\overline{\mathcal{A}}, \eta_{\overline{\mathcal{A}}}) \simeq (\overline{\mathcal{B}}, \eta_{\overline{\mathcal{B}}})$. This shows that $[y] \otimes \overline{\mathbb{F}}_p = \mathbf{x}_a$. To compute the value $t(\mathcal{B})$, we note that $P_{\mathbf{x}_a} = \alpha^t(e_{\bar{p}})$ and that the Weil pairing of $\overline{\mathcal{A}}$ induces $E_{\mathcal{B}}: \widehat{\mathcal{B}}[p^\infty] \times T_p(\overline{\mathcal{A}^t}) \rightarrow \widehat{\mathbf{G}}_m$ so that $E_{\mathcal{B}}(\beta(p^{-n}e_p), \alpha^t(e_{\bar{p}})) = \zeta_{p^n}^{\sqrt{-D_K}^{-1}}$ ([Kat81, page 150]). For a sufficiently large integer m , we have

$$\begin{aligned} t(\mathcal{B}) &= q_{\mathcal{B}}(\lambda_{\text{can}}^{-1}(P_{\mathbf{x}_a}), P_{\mathbf{x}_a}) = q_{\mathcal{B}}(\mathbf{N}(\mathfrak{a})^{-1}\alpha(e_{\bar{p}}), \alpha^t(e_{\bar{p}})) \\ &= E_{\mathcal{B}}("p^m" \alpha(p^{-m}e_{\bar{p}}), \alpha^t(e_{\bar{p}}))^{\mathbf{N}(\mathfrak{a})^{-1}}, \end{aligned}$$

where " p^m " : $\overline{\mathcal{A}}[p^m](\overline{\mathbb{F}}_p) \rightarrow \widehat{\mathcal{A}}$ is the Drinfeld lift map. To compute the lift, from the diagram

$$(3.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \boldsymbol{\mu}_{p^\infty} \otimes \mathbf{Z}_p e_p & \longrightarrow & \mathbf{Z}_p \otimes K/L_y & \longrightarrow & \mathbf{Q}_p/\mathbf{Z}_p \otimes \mathbf{Z}_p e_p \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ 0 & \longrightarrow & \widehat{\mathcal{B}}[p^\infty] & \longrightarrow & \mathcal{B}[p^\infty] & \longrightarrow & \mathbf{Q}_p/\mathbf{Z}_p \otimes T_p(\overline{\mathcal{A}}) \longrightarrow 0 \end{array}$$

we can see that the p^m -torsion point $\alpha(p^{-m}e_{\bar{p}}) \in p^{-m}\mathbf{Z}_p/\mathbf{Z}_p \otimes T_p(\overline{\mathcal{A}}) = \mathcal{A}[p^m](\overline{\mathbb{F}}_p)$ has a lift $\beta(p^{-m}e_{\bar{p}} - up^{-m-n}e_p) \in \mathcal{B}[p^\infty]$, so the Drinfeld lift " p^m " $\alpha(p^{-m}e_{\bar{p}})$ is given by $\beta(-up^{-n}e_p) \in \widehat{\mathcal{B}}[p^\infty]$. Hence, we obtain

$$t(\mathcal{B}) = E_{\mathcal{B}}(\beta(p^{-n}e_p), \alpha^t(e_{\bar{p}}))^{-u\mathbf{N}(\mathfrak{a})^{-1}} = \zeta_{p^n}^{-u\mathbf{N}(\mathfrak{a})^{-1}\sqrt{-D_K}^{-1}}. \quad \square$$

Proposition 3.3. *Let $f \in V_p(N, \mathcal{W})$ be a p -adic modular form with t -expansion $f(t)$ around $x_{\mathfrak{a}} \otimes \bar{\mathbb{F}}_p$. Put*

$$f_{\mathfrak{a}}(t) := f(t^{N(\mathfrak{a})^{-1}\sqrt{-D_K^{-1}}}).$$

If n is a positive integer and $\phi: (\mathbf{Z}/p^n\mathbf{Z})^\times \rightarrow \mathcal{O}_{\mathbf{C}_p}^\times$ is a primitive Dirichlet character, then

$$f_{\mathfrak{a}} \otimes \phi(x_{\mathfrak{a}}) = p^{-n} \mathfrak{g}(\phi) \sum_{u \in (\mathbf{Z}/p^n\mathbf{Z})^\times} \phi^{-1}(u) \cdot f(x_{\mathfrak{a}} * \mathbf{n}(up^{-n})).$$

PROOF. This follows from Lemma 3.1 combined with Lemma 3.2. \square

3.3. Anticyclotomic p -adic L -functions.

Hecke characters and p -adic Galois characters. A Hecke character $\chi: K^\times \backslash \mathbf{A}_K^\times \rightarrow \mathbf{C}^\times$ is called a Hecke character of infinity type (m, n) if $\chi_\infty(z) = z^m \bar{z}^n$, and is called anticyclotomic if χ is trivial on \mathbf{A}^\times .

For each prime \mathfrak{q} of \mathcal{O}_K , we let $\chi_{\mathfrak{q}}: K_{\mathfrak{q}} \rightarrow \mathbf{C}^\times$ denote the \mathfrak{q} -component of χ , and if χ has conductor \mathfrak{c} and \mathfrak{a} is any fractional ideal prime to \mathfrak{c} , we write $\chi(\mathfrak{a})$ for $\chi(a)$, where a is an idele with $a\widehat{\mathcal{O}}_K \cap K = \mathfrak{a}$ and $a_{\mathfrak{q}} = 1$ for all $\mathfrak{q} \mid \mathfrak{c}$.

Definition 3.4. The p -adic avatar $\widehat{\chi}: K^\times \backslash \widehat{K}^\times \rightarrow \mathbf{C}_p^\times$ of a Hecke character χ of infinity type (m, n) is defined by

$$\widehat{\chi}(z) = i_p \circ i_\infty^{-1}(\chi(z)) z_{\mathfrak{p}}^m z_{\bar{\mathfrak{p}}}^n$$

for $z \in \widehat{K}^\times$.

Via the reciprocity law map rec_K , each p -adic Galois character $\rho: G_K := \text{Gal}(\bar{\mathbf{Q}}/K) \rightarrow \mathbf{C}_p^\times$ will be implicitly regarded as a p -adic character $\rho: K^\times \backslash \widehat{K}^\times \rightarrow \mathbf{C}_p^\times$. We say that a p -adic Galois character ρ is *locally algebraic* if $\rho = \widehat{\rho}_{\mathbf{A}}$ is the p -adic avatar of some Hecke character $\rho_{\mathbf{A}}$. A locally algebraic character ρ is called of infinity type (m, n) if the associated Hecke character $\rho_{\mathbf{A}}$ is of infinity type (m, n) , and the conductor of ρ is defined to be the conductor of $\rho_{\mathbf{A}}$. Note that if $\rho_{\mathbf{A}}$ is unramified at \mathfrak{p} and of infinity type (m, n) , then ρ is crystalline at \mathfrak{p} as $\rho|_{G_{K_{\mathfrak{p}}}}$ is an unramified twist of the m -th power of the p -adic cyclotomic character.

Modular forms. In the remainder of this article, we fix $f \in S_{2r}^{\text{new}}(\Gamma_0(N))$ to be an elliptic newform (i.e. normalized eigenform for all Hecke operators) of weight $2r$ and level $N_f \mid N$. Let

$$f(q) = \sum_{n>0} \mathbf{a}_n(f) q^n$$

be the q -expansion of f at the infinity cusp. Let F be a finite extension of \mathbf{Q}_p containing the Hecke field of f , i.e. the field generated by $\{\mathbf{a}_n(f)\}_n$ over \mathbf{Q} . Let φ_f be the automorphic form attached to f , i.e. $\varphi_f: \mathbf{A}^\times \text{GL}_2(\mathbf{Q}) \backslash \text{GL}_2(\mathbf{A}) \rightarrow \mathbf{C}$ is the function satisfying

$$\varphi_f(g_\infty u) = J(g_\infty, i)^{-2r} f(g_\infty i), \quad \text{for } g_\infty \in \text{GL}_2(\mathbf{R})^+, u \in U_1(N_f),$$

and let $\pi = \otimes' \pi_{\mathfrak{q}}$ be the irreducible cuspidal automorphic representation on $\text{GL}_2(\mathbf{A})$ generated by φ_f . Note that π has trivial central character. Define the automorphic form φ_f^{\flat} by

$$(3.4) \quad \varphi_f^{\flat}(g) = \varphi_f(g) - \mathbf{a}_{\mathfrak{p}}(f) p^{-r} \varphi_f(g\gamma_{\mathfrak{p}}) + p^{-1} \varphi_f(g\gamma_{\mathfrak{p}}^2),$$

where $\gamma_{\mathfrak{p}} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in \text{GL}_2(\mathbf{Q}_{\mathfrak{p}}) \hookrightarrow \text{GL}_2(\widehat{\mathbf{Q}})$. Define the complex function $\mathbf{f}^{\flat}: \mathbb{H} \times \text{GL}_2(\widehat{\mathbf{Q}}) \rightarrow \mathbf{C}$ by

$$(3.5) \quad \mathbf{f}^{\flat}(\tau, g_f) = \varphi_f^{\flat}((g_\infty, g_f)) J(g_\infty, i)^{2r} |\det g_f|_{\mathbf{A}_f}^r, \\ (g_\infty \in \text{GL}_2(\mathbf{R})^+, g_\infty i = \tau).$$

Then there is a unique geometric modular form f^{\flat} of weight $2r$ and level $\Gamma_0(Np^2)$ defined over \mathcal{O}_F such that:

- $f^{\flat}(A_x, \eta_x, 2\pi i dw) = \mathbf{f}^{\flat}(x)$ for $x \in \mathbb{H} \times \text{GL}_2(\widehat{\mathbf{Q}})$,
- with Fourier expansion

$$f^{\flat}(\text{Tate}(q), \eta_{\text{can}}, \omega_{\text{can}}) = \sum_{p \nmid n} \mathbf{a}_n(f) q^n.$$

The p -adic avatar $\widehat{f}^b \in V_p(N, \mathcal{O}_F)$ of f^b introduced in §2.2 is a p -adic modular form of weight $2r$.

Explicit Waldspurger formula. We recall a result on the explicit calculation of toric period integrals in [Hsi14]. Let $c = c_o p^n$ with $p \nmid c_o$ and $n \geq 0$. Put

$$\text{Pic } \mathcal{O}_c := K^\times \backslash \widehat{K}^\times / \widehat{\mathcal{O}}_c^\times.$$

If $a \in \widehat{K}^{(c\mathfrak{p}^n)^\times}$ and $\mathfrak{a} = a\widehat{K} \cap \mathcal{O}_c$ is the corresponding fractional ideal of \mathcal{O}_c , we shall write $[a] = [\mathfrak{a}]$ for its class in $\text{Pic } \mathcal{O}_c$. Let $\chi: K^\times \backslash \mathbf{A}_K^\times / \widehat{\mathcal{O}}_c^\times \rightarrow \mathbf{C}^\times$ be an anticyclotomic Hecke character, and set

$$A(\chi) = \{\text{primes } q \mid D_K \text{ such that } \chi_q \text{ is unramified and } q \mid N_f.\}$$

We assume the following Heegner hypothesis:

(Heeg') N_f^- is a square-free product of primes ramified in K ,

and that (f, χ) satisfies the condition

(ST) $\mathfrak{a}_q(f)\chi(\mathfrak{q}) = -1$ for every $q \in A(\chi)$ ($q\mathcal{O}_K = \mathfrak{q}^2$).

Definition 3.5. Define the χ -toric period by

$$\begin{aligned} P_\chi(f^b) &:= \sum_{[a] \in \text{Pic } \mathcal{O}_c} \varphi_f^b((\varsigma_\infty, a\xi_c))\chi(a) \quad (\varsigma_\infty := \begin{pmatrix} \text{Im } \vartheta & \text{Re } \vartheta \\ 0 & 1 \end{pmatrix}) \\ &= (c \text{Im } \vartheta)^r \cdot \sum_{[a] \in \text{Pic } \mathcal{O}_c} \mathbf{f}^b(\vartheta, a\xi_c) \cdot \chi \cdot |\cdot|_{\mathbf{A}_K}^{-r}(a) \quad (\text{by (3.5)}). \end{aligned}$$

Let π_K be the automorphic representation of $\text{GL}_2(\mathbf{A}_K)$ obtained by the base change of π to K , and let $L(s, \pi_K \otimes \chi)$ be the automorphic L -function on $\text{GL}_2(\mathbf{A}_K)$ attached to π_K twisted by $\chi \circ \det^1$. If χ has infinity type $(r+m, -r-m)$ with $m \geq 0$, define the algebraic central value $L^{\text{alg}}(\frac{1}{2}, \pi_K \otimes \chi)$ by

$$L^{\text{alg}}\left(\frac{1}{2}, \pi_K \otimes \chi\right) = \frac{\Gamma(2r+m)\Gamma(m+1)}{(4\pi)^{2r+2m+1}(\text{Im } \vartheta)^{2r+2m}} \cdot \frac{L(\frac{1}{2}, \pi_K \otimes \chi)}{\Omega_K^{4(r+m)}},$$

and the p -adic multiplier $e_p(f, \chi)$ by

$$e_p(f, \chi) = \begin{cases} (1 - \mathfrak{a}_p(f)p^{-r}\chi_{\overline{p}}(p) + \chi_{\overline{p}}(p^2)p^{-1})^2 & \text{if } p \nmid c, \\ \varepsilon(\frac{1}{2}, \chi_p)^{-2} & \text{if } p \mid c. \end{cases}$$

Proposition 3.6. *Suppose that*

- (a) χ has infinity type $(r, -r)$ and $(c, N^+) = 1$,
- (b) (Heeg') and (ST) hold for (f, χ) .
- (c) The conductor of χ is $c\mathcal{O}_K$.

Then we have

$$\left(\frac{P_\chi(f^b)}{\Omega_K^{2r}}\right)^2 = L^{\text{alg}}\left(\frac{1}{2}, \pi_K \otimes \chi\right) \cdot e_p(f, \chi) \cdot \varepsilon\left(\frac{1}{2}, \chi_p\right)^2 \cdot 2^{\#A(\chi)+3} u_K^2 \sqrt{D_K} \cdot c(\text{Im } \vartheta)^{2r} \cdot \chi^{-1}(\mathfrak{N})\varepsilon(f),$$

where $u_K := \#(\mathcal{O}_K^\times)/2$ and $\varepsilon(f) := \prod_q \varepsilon(\frac{1}{2}, \pi_q)$ is the global root number of f .

PROOF. We will follow the notations in [Hsi14]. Let $W_\chi^b = W_{\chi, p}^b \prod_{v \neq p} W_{\chi, v}^b: \text{GL}_2(\mathbf{A}) \rightarrow \mathbf{C}$ be the Whittaker function defined in [Hsi14, §3.6], and let $\varphi_\chi^b: \text{GL}_2(\mathbf{Q}) \backslash \text{GL}_2(\mathbf{A}) \rightarrow \mathbf{C}$ be the associated automorphic form given by

$$\varphi_\chi(g) = \sum_{\alpha \in \mathbf{Q}} W_\chi^b\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Let $\varsigma = (\varsigma_\infty, \varsigma^{(\infty)}) \in \text{GL}_2(\mathbf{A})$ with $\varsigma^{(\infty)}$ as in §2.4, and define the toric period integral

$$P_\chi(\pi(\varsigma)\varphi_\chi) = \int_{K^\times \mathbf{A}^\times \backslash \mathbf{A}_K^\times} \varphi_\chi(t\varsigma)\chi(t) dt,$$

¹See [Jac72, Thm. 20.6] for the existence of the quadratic base change, and [JL70, §11] for the definition of L -functions on $\text{GL}(2)$.

where dt is the Tamagawa measure on $\mathbf{A}_K^\times/\mathbf{A}^\times$. Under the assumption (b), the explicit Waldspurger formula in [Hsi14, Theorem 3.14] implies that

$$(3.6) \quad P_\chi(\pi(\varsigma)\varphi_\chi)^2 = |D_K|^{-\frac{1}{2}} \cdot \frac{\Gamma(2r)}{(4\pi)^{2r+1}} \cdot e_{\mathfrak{p}}(f, \chi) \cdot L\left(\frac{1}{2}, \pi_K \otimes \chi\right) \cdot C'_\pi(\chi) N(\pi, \chi)^2,$$

where $N(\pi, \chi) = \prod_{q|c^-} L(1, \tau_{K_q/\mathbf{Q}_q})$ and $C'_\pi(\chi)$ is the constant

$$\begin{aligned} C'_\pi(\chi) &= 2^{\#(A(\chi))+3} (c^-)^{-1} \cdot \prod_{\mathfrak{q}|\mathfrak{C}\mathfrak{N}, \mathfrak{q} \neq \mathfrak{p}} \varepsilon\left(\frac{1}{2}, \pi_{\mathfrak{q}} \otimes \chi_{\bar{\mathfrak{q}}}\right) \\ &= 2^{\#(A(\chi))+3} (c^-)^{-1} \cdot \prod_{\mathfrak{q}|\mathfrak{N}} \varepsilon\left(\frac{1}{2}, \pi_{\mathfrak{q}}\right) \chi_{\mathfrak{q}}^{-1}(\mathfrak{N}) \prod_{\mathfrak{q}|\mathfrak{C}, \mathfrak{q} \neq \mathfrak{p}} \varepsilon\left(\frac{1}{2}, \chi_{\mathfrak{q}}\right)^{-2} \end{aligned}$$

In the last equality, we used the formulae

$$\varepsilon(s, \pi_{\mathfrak{q}} \otimes \chi_{\bar{\mathfrak{q}}}) = \varepsilon(s, \pi_{\mathfrak{q}} \otimes \chi_{\mathfrak{q}}^{-1}) = \begin{cases} \varepsilon(s, \pi_{\mathfrak{q}}) \chi_{\mathfrak{q}}^{-1}(\mathfrak{N}) & \text{if } \mathfrak{q} \mid N^+, \\ \varepsilon(1-s, \chi_{\mathfrak{q}})^{-2} & \text{if } \mathfrak{q} \mid c. \end{cases}$$

On the other hand, under assumption (a) one can verify that

$$\sum_{[u] \in \widehat{\mathcal{O}}_K^\times / \widehat{\mathcal{O}}_c^\times} \varphi_f^\flat(g(\varsigma_\infty, u\xi_c)) \chi(u) = \varphi_\chi(g\bar{c}\varsigma) \cdot \prod_{\mathfrak{q}|\mathfrak{C}} \mathfrak{g}(\chi_{\mathfrak{q}}^{-1}) \cdot c^- \prod_{q|c^-} (1+1/q)$$

by comparing the Whittaker functions of the automorphic forms φ_f^\flat and φ_χ on both sides, where $\mathbf{c} = (\mathbf{c}_q)_{\mathfrak{q}} \in \widehat{K}^\times$ is the idele with $\mathbf{c}_q = q^{\text{ord}_q(c)}$ if $\mathfrak{q} \mid \mathfrak{C}$ and $\mathbf{c}_q = 1$ if $\mathfrak{q} \nmid \mathfrak{C}$. From this equation, we obtain

$$\begin{aligned} P_\chi(\pi(\varsigma)\varphi_\chi) &= \frac{2}{\sqrt{D_K} u_K} \sum_{[a] \in \text{Pic } \mathcal{O}_K} \chi(a) \varphi_\chi(a\varsigma) \\ &= \frac{2N(\pi, \chi)}{c^- \sqrt{D_K} u_K} \cdot \prod_{\mathfrak{q}|\mathfrak{C}} \mathfrak{g}(\chi_{\mathfrak{q}}^{-1})^{-1} \cdot \sum_{[a] \in \text{Pic } \mathcal{O}_K} \chi(a\mathbf{c}^{-1}) \sum_{[u] \in \widehat{\mathcal{O}}_K^\times / \widehat{\mathcal{O}}_c^\times} \varphi_f^\flat(\varsigma_\infty, au\xi_c) \chi(u) \\ &= \frac{2N(\pi, \chi)}{c^- \sqrt{D_K} u_K} \cdot \prod_{\mathfrak{q}|\mathfrak{C}} \mathfrak{g}(\chi_{\mathfrak{q}}^{-1})^{-1} \chi_{\mathfrak{q}}^{-1}(q^{\text{ord}_q c}) \cdot \sum_{[a] \in \text{Pic } \mathcal{O}_c} \varphi_f^\flat(\varsigma_\infty, au\xi_c) \chi(u). \end{aligned}$$

We thus find

$$(3.7) \quad P_\chi(\pi(\varsigma)\varphi_\chi) = \frac{2N(\pi, \chi)}{c\sqrt{D_K} u_K} \cdot \prod_{\mathfrak{q}|\mathfrak{C}} \varepsilon(1, \chi_{\mathfrak{q}})^{-1} \cdot P_\chi(f^\flat).$$

It is clear that the theorem follows from (3.6) and (3.7). \square

Analytic construction of the p -adic L -function. Let $K_{p^\infty} = \cup_n K_{p^n}$ be the ring class field of conductor p^∞ , and let $\widetilde{\Gamma} := \text{Gal}(K_{p^\infty}/K)$. Then the Galois group Γ_K^- of the anticyclotomic \mathbf{Z}_p -extension is the maximal free quotient of $\widetilde{\Gamma}$. Denote by $\mathcal{C}(\widetilde{\Gamma}, \mathcal{O}_{\mathbf{C}_p})$ the space of continuous $\mathcal{O}_{\mathbf{C}_p}$ -valued functions on $\widetilde{\Gamma}$, and let $\mathfrak{X}_{p^\infty} \subset \mathcal{C}(\widetilde{\Gamma}, \mathcal{O}_{\mathbf{C}_p})$ be the set of locally algebraic p -adic characters $\rho : \widetilde{\Gamma} \rightarrow \mathcal{O}_{\mathbf{C}_p}^\times$.

Let $\text{rec}_{\mathfrak{p}} : \mathbf{Q}_p^\times = K_{\mathfrak{p}}^\times \rightarrow \text{Gal}(K^{\text{ab}}/K) \rightarrow \widetilde{\Gamma}$ be the local reciprocity law map. For $\rho \in \mathfrak{X}_{p^\infty}$, we define $\rho_{\mathfrak{p}} : \mathbf{Q}_p^\times \rightarrow \mathbf{C}_p^\times$ by

$$\rho_{\mathfrak{p}}(\beta) = \rho(\text{rec}_{\mathfrak{p}}(\beta)),$$

and for $\rho \in \mathcal{C}(\widetilde{\Gamma}, \mathcal{O}_{\mathbf{C}_p})$, we define $\rho|[\mathfrak{a}] : \mathbf{Z}_p^\times \rightarrow \mathcal{O}_{\mathbf{C}_p}$ by

$$\rho|[\mathfrak{a}](x) = \rho(\text{rec}_{\mathfrak{p}}(x)\sigma_{\mathfrak{a}}^{-1}) = \rho(\text{rec}_{\mathfrak{p}}(x) \text{rec}_K(a)).$$

For each $a \in \widehat{K}^{(c_o \mathfrak{N} p)^\times}$ with associated fractional ideal $\mathfrak{a} \subset \mathcal{O}_{c_o}$, let $(A_{\mathfrak{a}}, \eta_{\mathfrak{a}})$ be the CM elliptic curve with level structure introduced in §2.3. Let $t_{\mathfrak{a}}$ be the canonical Serre–Tate coordinate of \widehat{f}^\flat around $\mathbf{x}_{\mathfrak{a}} = [(A_{\mathfrak{a}}, \eta_{\mathfrak{a}})] \otimes_{\mathcal{W}} \overline{\mathbb{F}}_p$, and set

$$(3.8) \quad \widehat{f}_{\mathfrak{a}}^\flat(t_{\mathfrak{a}}) := \widehat{f}^\flat(t_{\mathfrak{a}}^{N(\mathfrak{a})^{-1} \sqrt{-D_K}^{-1}}) \in \mathcal{W}[[t-1]] \quad (N(\mathfrak{a}) = |a|_{\mathbf{A}_K}^{-1} c_o^{-1}).$$

Definition 3.7 (Analytic anticyclotomic p -adic L -functions). Let ψ be an anticyclotomic Hecke character of infinity type $(r, -r)$, and let $c_o\mathcal{O}_K$ be the prime-to- p part of the conductor of ψ . Define the p -adic measure $\mathcal{L}_{p,\psi}(f)$ on $\bar{\Gamma}$ by

$$\mathcal{L}_{p,\psi}(f)(\rho) = \sum_{[a] \in \text{Pic } \mathcal{O}_{c_o}} \psi(\mathbf{a})N(\mathbf{a})^{-r} \cdot \left(\hat{f}_a^b \otimes \psi_p \rho|_{[a]} \right) (A_a, \eta_a).$$

We shall also view $\mathcal{L}_{p,\psi}(f)$ as an element in the semi-local ring $\mathcal{W}[\bar{\Gamma}]$.

The p -adic measure $\mathcal{L}_{p,\psi}(f)$ satisfies the following interpolation formula at characters of infinity type $(m, -m)$ with $m \geq 0$. In what follows, we assume (Heeg'), (ST) for (f, ψ) and that $(c_o, pN^+) = 1$.

Proposition 3.8. *If $\hat{\phi} \in \mathfrak{X}_{p^\infty}$ is the p -adic avatar of a Hecke character ϕ of infinity type $(m, -m)$ with $m \geq 0$ and p -power conductor, then*

$$\left(\frac{\mathcal{L}_{p,\psi}(f)(\hat{\phi})}{\Omega_p^{2r+2m}} \right)^2 = L^{\text{alg}}\left(\frac{1}{2}, \pi_K \otimes \psi\phi\right) \cdot e_p(f, \psi\phi) \cdot \phi(\mathfrak{N}^{-1}) \cdot 2^{\#A(\psi)+3} c_o \varepsilon(f) \cdot u_K^2 \sqrt{D_K}.$$

PROOF. Suppose that $m = 0$. Then $\hat{\phi} = \phi$ is a finite order character, and $\chi := \psi\phi$ is an anticyclotomic Hecke character of infinity type $(r, -r)$. Let $c\mathcal{O}_K$ be the conductor of χ (so $c = c_o p^n$). Suppose that $n > 0$. By Definition 3.7 and Proposition 3.3, we have

$$\begin{aligned} \mathcal{L}_{p,\psi}(f)(\phi) &= c_o^r \sum_{[a] \in \text{Pic } \mathcal{O}_{c_o}} \left(\hat{f}_a^b \otimes \chi_p \right) (x_a) \chi|_{\mathbf{A}_K} (a) \\ (3.9) \quad &= p^{-n} \mathfrak{g}(\chi_p) c_o^r \sum_{[a] \in \text{Pic } \mathcal{O}_{c_o}} \chi|_{\mathbf{A}_K} (\bar{a}^{-1}) \sum_{u \in (\mathbf{Z}_p/p^n \mathbf{Z}_p)^\times} \hat{f}^b(x_a * \mathbf{n}(up^{-n})) \chi_p(u^{-1}). \end{aligned}$$

For $z \in \mathbf{Q}_p^\times$, we use $z_{\bar{p}}$ (resp. z_p) to denote the finite idele in \hat{K}^\times with z at \bar{p} (resp. p) and 1 at all the other places. Since f^b is of weight $2r$ and level $\Gamma_0(Np^2)$, a direct calculation shows that

$$\begin{aligned} \hat{f}^b(x_a * \mathbf{n}(up^{-n})) &= \mathbf{f}^b(\vartheta, \bar{a}^{-1} \xi_{c_o} \mathbf{n}(up^{-n})) \cdot \frac{\Omega_p^{2r}}{\Omega_K^{2r}} \\ &= \mathbf{f}^b(\vartheta, \bar{a}^{-1} u_{\bar{p}} p_{\bar{p}}^{-n} \xi_{c_o p^n}) \cdot \frac{\Omega_p^{2r}}{\Omega_K^{2r}} \quad (u \in \mathbf{Z}_p^\times), \end{aligned}$$

where (Ω_K, Ω_p) are the periods defined in §2.5. Note that we used (2.3) in the last equation. We thus find

$$\begin{aligned} \frac{\mathcal{L}_{p,\psi}(f)(\phi)}{\Omega_p^{2r}} &= \frac{p^{-n} \mathfrak{g}(\chi_p) c_o^r}{\Omega_K^{2r}} \cdot \sum_{[a] \in \text{Pic } \mathcal{O}_{c_o}} \sum_{u \in \hat{\mathcal{O}}_{c_o}^\times / \hat{\mathcal{O}}_c^\times} \chi|_{\mathbf{A}_K} (\bar{a}^{-1} u_{\bar{p}}) \mathbf{f}^b(\vartheta, \bar{a}^{-1} u_{\bar{p}} p_{\bar{p}}^{-n} \cdot \xi_c) \\ &= \frac{p^{-n} \mathfrak{g}(\chi_p) c_o^r}{\Omega_K^{2r}} \cdot \sum_{[a] \in \text{Pic } \mathcal{O}_c} \chi|_{\mathbf{A}_K} (a) \mathbf{f}^b(\vartheta, a p_{\bar{p}}^{-n} \cdot \xi_c) \\ &= \frac{\varepsilon(1, \chi_{\bar{p}}) \chi_{\bar{p}}(-1) c^r}{\Omega_K^{2r}} \cdot \sum_{[a] \in \text{Pic } \mathcal{O}_c} \chi|_{\mathbf{A}_K} (a) \mathbf{f}^b(\vartheta, a \xi_c) \quad (\text{by (1.1)}). \end{aligned}$$

Therefore, according to Definition 3.5 we obtain

$$\frac{\mathcal{L}_{p,\psi}(f)(\phi)}{\Omega_p^{2r}} = \frac{\varepsilon(\frac{1}{2}, \chi_p)^{-1}}{\Omega_K^{2r}} \cdot P_\chi(f^b) \cdot p^{-\frac{n}{2}} (\text{Im } \vartheta)^{-r}.$$

The proposition for the case $m = 0$ and $n > 0$ now follows from Theorem 3.6. If $n = 0$, i.e. $\chi_p = \mathbf{1}$ is the trivial character on \mathbf{Z}_p^\times , then one can use (3.4) and the fact that φ_f is a Hecke eigenform to show that $\hat{f}_a^b \otimes \chi_p(x_a) = \hat{f}_a^b(x_a)$, so (3.9) is still valid, and as above the proposition also follows in this case.

For general $m \geq 0$, comparing the interpolation formulas for $\mathcal{L}_{p,\psi}(f)$ and for the p -adic L -function $\mathcal{L}_p(\pi, \psi)$ constructed in [Hsi14, Thm. A] at p -ramified finite order characters ($m = 0$), we find that $\mathcal{L}_{p,\psi}(f) = u \cdot \mathcal{L}_p(\pi, \psi)$ with $u = 2^{\#A(\psi)+3} c_o \varepsilon(f) \sqrt{D_K}$, and hence the general interpolation formulae of $\mathcal{L}_{p,\psi}(f)$ can be deduced from those of $\mathcal{L}_p(\pi, \psi)$ in *loc.cit.*. We omit the details. \square

We now prove the nonvanishing of the p -adic L -function $\mathcal{L}_{\mathfrak{p},\psi}(f)$.

Theorem 3.9. *Suppose $(N_f, D_K) = 1$. For all but finitely many $\phi \in \mathfrak{X}_{p^\infty}$, we have $\mathcal{L}_{\mathfrak{p},\psi}(f)(\phi) \neq 0$.*

PROOF. Since f has conductor prime to D_K , f can not be a CM form arising from K , and hence the ℓ -adic representation $\rho_{f,\ell}$ is irreducible when restricted to G_K for every prime ℓ . Therefore, it is well-known that there exist infinitely many primes ℓ such that:

- ℓ is prime to $pND_K \prod_{q|c^-} (1+q)$,
- the residual Galois representation $\bar{\rho}_{f,\ell}|_{G_K}$ is absolutely irreducible.

By [Hsi14, Theorem C], the central L -values $\{L^{\text{alg}}(1/2, \pi_K \otimes \psi\phi)\}$ are non-zero modulo ℓ for all but finitely many finite order characters $\phi \in \mathfrak{X}_{p^\infty}$. (Note that the roles of p and ℓ have been switched here.) In particular, this implies that $\mathcal{L}_{\mathfrak{p},\psi}(f)$ does not vanish identically, and hence the theorem follows from p -adic Weierstrass preparation theorem. \square

4. GENERALIZED HEEGNER CYCLES

4.1. **Definitions.** We continue to let $f \in S_{2r}^{\text{new}}(\Gamma_0(N))$ be a newform of weight $2r$ and level N . We assume the (strong) Heegner condition

(Heeg) N is a product of primes split in K .

Thus (Heeg') and (ST) will automatically hold. Let $K = \mathbf{Q}(\sqrt{-D_K})$ be the imaginary quadratic field of discriminant $-D_K$. If $r > 1$, we further assume that

(can) either $D_K > 3$ is odd, or $8 \mid D_K$.

This assumption ensures the existence of canonical elliptic curves in the sense of Gross (see [Yan04, Thm. 0.1]). We shall fix a canonical elliptic curve A with CM by \mathcal{O}_K , which is characterized by the following properties:

- A is equipped with CM by $[\cdot]: \mathcal{O}_K \simeq \text{End } A$.
- There is a complex uniformization $\xi: \mathbf{C}/\mathcal{O}_K \simeq A(\mathbf{C})$.
- A is a \mathbf{Q} -curve defined over H_K^+ , where $H_K^+ = \mathbf{Q}(j(\mathcal{O}_K))$ is the real subfield of the Hilbert class field H_K of K .
- The conductor of A is only divisible by prime factors of D_K .

For each positive integer c , let $\mathcal{C}_c := \xi(c^{-1}\mathcal{O}_c/\mathcal{O}_K) \subset A$ be a cyclic subgroup of order c . The elliptic curve A/\mathcal{C}_c is defined over the real subfield $\mathbf{Q}(j(\mathcal{O}_c))$ of the ring class field K_c of conductor c . Let $\varphi_c: A/K_c \rightarrow A_c/K_c$ be the isogeny given by the natural quotient map. Then A/\mathcal{C}_c is equipped with the complex uniformization $A_c \simeq \mathbf{C}/\mathcal{O}_c$ such that $\varphi_c: \mathbf{C}/\mathcal{O}_K \rightarrow \mathbf{C}/\mathcal{O}_c$ is given by $z \mapsto cz$. Thus we see that the elliptic curve A_c introduced in §2.3 descends to the elliptic curve A/\mathcal{C}_c , still denoted by A_c in the sequel.

For any ideal \mathfrak{a} of \mathcal{O}_c , in this section we always assume that $\mathfrak{a}\mathcal{O}_K$ is prime to $cD_K\mathfrak{p}\mathfrak{N}$. Let \mathfrak{a} be an ideal of \mathcal{O}_c and recall that $\sigma_{\mathfrak{a}} \in \text{Gal}(K^{\mathfrak{a}}/K)$ is the image of \mathfrak{a} under the Artin map, where $K^{\mathfrak{a}}$ is the maximal abelian \mathfrak{a} -ramified extension of K . Then, by the main theorem of complex multiplication (cf. [dS87, Prop. 1.5, p.42]), we have $A_{\mathfrak{a}} = A_c^{\sigma_{\mathfrak{a}}}$ and the isogeny $\lambda_{\mathfrak{a}}: A_c \rightarrow A_{\mathfrak{a}}$ in §2.3 is actually defined over K_c and characterized by the rule

$$(4.1) \quad \lambda_{\mathfrak{a}}(x) = \sigma_{\mathfrak{a}}(x) \quad \text{for all } x \in A[m], (m, N(\mathfrak{a})) = 1.$$

Define the isogeny

$$\varphi_{\mathfrak{a}} := \lambda_{\mathfrak{a}} \circ \varphi_c: A/K_c \longrightarrow A_{\mathfrak{a}}/K_c,$$

and let $\Gamma_{\mathfrak{a}}$ be the graph

$$\Gamma_{\mathfrak{a}} = \{\varphi_{\mathfrak{a}}(z), z \mid z \in A\} \subset A_{\mathfrak{a}} \times A.$$

Let $x_{\mathfrak{a}} = [(A_{\mathfrak{a}}, \eta_{\mathfrak{a}})] \in Y_1(N)(K_c)$ be the CM point associated to \mathfrak{a} as in the last paragraph of §2.3, and let \mathcal{A} be the universal elliptic curve over $Y_1(N)$. Then $x_{\mathfrak{a}}$ determines an embedding $i_{x_{\mathfrak{a}}}: A_{\mathfrak{a}} \rightarrow \mathcal{A}$, and we define

$$\mathcal{Y}_{\mathfrak{a}} = (i_{x_{\mathfrak{a}}} \times \text{id})(\Gamma_{\mathfrak{a}}) = \{(i_{x_{\mathfrak{a}}}(\varphi_{\mathfrak{a}}(z)), z) \mid z \in A\} \subset \mathcal{A} \times A.$$

Denote by W_{2r-2} the Kuga–Sato variety of dimension $2r-1$ (cf. [BDP13, p.1056]). Following [BDP13, p.1063], define the cycle $\Upsilon_{\mathfrak{a}}$ in the generalized Kuga–Sato variety $X_{2r-2} := W_{2r-2} \times A^{2r-2}$ by

$$\Upsilon_{\mathfrak{a}} = \mathcal{Y}_{\mathfrak{a}}^{2r-2} \subset (i_{x_{\mathfrak{a}}}(A_{\mathfrak{a}}) \times A)^{2r-2} \hookrightarrow X_{2r-2}.$$

Let $\epsilon_X = \epsilon_W \times \epsilon_A$, with $\epsilon_W \in \mathbf{Z}[\mathrm{Aut}(W_{2r-2})]$ and $\epsilon_A \in \mathbf{Z}[\mathrm{Aut}(A^{2r-2})]$ the idempotents defined in [BDP13, (2.1.2), (1.4.4)]. The following definition is given in [BDP13, p.1063].

Definition 4.1. The *generalized Heegner cycle* $\Delta_{\varphi_{\mathfrak{a}}}$ associated to the isogeny $\varphi_{\mathfrak{a}}$ is

$$\Delta_{\varphi_{\mathfrak{a}}} := \epsilon_X[\Upsilon_{\mathfrak{a}}] \in \mathrm{CH}^{2r-1}(X_{2r-2}/K_c)_{0, \mathbf{Q}}.$$

4.2. Generalized Heegner classes (I). Let p be a prime with $p \nmid 2(2r-1)!N\varphi(N)$. Let F be a finite extension of \mathbf{Q}_p containing the Hecke field of f . Let V_f be the two-dimensional p -adic representation of $G_{\mathbf{Q}}$ over F attached to the newform f by Deligne, and denote by $V_f(r)$ the Tate twist $V_f \otimes \varepsilon_{\mathrm{cyc}}^r$, where $\varepsilon_{\mathrm{cyc}}$ is the p -adic cyclotomic character. Following [BDP13, §3.1], we consider the p -adic Abel–Jacobi map

$$\begin{aligned} \Phi_{\acute{e}t, f} : \mathrm{CH}^{2r-1}(X_{2r-2}/K_c)_0 &\longrightarrow H^1(K_c, \epsilon_X H_{\acute{e}t}^{4r-3}(X_{2r-2}/\overline{\mathbf{Q}}, \mathbf{Z}_p)(2r-1)) \\ &\longrightarrow H^1(K, \epsilon_W H^{2r-1}(W_{2r-2}/\overline{\mathbf{Q}}, \mathbf{Z}_p)(r)) \otimes \mathrm{Sym}^{2r-2} H_{\acute{e}t}^1(A/\overline{\mathbf{Q}}, \mathbf{Z}_p)(r-1) \\ &\longrightarrow H^1(K_c, T \otimes S^{r-1}(A)), \end{aligned}$$

where T is the Galois stable \mathcal{O}_F -lattice in $V_f(r)$ in [Nek92, §3], and $S^{r-1}(A)$ is the G_{H_K} -module

$$S^{r-1}(A) := \mathrm{Sym}^{2r-2} T_p(A)(1-r)$$

with $T_p(A)$ the p -adic Tate module of A . For every ideal \mathfrak{a} of \mathcal{O}_c , define the *generalized Heegner class* $z_{f, \mathfrak{a}}$ associated to \mathfrak{a} by

$$(4.2) \quad z_{f, \mathfrak{a}} := \Phi_{\acute{e}t, f}(\Delta_{\varphi_{\mathfrak{a}}}) \in H^1(K_c, T \otimes S^{r-1}(A)).$$

In the following, we shall simply write $z_{f, c}$ for z_{f, \mathcal{O}_c} .

4.3. Norm relations.

Lemma 4.2. *If $D \subset (A_{\mathfrak{a}} \times A)^{2r-2}$ is a cycle of codimension $r-1$ such that D is zero in the Néron–Severi group of $\mathrm{NS}(A_{\mathfrak{a}} \times A)^{2r-2}$, then the p -adic Abel–Jacobi image of $\epsilon_X(i_{x_{\mathfrak{a}}})_*(D)$ in $H^1(K_c, T \otimes S^{r-1}(A))$ is also trivial.*

PROOF. This follows from the fact that the Abel–Jacobi image of $\epsilon_X(i_{x_{\mathfrak{a}}})_*(D)$ lies in the image of the map

$$H^1(K_c, \epsilon_X H_{\acute{e}t}^{4r-5}(A_{\mathfrak{a}}^{2r-2} \times A_{\overline{\mathbf{Q}}}^{2r-2}, \mathbf{Z}_p)) \xrightarrow{i_{x_{\mathfrak{a}}}} H^1(K_c, \epsilon_X H_{\acute{e}t}^{4r-3}(X_{2r-2}/\overline{\mathbf{Q}}, \mathbf{Z}_p))$$

and $\epsilon_X H_{\acute{e}t}^{4r-5}(\bar{A}_{\mathfrak{a}}^{2r-2} \times \bar{A}_{\overline{\mathbf{Q}}}^{2r-2}, \mathbf{Z}_p) = 0$. \square

We refer to §4.4 for the definition of the character $\tilde{\kappa}_A$ appearing in the next result.

Lemma 4.3. *Suppose $\mathfrak{a}\mathcal{O}_K$ is trivial in $\mathrm{Pic} \mathcal{O}_K$, and let $\alpha := \tilde{\kappa}_A(\mathfrak{a}) \in K^\times$. Then for every ideal \mathfrak{b} of \mathcal{O}_c prime to cND , we have*

$$(\mathrm{id} \times [\alpha])^* \Delta_{\varphi_{\mathfrak{b}}}^{\sigma_{\mathfrak{a}}} = \Delta_{\varphi_{\mathfrak{ab}}}.$$

PROOF. Let $\sigma = \sigma_{\mathfrak{a}} \in \mathrm{Gal}(K^{ab}/H_K^+K)$. By definition, $A^\sigma = A$ and $A_{\mathfrak{a}}^\sigma = A_{\mathfrak{ab}}$. Note that for any $t \in A[m]$ with $(m, N(\mathfrak{ab})) = 1$, we have $\sigma(t) = \lambda_{\mathfrak{a}\mathcal{O}}(t) = [\alpha](t)$ and

$$\varphi_c^\sigma \circ [\alpha](t) = \varphi_n^\sigma(\sigma(t)) = \sigma(\varphi_c(t)) = \lambda_{\mathfrak{a}}(\varphi_n(t)).$$

This implies that $\varphi_c^\sigma \circ [\alpha] = \lambda_{\mathfrak{a}} \circ \varphi_c$. Therefore,

$$[\alpha] \circ \varphi_{\mathfrak{b}}^\sigma = \varphi_{\mathfrak{b}}^\sigma \circ [\alpha] = \lambda_{\mathfrak{b}}^\sigma \circ \varphi_c^\sigma \circ [\alpha] = \lambda_{\mathfrak{b}}^\sigma \circ \lambda_{\mathfrak{a}} \circ \varphi_c = \varphi_{\mathfrak{ab}},$$

and

$$(\mathrm{id} \times [\alpha])^* \Gamma_{\mathfrak{b}}^{\sigma_{\mathfrak{a}}} = ([\alpha] \times \mathrm{id})_* \Gamma_{\mathfrak{b}}^{\sigma_{\mathfrak{a}}} = \Gamma_{[\alpha] \circ \varphi_{\mathfrak{b}}^\sigma} = \Gamma_{\varphi_{\mathfrak{ab}}} = \Gamma_{\mathfrak{ab}}.$$

The lemma thus follows immediately from $x_{\mathfrak{b}}^{\sigma_{\mathfrak{a}}} = x_{\mathfrak{ab}}$. \square

Let $G_n := \text{Gal}(K_{cp^n}/K_{cp^{n-1}})$, which is identified with $\ker \{\text{Pic } \mathcal{O}_{cp^n} \rightarrow \text{Pic } \mathcal{O}_{cp^{n-1}}\}$ via the Artin isomorphism. The usual Hecke correspondence T_q associated with a prime $q \nmid N$ on the Kuga–Sato variety W_{2r-2} (see [Sch90, §4]) induces the Hecke correspondence $T_q \times \text{id}$ on the generalized Kuga–Sato variety $X_{2r-2} = W_{2r-2} \times A^{2r-2}$. In what follows, we shall still write T_q for $T_q \times \text{id}$ if no confusion arises.

Proposition 4.4. *Assume that $p \nmid c$. If $p = \mathfrak{p}\bar{\mathfrak{p}}$ is split in K , then for all $n > 1$ we have*

$$T_p z_{f, cp^{n-1}} = p^{2r-2} \cdot z_{f, cp^{n-2}} + \text{cor}_{K_{cp^n}/K_{cp^{n-1}}}(z_{f, cp^n}),$$

where $u_c = \#(\mathcal{O}_c^\times)$, and $\sigma_{\mathfrak{p}}, \sigma_{\bar{\mathfrak{p}}} \in \text{Gal}(K_c/K)$ are the Frobenius elements of \mathfrak{p} and $\bar{\mathfrak{p}}$. Moreover, if $\ell \nmid c$ is inert in K , then

$$T_\ell z_{f, c} = \text{cor}_{K_{c\ell}/K_c}(z_{f, c\ell}).$$

PROOF. Let $\mathcal{L} \subset \mathcal{O}_{cp^{n-1}}$ be a sublattice of $\mathcal{O}_{cp^{n-1}}$ with index p , and let $A_{\mathcal{L}} = \mathbf{C}/\mathcal{L}$. Let $\psi_{\mathcal{L}}: A_{\mathcal{L}} \rightarrow A_{cp^{n-1}}$ be the isogeny induced by $\mathcal{L} \hookrightarrow \mathcal{O}_{cp^{n-1}}$. We have two cases:

Case (i): \mathcal{L} is an \mathcal{O}_{cp^n} -ideal and the class $[\mathcal{L}]$ is trivial in $\text{Pic } \mathcal{O}_{cp^{n-1}}$, so we can write $\mathcal{L} = \alpha \mathfrak{a}^{-1}$ for some integral \mathcal{O}_{cp^n} -ideal \mathfrak{a} with $\alpha = \tilde{\kappa}_A(\mathfrak{a})$. Then we have $A_{\mathcal{L}} \simeq A_{\mathfrak{a}}$ and

$$\psi_{\mathcal{L}} \circ \varphi_{\mathfrak{a}} = [p\alpha] \circ \varphi_{cp^{n-1}}.$$

Denote by T_x the translation map by a torsion point $x \in A_{\mathfrak{a}} \times A$. Then we have

$$\begin{aligned} \bigsqcup_{z \in \ker \psi_{\mathcal{L}}} T_{(z,0)}^* \Gamma_{\mathfrak{a}} &= \{(x, y) \mid \psi_{\mathcal{L}}(x) = \psi_{\mathcal{L}}(\varphi_{\mathfrak{a}}(y))\} \\ &= \{(x, y) \mid \psi_{\mathcal{L}}(x) = \varphi_{cp^{n-1}}(p\alpha y)\} \\ &= (\text{id} \times [p\alpha])^* \psi_{\mathcal{L}}^* \Gamma_{\mathcal{O}_{cp^{n-1}}}. \end{aligned}$$

This implies that $p \cdot \Gamma_{\mathfrak{a}}$ and $p \cdot (\text{id} \times [\alpha])^* \psi_{\mathcal{L}}^* \Gamma_{\mathcal{O}_{cp^{n-1}}}$ are equal in the Néron–Severi group $\text{NS}(A_{\mathfrak{a}} \times A)$, and hence by Lemma 4.2 we have

$$z_{f, \mathfrak{a}} = (\text{id} \times [\alpha])^* \psi_{\mathcal{L}}^* z_{f, cp^{n-1}}.$$

Using Lemma 4.3 and the projection formula $(\text{id} \times [\alpha])_* (\text{id} \times [\alpha])^* = N(\mathfrak{a}\mathcal{O}_K)^{2r-2}$, we conclude that

$$(4.3) \quad \psi_{\mathcal{L}}^* z_{f, cp^{n-1}} = z_{f, cp^n}^{\sigma_{\mathfrak{a}}}.$$

Case (ii): $\mathcal{L} = p\mathcal{O}_{cp^{n-2}}$ and p is split in K . Then $A_{\mathcal{L}} \simeq A_{cp^{n-2}}$, and

$$\psi_{\mathcal{L}} \circ \varphi_{cp^{n-2}} = \varphi_{cp^{n-1}}.$$

Note that

$$\psi_{\mathcal{L}}^* \Gamma_{\mathcal{O}_{cp^{n-1}}} = \bigsqcup_{z \in \ker \psi_{\mathcal{L}}} T_{(z,0)}^* (\Gamma_{\mathcal{O}_{cp^{n-2}}}),$$

so $\psi_{\mathcal{L}}^* \Gamma_{\mathcal{O}_{cp^{n-1}}}$ and $p \cdot \Gamma_{\mathcal{O}_{cp^{n-2}}}$ are equal in the Néron–Severi group $\text{NS}(A_{cp^{n-2}} \times A)$. By Lemma 4.2, we have

$$(4.4) \quad \psi_{\mathcal{L}}^* z_{f, cp^n} = p^{2r-2} \cdot z_{f, cp^{n-2}}.$$

Choose a set Ξ of representatives of fractional \mathcal{O}_{cp^n} -ideals of $\ker \{\text{Pic } \mathcal{O}_{cp^n} \rightarrow \text{Pic } \mathcal{O}_{cp^{n-1}}\}$, and let

$$\Xi^* := \{\alpha^{-1} \mathfrak{a} \subset \mathcal{O}_{cp^n} \mid \mathfrak{a} \in \Xi, \alpha = \tilde{\kappa}_A(\mathfrak{a})\}.$$

If p is split, then

$$\{\mathcal{L} \subset \mathcal{O}_{cp^{n-1}} \mid [\mathcal{O}_{cp^{n-1}} : \mathcal{L}] = p\} = \Xi^* \sqcup \{p\mathcal{O}_{cp^{n-2}}\},$$

and thus by (4.3) and (4.4) we see that

$$T_p z_{f, cp^{n-1}} = \sum_{\substack{L \subset \mathcal{O}_{cp^{n-1}}, \\ [\mathcal{O}_{cp^{n-1}} : L] = p}} \psi_L^* z_{f, cp^{n-1}} = p^{2r-2} \cdot z_{f, cp^{n-2}} + \sum_{\sigma \in G_n} z_{f, cp^n}^{\sigma}.$$

If ℓ is inert and $n = 1$, then

$$\{\mathcal{L} \subset \mathcal{O}_c \mid [\mathcal{O}_c : \mathcal{L}] = \ell\} = \{\alpha^{-1} \mathfrak{a} \subset \mathcal{O}_c \mid \mathfrak{a} \text{ ideal of } \mathcal{O}_{c\ell}, \alpha = \tilde{\kappa}_A(\mathfrak{a})\},$$

and hence

$$T_\ell z_{f,c} = \sum_{\sigma \in \text{Gal}(K_{c\ell}/K_c)} z_{f,c\ell}^\sigma.$$

This completes the proof. \square

4.4. Generalized Heegner classes (II). Let c_o be a positive integer with $(c_o, pN) = 1$, and let $\chi: \text{Gal}(K_{c_o p^\infty}/K) \rightarrow \mathcal{O}_F^\times$ be a locally algebraic anticyclotomic character of infinity type $(j, -j)$ with $-r < j < r$ and conductor $c_o p^s \mathcal{O}_K$. The aim of this section is to construct classes $z_{f,\chi,c} \in H^1(K_c, T \otimes \chi)$ by taking the corestriction of $z_{f,c}$ for every c divisible by $c_o p^s$. However, note that the CM elliptic curve A is only defined over the Hilbert class field H_K , so the group $\text{Gal}(K_c/K)$ does not act on $z_{f,c}$ in general. In order to get a natural Galois action, we consider

$$B/K := \text{Res}_{H_K/K} A,$$

the abelian variety obtained by restriction of scalars. As is well-known, B is a CM abelian variety over K and $M := \mathbf{Q} \otimes_{\mathbf{Z}} \text{End}_K B$ is a product of CM fields over K with $\dim B = [M : K] = [H_K : K]$ (see [Rub81, Prop. (1.2)]).

Let $I(D_K)$ be the group of prime-to- D_K fractional ideals of K , and let

$$\tilde{\kappa}_A : I(D_K) \longrightarrow M^\times$$

be the CM character associated to B with the following properties (cf. [Rub81, Lemma, p.457]):

- $\tilde{\kappa}_A(\alpha \mathfrak{a}) = \pm \alpha \cdot \tilde{\kappa}_A(\mathfrak{a})$ for all $\alpha \in K^\times$ with α prime to D_K and $\mathfrak{a} \in I(D_K)$.
- For all $\mathfrak{a} \in I(D_K)$ and $t \in B[m]$ with $(m, N(\mathfrak{a})) = 1$, we have

$$\tilde{\kappa}_A(\mathfrak{a})(t) = \sigma_{\mathfrak{a}}(t);$$

and if $\sigma_{\mathfrak{a}}$ is trivial on H_K (or equivalently, if \mathfrak{a} is the norm of an ideal of H_K), then $\tilde{\kappa}_A(\mathfrak{a}) \in K^\times$ and $\sigma_{\mathfrak{a}}(t) = [\tilde{\kappa}_A(\mathfrak{a})]t$ for all $t \in A[m]$.

Define the G_K -module

$$S^{r-1}(B) := \text{Sym}^{2r-2} T_p(B)(1-r) \otimes_{\mathbf{Z}_p} \mathcal{O}_F \simeq \text{Ind}_{G_{H_K}}^{G_K} S^{r-1}(A) \otimes_{\mathbf{Z}_p} \mathcal{O}_F.$$

Enlarge F so that $M \subset F$, and let $\kappa_A : G_K \rightarrow \mathcal{O}_F^\times$ be the p -adic avatar of $\tilde{\kappa}_A$. By the above properties of the CM character $\tilde{\kappa}_A$, we have

$$T_p(B) \otimes_{\mathbf{Q}_p} F = \bigoplus_{\rho \in \text{Hom}(M, F)} {}^\rho \kappa_A,$$

where ${}^\rho \kappa_A(\sigma) := \rho(\kappa_A(\sigma))$. It follows that if κ_A^τ is the p -adic character of G_K defined by $\kappa^\tau(\sigma) := \kappa(\tau\sigma\tau^{-1})$, where τ is the complex conjugation, then $(\kappa_A^\tau/\kappa_A)^j$ has infinity type $(j, -j)$ and is a direct summand of $S^{r-1}(B)$ as G_K -modules. Therefore, there exists a finite order anticyclotomic character χ_t such that χ is realized as a direct summand of $S^{r-1}(B) \otimes \chi_t$ as G_K -modules, and let

$$(4.5) \quad e_\chi : S^{r-1}(B) \otimes \chi_t \longrightarrow \chi$$

be the corresponding G_K -equivariant projection. Note that χ_t is unique up to multiplication by a character of $\text{Gal}(H_K/K)$, and that it has the same conductor as χ . In view of the decomposition

$$T_p(B) = \bigoplus_{\rho \in \text{Gal}(H_K/K)} T_p(A^\rho) \simeq \text{Ind}_{G_{H_K}}^{G_K} (T_p(A)),$$

we shall regard the classes $z_{f,\mathfrak{a}}$ of (4.2) as elements $z_{f,\mathfrak{a}} \in H^1(K_c, T \otimes S^{r-1}(B))$ via the natural inclusion $T_p(A) \rightarrow T_p(B)$ for c divisible by $c_o p^s$ and \mathfrak{a} an \mathcal{O}_c -ideal.

Proposition 4.5. *Let \mathfrak{a} be an \mathcal{O}_c -ideal with $(\mathfrak{a}, cND_K) = 1$. Then*

$$\chi_t(\sigma_{\mathfrak{a}}) \cdot (\text{id} \otimes e_\chi) z_{f,c}^\sigma = \chi \varepsilon_{\text{cyc}}^{1-r}(\sigma_{\mathfrak{a}}) \cdot (\text{id} \otimes e_\chi) z_{f,\mathfrak{a}}.$$

PROOF. We write $\sigma_{\mathfrak{a}} = \text{rec}_K(a^{-1})$, where $a \in \widehat{K}^{(c)\times}$ is such that $a\widehat{\mathcal{O}}_c \cap K = \mathfrak{a}$, and let $\sigma = \sigma_{\mathfrak{a}} \in \text{Gal}(K_c/K)$. One easily verifies that

$$\begin{aligned} (\text{id} \times \lambda_{\mathfrak{a}\mathcal{O}_K})_* \Delta_{\varphi_{\mathfrak{a}}} &= \{(\lambda_{\mathfrak{a}}(\varphi_c(z)), \lambda_{\mathfrak{a}\mathcal{O}_K}(z) \mid z \in A\} \\ &= \{(\varphi_c^{\sigma_{\mathfrak{a}}}(\lambda_{\mathfrak{a}\mathcal{O}_K}(z)), \lambda_{\mathfrak{a}\mathcal{O}_K}(z) \mid z \in A\} \\ &= \Delta_c^{\sigma_{\mathfrak{a}}}. \end{aligned}$$

We have the following fact:

$$\tilde{\kappa}_A(\mathfrak{a}\mathcal{O}_K) = (\lambda_{\mathfrak{a}\mathcal{O}_K}^{\rho})_{\rho} \in \bigoplus_{\rho \in \text{Gal}(H_K/K)} \text{Hom}(A^{\rho}, A^{\rho\sigma_{\mathfrak{a}}}) \subset \text{End}(B).$$

This can be checked, for instance, by comparing the action of both sides on the p -adic Tate module of B (see Eq. (4.1)). By the above fact, we find that

$$z_{f,c}^{\sigma} = \Phi_{\acute{e}t,f}(\Delta_c^{\sigma_{\mathfrak{a}}}) = (\text{id} \times \lambda_{\mathfrak{a}\mathcal{O}_K})_* z_{f,\mathfrak{a}} = [\tilde{\kappa}_A(\mathfrak{a}\mathcal{O}_K)]_*(z_{f,\mathfrak{a}}),$$

where $[\tilde{\kappa}_A(\mathfrak{a}\mathcal{O}_K)]_*$ denotes the push-forward of $\tilde{\kappa}_A(\mathfrak{a}\mathcal{O}_K)$ acting on $\text{Sym}^{2r-2} H_{\acute{e}t}^1(B/\overline{\mathbf{Q}}, \mathbf{Z}_p)$. Note that $[\tilde{\kappa}_A(\mathfrak{a}\mathcal{O}_K)]_*$ induces the Galois action $\sigma_{\mathfrak{a}}$ on $H_{\acute{e}t}^1(B/\overline{\mathbf{Q}}, \mathbf{Z}_p) (\simeq T_p(\text{Pic}_{\overline{\mathbf{Q}}}^0))$ and that

$$e_{\chi}(\sigma \otimes \varepsilon_{\text{cyc}}^{r-1}(\sigma) \otimes \chi_t(\sigma)t) = \chi(\sigma)e_{\chi}t$$

for every $t \in S^{r-1}(B) \otimes \chi_t = \text{Sym}^{2r-2} H_{\acute{e}t}^1(B/\overline{\mathbf{Q}}, \mathbf{Z}_p)(r-1) \otimes \chi_t$ by the definition of e_{χ} . We thus find that

$$\begin{aligned} (\text{id} \otimes e_{\chi})z_{f,c}^{\sigma} &= e_{\chi}([\tilde{\kappa}_A(\mathfrak{a}\mathcal{O}_K)]_* z_{f,\mathfrak{a}}) \\ &= \chi_t^{-1} \varepsilon_{\text{cyc}}^{1-r}(\sigma_{\mathfrak{a}}) \cdot e_{\chi}(\sigma_{\mathfrak{a}} \otimes \varepsilon_{\text{cyc}}^{r-1}(\sigma_{\mathfrak{a}}) \otimes \chi_t(\sigma_{\mathfrak{a}}) \cdot z_{f,\mathfrak{a}}) \\ &= \chi_t^{-1} \varepsilon_{\text{cyc}}^{1-r}(\sigma_{\mathfrak{a}}) \chi(\sigma_{\mathfrak{a}}) \cdot (\text{id} \otimes e_{\chi})z_{f,\mathfrak{a}}, \end{aligned}$$

and the proposition follows. \square

For each integer c divisible by the conductor of χ , put $z_{f,c} \otimes \chi_t := z_{f,c} \in H^1(K_c, T \otimes S^{r-1}(B) \otimes \chi_t)$, and let $z_{f,\chi,c}$ be the χ -component of the class $z_{f,c}$ defined by

$$(4.6) \quad z_{f,\chi,c} := (\text{id} \otimes e_{\chi})(z_{f,c} \otimes \chi_t) \in H^1(K_c, T \otimes \chi).$$

We finish this section with the proof of two lemmas which will be used in §7. Recall that we have fixed a decomposition $N\mathcal{O}_K = \mathfrak{N}\overline{\mathfrak{N}}$.

Lemma 4.6. *Let τ be the complex conjugation. Then*

$$(z_{f,\chi,c})^{\tau} = w_f \cdot \chi(\sigma_{\mathfrak{N}}) \cdot (z_{f,\chi^{-1},c})^{\sigma_{\overline{\mathfrak{N}}}},$$

where $w_f \in \{\pm 1\}$ is the Atkin–Lehner eigenvalue of f .

PROOF. We begin by noting that complex conjugation does indeed act on $z_{f,c}$, since the elliptic curve A is defined over the real field H_K^+ . Let w_N be the Atkin–Lehner involution, and set $\mathfrak{N}_c := \mathfrak{N} \cap \mathcal{O}_c$. We have the relations $w_N(\tau(x_c)) = x_{\overline{\mathfrak{N}}}$ and $w_N^*[\Gamma_{\overline{\mathfrak{N}}}] = N \cdot [\Gamma_c]$ in $\text{NS}(A_c \times A)$ (cf. [Shn16, Lemma 20]), from which we find that

$$(w_N \times \text{id})^* \Delta_{\overline{\mathfrak{N}}} = N^{r-1} \cdot \Delta_c^{\tau}.$$

Combined with Lemma 4.5, the above equation yields the lemma. \square

Lemma 4.7. *Let $\ell \nmid cND_K$ be a prime inert in K . Let $\bar{\lambda}$ be a prime of $\overline{\mathbf{Q}}$ above ℓ , and let $\lambda_{c\ell}$ and λ_c be the primes of $K_{c\ell}$ and K_c below $\bar{\lambda}$. Denote by $K_{\lambda_{c\ell}}$ and K_{λ_c} be the completions of $K_{c\ell}$ and K_c at $\lambda_{c\ell}$ and λ_c , respectively. Then*

$$\text{res}_{K_{\lambda_{c\ell}}, K_{\lambda_c}}(\text{loc}_{\lambda}(z_{f,\chi,c})^{\text{Frob}_{\ell}}) = \text{loc}_{\lambda_{c\ell}}(z_{f,\chi,c\ell}),$$

where $\text{Frob}_{\ell} \in \text{Gal}(\mathbf{Q}_{\ell}^{\text{ur}}/\mathbf{Q}_{\ell})$ is the Frobenius element of ℓ .

PROOF. Since χ is anticyclotomic and ℓ is inert, χ is a trivial character of $G_{K_{\lambda_c}}$, and hence Frob_ℓ acts naturally on $H^1(K_{\lambda_c}, T \otimes \chi) = H^1(K_{\lambda_c}, T)$. The natural isogeny $A_c \rightarrow A_{c\ell}$ reduces to the Frobenius map Frob_ℓ modulo $\bar{\lambda}$, so we find that

$$(\text{Frob}_\ell \times 1)(\tilde{\Delta}_c) = \tilde{\Delta}_{c\ell},$$

where $\tilde{\Delta}_?$ denotes the reduction of $\Delta_?$ modulo $\bar{\lambda}$. The lemma follows. \square

4.5. The p -adic Gross–Zagier formula of Bertolini–Darmon–Prasanna. The purpose of this section is to give a mild extension of the p -adic Gross–Zagier formula in [BDP13, Thm. 5.13], which relates the Bloch–Kato logarithm of generalized Heegner classes to the values of the p -adic L -function $\mathcal{L}_{p,\psi}(f)$ at characters outside the range of interpolation. We keep the notation as in §4.4.

Some notation for p -adic representations. Let L be a finite extension of \mathbf{Q}_p , and let V be a finite dimensional F -vector space with a continuous F -linear action of G_L . Recall that $\mathbf{D}_{\text{dR},L}(V)$ denotes the filtered $(L \otimes_{\mathbf{Q}_p} F)$ -module $(\mathbf{B}_{\text{dR}} \otimes_{\mathbf{Q}_p} V)^{G_L}$, where \mathbf{B}_{dR} is Fontaine’s ring of p -adic periods. If V is a de Rham representation (i.e. $\dim_L \mathbf{D}_{\text{dR},L}(V) = \dim_F V$), then there is a canonical isomorphism $D_{\text{dR},E}(V) = E \otimes_L \mathbf{D}_{\text{dR},L}(V)$ for any finite extension E/L . Denote by $\langle \cdot, \cdot \rangle$ the de Rham pairing

$$\langle \cdot, \cdot \rangle : \mathbf{D}_{\text{dR},L}(V) \times \mathbf{D}_{\text{dR},L}(V^*(1)) \longrightarrow L \otimes_{\mathbf{Q}_p} F \longrightarrow \mathbf{C}_p,$$

where $V^* = \text{Hom}_F(V, F)$. Let $\mathbf{B}_{\text{cris}} \subset \mathbf{B}_{\text{dR}}$ be the crystalline period ring and define $\mathbf{D}_{\text{cris},L}(V) := (\mathbf{B}_{\text{cris}} \otimes_{\mathbf{Q}_p} V)^{G_L}$. Then $\mathbf{D}_{\text{cris},L}(V)$ is an $(L_0 \otimes_{\mathbf{Q}_p} F)$ -module equipped with the action of crystalline Frobenius Φ , where L_0 is the maximal unramified subfield of L . When $L = \mathbf{Q}_p$, we write $\mathbf{D}_{\text{dR}}(V) = \mathbf{D}_{\text{dR},\mathbf{Q}_p}(V)$ and $\mathbf{D}_{\text{cris}}(V) = \mathbf{D}_{\text{cris},\mathbf{Q}_p}(V)$. If V is a crystalline representation (i.e. $\dim_{L_0} \mathbf{D}_{\text{cris},L}(V) = \dim_F V$), then we have a canonical isomorphism $L \otimes_{L_0} \mathbf{D}_{\text{cris},L}(V) = \mathbf{D}_{\text{dR},L}(V)$.

Let $H_e^1(L, V)$ be the image of the Bloch–Kato exponential map

$$\exp_{L,V} : \frac{\mathbf{D}_{\text{dR},L}(V)}{\text{Fil}^0 \mathbf{D}_{\text{dR},L}(V) + \mathbf{D}_{\text{cris},L}(V)^{\Phi=1}} \hookrightarrow H^1(L, V),$$

and $H_f^1(L, V) \subset H^1(L, V)$ be the Bloch–Kato ‘finite’ subspace. If $\mathbf{D}_{\text{cris},L}(V)^{\Phi=1} = 0$, then the natural inclusion $H_e^1(L, V) \subset H_f^1(L, V)$ is an equality (see for example [BK90, Cor. 3.8.4]), and we define the Bloch–Kato logarithm map

$$\log := \log_{L,V} : H_f^1(L, V) \longrightarrow \frac{\mathbf{D}_{\text{dR},L}(V)}{\text{Fil}^0 \mathbf{D}_{\text{dR},L}(V)} = (\text{Fil}^0 \mathbf{D}_{\text{dR},L}(V^*(1)))^\vee$$

to be the inverse of the Bloch–Kato exponential. We also let \exp^* be the dual exponential map

$$\exp^* := \exp_{L,V}^* : H^1(L, V^*(1)) \longrightarrow \text{Fil}^0 \mathbf{D}_{\text{dR},L}(V^*(1)),$$

obtained by dualizing $\exp_{L,V}$ with respect to the de Rham and local Tate pairings (cf. [LZ14, §2.4]).

Recall that we assumed $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K , with \mathfrak{p} induced by the fixed embedding $\iota_p : \bar{\mathbf{Q}} \rightarrow \mathbf{C}_p$. If E is a finite extension of K , we denote by $E_{\mathfrak{p}}$ the completion of E at the prime induced by ι_p . With a slight abuse of notation, we call $E_{\mathfrak{p}}$ the \mathfrak{p} -adic completion of E , and for any G_E -module V , we let

$$\text{loc}_{\mathfrak{p}} : H^1(E, V) \longrightarrow H^1(E_{\mathfrak{p}}, V)$$

denote the localization map.

Some de Rham cohomology classes. By the work of Scholl [Sch90], it is known that V_f can be realised as a quotient of $H_{\text{ét}}^{2r-1}(W_{2r-2/\bar{\mathbf{Q}}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} F$, and we get the composite quotient map

$$H_{\text{dR}}^{2r-1}(W_{2r-2}/F) \cong \mathbf{D}_{\text{dR}}(H_{\text{ét}}^{2r-1}(W_{2r-2/\bar{\mathbf{Q}}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} F) \longrightarrow \mathbf{D}_{\text{dR}}(V_f)$$

by applying the comparison isomorphism [Tsu99]. Let $\tilde{\omega}_f \in H_{\text{dR}}^{2r-1}(W_{2r-2}/F)$ be the differential form attached to the newform f via the rule in [BDP13, Cor. 2.3], and let $\omega_f \in \mathbf{D}_{\text{dR}}(V_f)$ be the image of $\tilde{\omega}_f$.

Let $L = H_{K,\mathfrak{p}}$ be the \mathfrak{p} -adic completion of H_K . The \mathcal{O}_K -action on A/L gives rise to a canonical decomposition of the de Rham cohomology group $H_{\text{dR}}^1(A/L) = H_{\text{dR}}^{1,0}(A/L) \oplus H_{\text{dR}}^{0,1}(A/L)$. Recall our fixed choice of Néron differential $\omega_A \in H_{\text{dR}}^{1,0}(A/L)$, which determines $\eta_A \in H_{\text{dR}}^{0,1}(A/L)$ by the requirement

that $\langle \omega_A, \eta_A \rangle = 1$ (cf. [BDP13, page 1051]). We shall view ω_A, η_A as elements in $\mathbf{D}_{\text{dR},L}(H_{\text{ét}}^1(A/\overline{\mathbf{Q}}, \mathbf{Q}_p))$ by the comparison isomorphism, and let

$$\omega_A^{r-1+j} \eta_A^{r-1-j} \quad (-r < j < r)$$

be the resulting basis for $\mathbf{D}_{\text{dR},L}(\text{Sym}^{2r-2} H_{\text{ét}}^1(A/\overline{\mathbf{Q}}, \mathbf{Q}_p))$, where $\omega_A^{r-1+j} \eta_A^{r-1-j}$ is as in [BDP13, (1.4.6)].

p-adic Gross–Zagier formula. Define the generalized Heegner class $z_{f,\chi}$ attached to (f, χ) by

$$\begin{aligned} z_{f,\chi} &:= \text{cor}_{K_{c_o p^s}/K}(z_{f,\chi, c_o p^s}) \\ &= \sum_{\sigma \in \text{Gal}(K_{c_o p^s}/K)} \chi_t(\sigma) \cdot (\text{id} \otimes e_\chi) z_{f, c_o p^s}^\sigma \\ (4.7) \quad &= \sum_{[\mathfrak{a}] \in \text{Pic } \mathcal{O}_{c_o p^s}} \chi^{\varepsilon_{\text{cyc}}^{1-r}}(\sigma_{\mathfrak{a}}) \cdot (\text{id} \otimes e_\chi) z_{f, \mathfrak{a}}, \end{aligned}$$

where $c_o p^s \mathcal{O}_K$ is the conductor of χ .

Remark 4.8. By [Nek00, Thm. 3.3.1], the classes $z_{f,\chi,c}$ from §4.4 lie in the Bloch–Kato Selmer group $\text{Sel}(K_c, T \otimes \chi) \subset H^1(K_c, T \otimes \chi)$; in particular, $\text{loc}_{\mathfrak{p}}(z_{f,\mathfrak{a}}) \in H_f^1(K_{c,\mathfrak{p}}, T \otimes S^{r-1}(A))$ and $\text{loc}_{\mathfrak{p}}(z_{f,\chi}) \in H_f^1(K_{\mathfrak{p}}, T \otimes \chi)$.

Theorem 4.9. *Suppose $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K . Let ψ be an anticyclotomic Hecke character of infinity type $(r, -r)$ and conductor $c_o \mathcal{O}_K$ with $(c_o, Np) = 1$. If $\widehat{\phi} \in \mathfrak{X}_{p^\infty}$ is the p -adic avatar of an anticyclotomic Hecke character of infinity type $(r+j, -j-r)$ with $-r < j < r$ and conductor $p^n \mathcal{O}_K$ with $n > 1$, then*

$$\frac{\mathcal{L}_{\mathfrak{p},\psi}(f)(\widehat{\phi}^{-1})}{\Omega_p^{-2j}} = \frac{\mathfrak{g}(\phi_p^{-1}) \phi_{\mathfrak{p}}(p^n) c_o^{1-r} \widehat{\psi}_{\mathfrak{p}}^{-1}(p^n)}{(r-1+j)!} \cdot \langle \log_{\mathfrak{p}}(z_{f,\chi}), \omega_f \otimes \omega_A^{r-1+j} \eta_A^{r-1-j} \rangle,$$

where $\chi := \widehat{\psi}^{-1} \widehat{\phi}$ and $\log_{\mathfrak{p}} := \log \circ \text{loc}_{\mathfrak{p}}$.

PROOF. Let $t_{\mathfrak{a}}$ be the Serre–Tate coordinate of $\mathbf{x}_{\mathfrak{a}} := [(A_{\mathfrak{a}}, \eta_{\mathfrak{a}})] \otimes \overline{\mathbb{F}}_p$. Since the Fourier coefficients $\mathbf{a}_n(f^{\flat})$ of f^{\flat} vanish for n divisible by p , we have

$$U_p f^{\flat}(t) := \sum_{\zeta^p=1} \widehat{f}^{\flat}(t\zeta) = 0.$$

This implies that the associated measure $d\widehat{f}_{\mathfrak{a}}^{\flat}$ is supported on \mathbf{Z}_p^{\times} , and hence by Lemma 3.1, that

$$\left(\widehat{f}_{\mathfrak{a}}^{\flat} \otimes \psi_{\mathfrak{p}} \widehat{\phi}^{-1} | [\mathfrak{a}] \right) (t_{\mathfrak{a}}) = \phi(\sigma_{\mathfrak{a}}) \mathbf{N}(\mathfrak{a})^{r+j} \cdot \left(\theta^{-j-r} \widehat{f}^{\flat} \otimes \psi_{\mathfrak{p}} \phi_{\mathfrak{p}}^{-1} \right) (t_{\mathfrak{a}}),$$

where θ is the operator acting on $\mathcal{O}_{\widehat{S}_{\mathbf{x}_{\mathfrak{a}}}}$ as $t_{\mathfrak{a}} \frac{d}{dt_{\mathfrak{a}}}$. Put $\xi := \psi^{-1} \phi$. By Proposition 3.3, we thus find that

$$\begin{aligned} \mathcal{L}_{\mathfrak{p},\psi}(f)(\widehat{\phi}^{-1}) &= \sum_{[\mathfrak{a}] \in \text{Pic } \mathcal{O}_{c_o}} \psi(\mathfrak{a}) \mathbf{N}(\mathfrak{a})^{-r} \cdot \left(\widehat{f}_{\mathfrak{a}}^{\flat} \otimes \psi_{\mathfrak{p}} \widehat{\phi}^{-1} | [\mathfrak{a}] \right) (x_{\mathfrak{a}}) \\ &= \sum_{[\mathfrak{a}] \in \text{Pic } \mathcal{O}_{c_o}} \xi^{-1}(\mathfrak{a}) \mathbf{N}(\mathfrak{a})^j \cdot \left(\theta^{j-r} \widehat{f}^{\flat} \otimes \xi_{\mathfrak{p}}^{-1} \right) (x_{\mathfrak{a}}) \\ &= p^{-n} \mathfrak{g}(\xi_{\mathfrak{p}}^{-1}) \cdot \sum_{[\mathfrak{a}] \in \text{Pic } \mathcal{O}_{c_o}} \xi^{-1}(\mathfrak{a}) \mathbf{N}(\mathfrak{a})^j \cdot \sum_{u \in (\mathbf{Z}/p^n \mathbf{Z})^{\times}} \theta^{-j-r} \widehat{f}^{\flat}(x_{\mathfrak{a}} * \mathbf{n}(up^{-n})) \xi_{\mathfrak{p}}(u). \end{aligned}$$

Since $\theta^{-j-r} \widehat{f}^{\flat}$ is a p -adic modular form of weight $-2j$, we deduce from (2.1) together with (2.5) that

$$\theta^{-j-r} \widehat{f}^{\flat}(x_{\mathfrak{a}} * \mathbf{n}(up^{-n})) = \theta^{-j-r} \widehat{f}^{\flat}(\text{rec}_K(a^{-1} u_{\mathfrak{p}} p_{\mathfrak{p}}^{-n}) x_{c_o p^n}) u^{2j}.$$

From the relations

$$\chi(\text{rec}_K(a)) = \widehat{\xi}(\text{rec}_K(a)) = \xi(a) a_{\mathfrak{p}}^j a_{\overline{\mathfrak{p}}}^{-j}, \quad \varepsilon_{\text{cyc}}(a) = |a|_{\mathbf{A}_K}, \quad \varepsilon_{\text{cyc}}(u_{\mathfrak{p}} p_{\mathfrak{p}}^n) = u,$$

it follows that

$$(4.8) \quad \begin{aligned} \mathcal{L}_{\mathbf{p},\psi}(f)(\widehat{\phi}^{-1}) &= p^{-n} \mathbf{g}(\xi_{\mathbf{p}}^{-1}) c_o^{-j} \cdot \sum_{[a] \in \text{Pic}(\mathcal{O}_{c_o p^n})} \chi_{\text{cyc}}^j(\text{rec}_K(a)) \cdot \theta^{-j-r} \widehat{f}^{\flat}(\text{rec}_K(a) x_{c_o p^n}) \chi_{\mathbf{p}}(p^n) \\ &= \mathbf{g}(\xi_{\mathbf{p}}^{-1}) c_o^j p^{n(j-1)} \chi_{\mathbf{p}}(p^n) \cdot \sum_{\sigma \in \text{Gal}(K_{c_o p^n}/K)} \chi_{\text{cyc}}^j(\sigma) \cdot \theta^{-j-r} \widehat{f}^{\flat}(x_{c_o p^n}^{\sigma}). \end{aligned}$$

On the other hand, if $\sigma = \sigma_{\mathbf{a}}$ with $a \in \widehat{K}^{(pc_o N) \times}$ and $\mathbf{a} = a \widehat{\mathcal{O}}_{c_o p^n}^{\times} \cap K$, then

$$\theta^{-j-r} \widehat{f}^{\flat}(x_{c_o p^n}^{\sigma}) = \theta^{-j-r} \widehat{f}^{\flat}(x_{\mathbf{a}}) = \theta^{-j-r} f^{\flat}(x_{\mathbf{a}}, \widehat{\omega}(\eta_{\mathbf{a},p})),$$

where $\widehat{\omega}(\eta_{\mathbf{a},p})$ is the differential form induced from the p^{∞} -level structure $\eta_{\mathbf{a},p}$ defined in §2.3. For the isogeny $\varphi_{\mathbf{a}} : A \rightarrow A_{\mathbf{a}}$, one can verify that $\deg \varphi_{\mathbf{a}} = c_o p^n |a|_{\mathbf{A}_K}^{-1}$ and

$$\varphi_{\mathbf{a}}^*(\widehat{\omega}(\eta_{\mathbf{a},p})) = c_o \cdot \widehat{\omega}(\eta_{\mathcal{O}_K,p}) = \frac{c_o}{\Omega_p} \cdot \omega_A.$$

Thus following the calculations in Proposition 3.24, Lemma 3.23, and Lemma 3.22 of [BDP13], we see that

$$(4.9) \quad \theta^{-j-r} \widehat{f}^{\flat}(x_{c_o p^n}^{\sigma}) = \frac{(c_o p^n |a|_{\mathbf{A}_K}^{-1})^{-j-r-1}}{(r_1 + j)!} \cdot \left(\frac{c_o}{\Omega_p} \right)^{2j} \cdot \langle \log_{\mathbf{p}}(z_{f,\mathbf{a}}^{\flat}), \omega_f \otimes \omega_A^{r-1+j} \eta_A^{r-1-j} \rangle,$$

where

$$z_{f,\mathbf{a}}^{\flat} := z_{f,\mathbf{a}} - \mathbf{a}_p(f) p^{2j} \cdot z_{f,\mathbf{a} \mathcal{O}_{c_o p^{n-1}}} - p^{2j+1} \cdot z_{f,\mathbf{a} \mathcal{O}_{c_o p^{n-2}}}.$$

Substituting (5.6) into (4.8), and using that ϕ has the exact conductor p^n ($n > 1$) and $\psi_{\mathbf{p}}$ is unramified, we conclude that

$$\begin{aligned} \frac{\mathcal{L}_{\mathbf{p},\psi}(f)(\widehat{\phi}^{-1})}{\Omega_p^{2j}} &= \frac{\mathbf{g}(\xi_{\mathbf{p}}^{-1}) c_o^{1-r} p^{-nr} \chi_{\mathbf{p}}(p^n)}{(r-1+j)!} \cdot \sum_{\sigma \in \text{Gal}(K_{c_o p^n}/K)} \chi_{\text{cyc}}^{1-r}(\sigma_{\mathbf{a}}) \cdot \langle \log_{\mathbf{p}}(z_{f,\mathbf{a}}^{\flat}), \omega_f \otimes \omega_A^{r-1+j} \eta_A^{r-1-j} \rangle \\ &= \frac{\mathbf{g}(\phi_{\mathbf{p}}^{-1}) \phi_{\mathbf{p}}(p^n) c_o^{1-r} \widehat{\psi}_{\mathbf{p}}^{-1}(p^n)}{(r-1+j)!} \cdot \sum_{[a] \in \text{Pic} \mathcal{O}_c} \chi_{\text{cyc}}^{1-r}(\sigma_{\mathbf{a}}) \cdot \langle \log_{\mathbf{p}}(z_{f,\mathbf{a}}), \omega_f \otimes \omega_A^{r-1+j} \eta_A^{r-1-j} \rangle \\ &= \frac{\mathbf{g}(\phi_{\mathbf{p}}^{-1}) \phi_{\mathbf{p}}(p^n) c_o^{1-r} \widehat{\psi}_{\mathbf{p}}^{-1}(p^n)}{(r-1+j)!} \cdot \langle \log_{\mathbf{p}}(z_{f,\chi}), \omega_f \otimes \omega_A^{r-1+j} \eta_A^{r-1-j} \rangle \end{aligned}$$

as was to be shown. \square

5. EXPLICIT RECIPROCITY LAW

5.1. The Perrin-Riou big logarithm. In this section we deduce from the main result of [LZ14] the construction of a variant of the Perrin-Riou logarithm map for certain relative height one Lubin–Tate extensions.

For any commutative compact p -adic Lie group G and any complete discretely valued extension E of \mathbf{Q}_p , we let $\Lambda_{\mathcal{O}_E}(G) := \varprojlim_n \mathcal{O}_E[G/G^{p^n}]$, $\Lambda_E(G) := \Lambda_{\mathcal{O}_E}(G) \otimes_{\mathcal{O}_E} E$, and $\mathcal{H}_E(G)$ be the ring of tempered p -adic distributions on G valued in E . If L is a finite extension of \mathbf{Q}_p and G is the Galois group of a p -adic Lie extension of $L_{\infty} = \cup_n L_n$ of L with L_n/L finite and Galois, we define

$$H_{\text{Iw}}^i(L_{\infty}, V) := \left(\varprojlim_n H^i(L_n, T) \right) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

where T is any G_L -stable lattice in V (this is independent of the choice of T).

In the following, we let L be a finite unramified extension of \mathbf{Q}_p with ring of integers \mathcal{O}_L , and let \widehat{F}^{ur} denote the composite of $\widehat{\mathbf{Q}}_p^{\text{ur}}$ with a finite extension F of \mathbf{Q}_p . We also let $t \in \mathbf{B}_{\text{dR}}$ be Fontaine's p -adic analogue of $2\pi i$ associated with the compatible system $\{\zeta_{p^n}\}_{n=1,2,\dots}$ of p -power roots of unity.

Theorem 5.1. *Let V be a crystalline F -representation of G_L with non-negative Hodge–Tate weights, and assume that V has no quotient isomorphic to the trivial representation. Let \mathfrak{F} be a relative height*

one Lubin–Tate formal group over $\mathcal{O}_L/\mathbf{Z}_p$, and let $\Gamma := \text{Gal}(L(\mathfrak{F}_{p^\infty})/L) \simeq \mathbf{Z}_p^\times$. If $V^{G_{L(\mathfrak{F}_{p^\infty})}} = 0$, there exists a $\Lambda_{\mathbf{Z}_p}(\Gamma)$ -linear map

$$\mathcal{L}_V : H_{\text{Iw}}^1(L(\mathfrak{F}_{p^\infty}), V) \longrightarrow \mathcal{H}_{\widehat{F}_{\text{ur}}}(\Gamma) \otimes_L \mathbf{D}_{\text{cris}, L}(V)$$

with the following interpolation property: for any $\mathbf{z} \in H_{\text{Iw}}^1(L(\mathfrak{F}_{p^\infty}), V)$ and any locally algebraic character $\chi : \Gamma \rightarrow \overline{\mathbf{Q}}_p^\times$ of Hodge–Tate weight j and conductor p^n , we have

$$\mathcal{L}_V(\mathbf{z})(\chi) = \varepsilon(\chi^{-1}) \cdot \frac{\Phi^n P(\chi^{-1}, \Phi)}{P(\chi, p^{-1}\Phi^{-1})} \cdot \begin{cases} \frac{(-1)^{-j-1}}{(-j-1)!} \cdot \log_{L, V(\chi^{-1})}(\mathbf{z}^{\chi^{-1}}) \otimes t^{-j} & \text{if } j < 0, \\ j! \cdot \exp_{L, V(\chi^{-1})^*(1)}^*(\mathbf{z}^{\chi^{-1}}) \otimes t^{-j} & \text{if } j \geq 0, \end{cases}$$

where

- $\varepsilon(\chi^{-1})$ and $P(\chi^\pm, X)$ are the epsilon-factor and the L -factor for Galois characters χ and χ^\pm , respectively (see [LZ14, §2.8] for the definitions).
- Φ is the crystalline Frobenius operator on $\overline{\mathbf{Q}}_p \otimes_L \mathbf{D}_{\text{cris}, L}(V)$ acting trivially on the first factor.
- $\mathbf{z}^{\chi^{-1}} \in H^1(L, V(\chi^{-1}))$ is the specialisation of \mathbf{z} at χ^{-1} .

PROOF. Let $K_\infty \subset L \cdot \mathbf{Q}_p^{\text{ab}}$ be a p -adic Lie extension of F containing $\widehat{F}_{\text{ur}} \cdot L(\mathfrak{F}_{p^\infty})$, and set $G := \text{Gal}(K_\infty/L)$. By [LZ14, Thm. 4.7] there exists a $\Lambda_{\mathcal{O}_L}(G)$ -linear map

$$\mathcal{L}_V^G : H_{\text{Iw}}^1(K_\infty, V) \longrightarrow \mathcal{H}_{\widehat{F}_{\text{ur}}}(G) \otimes_L \mathbf{D}_{\text{cris}, L}(V)$$

satisfying the above interpolation formula for all continuous characters χ of G (see [*loc. cit.*, Thm. 4.15]). Let \mathcal{J} be the kernel of the natural projection $\mathcal{H}_{\widehat{F}_{\text{ur}}}(G) \rightarrow \mathcal{H}_{\widehat{F}_{\text{ur}}}(\Gamma)$. The corestriction map

$$H_{\text{Iw}}^1(K_\infty, V)/\mathcal{J} \longrightarrow H_{\text{Iw}}^1(L(\mathfrak{F}_{p^\infty}), V)$$

is injective, and its cokernel is $H_{\text{Iw}}^2(K_\infty, V)[\mathcal{J}]$, which vanishes if $V^{G_{L(\mathfrak{F}_{p^\infty})}} = 0$. Thus quotienting \mathcal{L}_V^G by \mathcal{J} we obtain a map

$$H_{\text{Iw}}^1(L(\mathfrak{F}_{p^\infty}), V) \cong H_{\text{Iw}}^1(K_\infty, V)/\mathcal{J} \longrightarrow \mathcal{H}_{\widehat{F}_{\text{ur}}}(\Gamma) \otimes_L \mathbf{D}_{\text{cris}, L}(V)$$

with the desired properties. \square

5.2. Iwasawa cohomology classes. Keep the notations from §4.1, and for any positive integer c , let $\Sigma = \Sigma_c$ be a finite set of places of K containing the primes above pNc . Recall the Heegner classes $z_{f, \mathfrak{a}} \in H^1(K_c, T \otimes S^{r_1}(A))$ of (4.2) attached to every integral \mathcal{O}_c -ideal \mathfrak{a} .

In this section we further assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K and that the newform f is ordinary at p , i.e. the p -th Fourier coefficient $\mathfrak{a}_p(f) \in \mathcal{O}_F^\times$. The latter assumption will be crucial to construct, out of the classes $z_{f, cp^n} = z_{f, \mathcal{O}_{cp^n}}$ for varying n , elements in the Iwasawa cohomology groups

$$H_{\text{Iw}}^1(K_{cp^\infty}, T) := \varprojlim_n H^1(\text{Gal}(K^\Sigma/K_{cp^n}), T),$$

where K^Σ is the maximal extension of K unramified outside Σ .

Definition 5.2. Let α be the p -adic unit root of $X^2 - \mathfrak{a}_p(f)X + p^{2r-1}$. The α -stabilized Heegner class $z_{f, \mathfrak{a}, \alpha} \in H^1(K_c, T \otimes S^{r-1}(A))$ is given by

$$z_{f, \mathfrak{a}, \alpha} := \begin{cases} z_{f, \mathfrak{a}} - \frac{p^{2r-2}}{\alpha} \cdot z_{f, \mathfrak{a}\mathcal{O}_{c/p}} & \text{if } p \mid c, \\ \frac{1}{u_c} \left(1 - \frac{p^{r-1}}{\alpha} \sigma_{\mathfrak{p}}\right) \left(1 - \frac{p^{r-1}}{\alpha} \sigma_{\bar{\mathfrak{p}}}\right) \cdot z_{f, \mathfrak{a}} & \text{if } p \nmid c, \end{cases}$$

where $u_c = \#\mathcal{O}_c^\times$ and $\sigma_{\mathfrak{p}}, \sigma_{\bar{\mathfrak{p}}} \in \text{Gal}(K_c/K)$ are the Frobenius elements of \mathfrak{p} and $\bar{\mathfrak{p}}$.

Lemma 5.3. For all $c \geq 1$, we have

$$\text{cor}_{K_{cp}/K_c}(z_{f, cp, \alpha}) = \alpha \cdot z_{f, c, \alpha}.$$

PROOF. This follows from a straightforward computation using Proposition 4.4. \square

Now let $z_{f,a}^o$ denote the image of $z_{f,a}$ under the natural map

$$\mathrm{id} \otimes e^o : H^1(K_c, T \otimes S^{r-1}(B)) \longrightarrow H^1(K_c, T),$$

where $e^o = e_1$ is the projection (4.5) attached to the trivial character (so $\chi = \chi_t = \mathbf{1}$). Similarly as before, we shall simply write $z_{f,c,\alpha}^o$ for $z_{f,\mathcal{O}_{c,\alpha}}^o$. In view of Lemma 5.3, the classes $\alpha^{-n} \cdot z_{f,cp^n,\alpha}^o$ are compatible under corestriction, thus defining the Iwasawa cohomology class

$$(5.1) \quad \mathbf{z}_{f,c,\alpha} := \varprojlim_n \alpha^{-n} \cdot z_{f,cp^n,\alpha}^o \in H_{\mathrm{Iw}}^1(K_{cp^\infty}, T).$$

For any character χ of $\mathrm{Gal}(K_{cp^\infty}/K_c)$ we may consider the twist of $\mathbf{z}_{f,c,\alpha}$ in $H^1(K_c, T \otimes \chi)$. The next lemma compares the resulting classes, for characters χ of *finite* order, to the classes $z_{f,\chi,c}$ of §4.4.

Lemma 5.4. Suppose that $p \nmid c$. Let $\chi : \mathrm{Gal}(K_{cp^\infty}/K_c) \rightarrow \mathcal{O}_{\mathbf{C}_p}^\times$ be a nontrivial finite order character of conductor cp^n , and let $\mathbf{z}_{f,c,\alpha}^\chi$ be the image of $\mathbf{z}_{f,c,\alpha}$ under the χ -specialization map

$$H_{\mathrm{Iw}}^1(K_{cp^\infty}, T) \longrightarrow H^1(K_c, T \otimes \chi).$$

Then

$$\mathbf{z}_{f,c,\alpha}^\chi = \alpha^{-n} \cdot z_{f,\chi,c}.$$

PROOF. Directly from the definition of $\mathbf{z}_{f,c,\alpha}$, by [Rub00, Lemma 2.4.3] we see that

$$\mathbf{z}_{f,c,\alpha}^\chi = \alpha^{-n} \sum_{\sigma \in \mathrm{Gal}(K_{cp^n}/K_c)} \chi(\sigma) (z_{f,cp^n,\alpha}^o)^\sigma,$$

and since χ is nontrivial, we may replace $z_{f,cp^n,\alpha}^o$ by z_{f,cp^n}^o in this equation. By Proposition 4.5 (noting that e_χ can be taken to be e^o with $\chi_t = \chi$), the result follows from the definition (4.6) of $z_{f,\chi,c}$. \square

5.3. Explicit reciprocity law for generalized Heegner cycles. We now specialize the local machinery of §5.1 to the global setting in §5.2. In particular, we assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K and that the newform $f \in S_{2r}^{\mathrm{new}}(\Gamma_0(N))$ is ordinary at p .

Let ψ be an anticyclotomic Hecke character of infinity type $(r, -r)$ and conductor $c_o \mathcal{O}_K$ with $p \nmid c_o$. Recall that the p -adic avatar $\widehat{\psi}$ is a p -adic character of $\mathrm{Gal}(K_{c_o p^\infty}/K)$ valued in some finite extension \mathbf{Q}_p which by the hypothesis on the conductor is crystalline at the primes above p . Let F be a finite extension of \mathbf{Q}_p containing the Fourier coefficients of f and the values of $\widehat{\psi}$, and let $V_f \cong F^2$ be the Galois representation associated to f . We assume throughout that $p \nmid N$, so that $V_f|_{G_{\mathbf{Q}_p}}$ is crystalline.

By p -ordinarity, there is an exact sequence of $G_{\mathbf{Q}_p}$ -modules

$$0 \longrightarrow \mathcal{F}^+ V_f \longrightarrow V_f \longrightarrow \mathcal{F}^- V_f \longrightarrow 0$$

with $\mathcal{F}^\pm V_f \cong F$ and with the $G_{\mathbf{Q}_p}$ -action on $\mathcal{F}^+ V_f$ being unramified (see [Wil88, Thm. 2.1.4]). Let $T \subset V_f(r)$ be a $G_{\mathbf{Q}_p}$ -stable lattice as in §4.2, and set $\mathcal{F}^+ T := \mathcal{F}^+ V_f(r) \cap T$. Let

$$V := V_f(r) \otimes \widehat{\psi}^{-1}|_{G_{K_p}}, \quad \widehat{\psi}_p := \widehat{\psi}|_{G_{K_p}}.$$

The dual representation V^* is $\mathrm{Hom}_F(V, F) = V_f(r-1) \otimes \widehat{\psi}_p$. Define

$$\mathcal{F}^\pm V := \mathcal{F}^\pm V_f(r) \otimes \widehat{\psi}_p^{-1}, \quad \mathcal{F}^\mp V^* := \mathrm{Hom}_F(\mathcal{F}^\pm V, F).$$

We next introduce an element $\omega_{f,\psi} \in \mathbf{D}_{\mathrm{cris},L}(\mathcal{F}^- V^*)$. Recall that A is the canonical CM elliptic curve over the Hilbert class field H_K fixed in §4.1. Let $\kappa_A : G_{H_K} \rightarrow \mathrm{Aut} T_p(A) \cong \mathbf{Z}_p^\times$ be the character describing the Galois action on the \mathfrak{p} -adic Tate module of A . Thus $H_{\mathrm{et}}^1(A/\bar{\mathbf{Q}}, \mathbf{Q}_p) \cong \kappa_A^{-1} \oplus \kappa_A \varepsilon_{\mathrm{cyc}}^{-1}$ as G_{H_K} -modules. Recall that $t \in \mathbf{B}_{\mathrm{dR}}$ denotes Fontaine's p -adic analogue of $2\pi i$ and set

$$t_A := \Omega_p t,$$

where Ω_p is the p -adic CM period defined in §2.5. Then t_A generates $\mathbf{D}_{\mathrm{cris},F}(\kappa_A^{-1})$, and according to the discussion in [dS87, §II.4.3] we have

$$(5.2) \quad \omega_A = t_A, \quad \eta_A = t_A^{-1} t.$$

On the other hand, note that the character $\widehat{\psi}_{\mathfrak{p}} \varepsilon_{\text{cyc}}^{-r}$ is trivial on the inertia group, and $\mathbf{D}_{\text{cris}}(\widehat{\psi}_{\mathfrak{p}}(-r)) = F\omega_{\psi}$ is a one-dimensional F -vector space with generator ω_{ψ} . Define the class

$$\omega_{f,\psi} := \omega_f \otimes t^{1-2r} \otimes \omega_{\psi} \in \mathbf{D}_{\text{cris}}(V^*).$$

With a slight abuse of notation, we shall still denote by $\omega_{f,\psi}$ its image under the natural projection $\mathbf{D}_{\text{cris}}(V^*) \rightarrow \mathbf{D}_{\text{cris}}(\mathcal{F}^- V^*)$, which is nonzero by weak-admissibility ([Fon94, §3.3]). Moreover, since the periods of unramified characters lie in $F^{\text{ur}} := \widehat{\mathbf{Q}}_p^{\text{ur}} F \subset \mathbf{B}_{\text{cris}}$, there exists a non-zero element $\Omega_{\psi} \in \widehat{F}^{\text{ur}}$ such that, for all $x \in \mathbf{D}_{\text{cris}}(\mathcal{F}^+ V)$, we have

$$(5.3) \quad \langle x, \omega_{f,\psi} \rangle = \langle x, \omega_f \otimes t^{1-2r} \rangle \Omega_{\psi},$$

and the action of the crystalline Frobenius Φ is given by

$$(5.4) \quad \langle x, \Phi \omega_{f,\psi} \rangle = \alpha^{-1} \widehat{\psi}_{\mathfrak{p}}(p) \cdot \langle x, \omega_{f,\psi} \rangle.$$

Let L_{∞}/L denote the \mathfrak{p} -adic completion of $K_{c_o p^{\infty}}/K_{c_o}$ and let $\Gamma := \text{Gal}(L_{\infty}/L)$. Let h_p be the order of \mathfrak{p} in $\text{Pic}(\mathcal{O}_{c_o})$, and write $\mathfrak{p}^{h_p} = (\pi)$ with $\pi \in \mathcal{O}_{c_o}$. Then L is the unramified extension of \mathbf{Q}_p of degree h_p . By local class field theory, L_{∞} is contained in the extension $L(\mathfrak{F}_{p^{\infty}})$ obtained by adjoining to L the torsion points of the relative height one Lubin–Tate formal group \mathfrak{F} attached to the uniformizer $\pi/\bar{\pi}$ (see [Shn16, Prop. 37] for details). Note that the element $\text{rec}_{\mathfrak{p}}(\pi/\bar{\pi})$ fixes $L(\mathfrak{F}_{p^{\infty}})$ and acts on $\mathcal{F}^+ V$ by a multiplication by $(\frac{\pi}{\bar{\pi} p^{h_p}})^r \alpha^{h_p}$, which is not 1 by Ramanujan’s conjecture for f [Del71], [Del80]. This implies that $(\mathcal{F}^+ V)^{G_{L(\mathfrak{F}_{p^{\infty}})}} = 0$, and hence we may consider the big logarithm map $\mathcal{L}_{\mathcal{F}^+ V}$ of Theorem 5.1 over the extension L_{∞}/L .

Lemma 5.5. The composition of $\mathcal{L}_{\mathcal{F}^+ V}$ with the natural pairing

$$\langle -, \omega_{f,\psi} \rangle : \mathcal{H}_{\widehat{F}^{\text{ur}}}(\Gamma) \otimes \mathbf{D}_{\text{cris}}(\mathcal{F}^+ V) \times \mathbf{D}_{\text{cris}}(\mathcal{F}^- V^*) \longrightarrow \mathcal{H}_{\widehat{F}^{\text{ur}}}(\Gamma)$$

has image contained in the Iwasawa algebra $\Lambda_{\widehat{F}^{\text{ur}}}(\Gamma)$.

PROOF. This follows easily from the Frobenius eigenvalue formula (5.4) and [LZ14, Prop. 4.8]. \square

In what follows, we make the identification $\text{Gal}(K_{c_o p^{\infty}}/K_{c_o}) \simeq \Gamma = \text{Gal}(L_{\infty}/L)$ via the restriction map. Let $\rho : \Gamma \rightarrow \mathcal{W}^{\times}$ be a continuous character, where \mathcal{W} is the ring of the integers in \widehat{F}^{ur} . For every $\mathbf{z} \in H_{\text{Iw}}^1(K_{c_o p^{\infty}}, T)$, denote by $\mathbf{z} \otimes \rho \in H_{\text{Iw}}^1(K_{c_o p^{\infty}}, T \otimes \rho)$ the ρ -twist of \mathbf{z} . By definition, for any $\chi : \Gamma \rightarrow \mathcal{W}^{\times}$, we have

$$(\mathbf{z} \otimes \rho)^{\chi} = \mathbf{z}^{\rho \chi} \in H^1(K_{c_o}, T \otimes \rho \chi).$$

As shown in [LZ16, Prop. 2.4.2], there is an isomorphism $H_{\text{Iw}}^1(K_{c_o p^{\infty}}, T) \simeq H^1(K_{c_o}, T \otimes \Lambda_{\mathcal{O}_F}(\Gamma))$. Thus letting $\widetilde{\Gamma}_{c_o} := \text{Gal}(K_{c_o p^{\infty}}/K)$ we may view $\mathbf{z}_{f,c_o,\alpha}$ as an element in $H^1(K_{c_o}, \Lambda_{\mathcal{O}_F}(\widetilde{\Gamma}_{c_o}))$ via

$$H_{\text{Iw}}^1(K_{c_o p^{\infty}}, T) \simeq H^1(K_{c_o}, \Lambda_{\mathcal{O}_F}(\Gamma)) \longrightarrow H^1(K_{c_o}, \Lambda_{\mathcal{O}_F}(\widetilde{\Gamma}_{c_o})),$$

and define

$$(5.5) \quad \mathbf{z}_f := \text{cor}_{K_{c_o}/K}(\mathbf{z}_{f,c_o,\alpha}) \in H^1(K, T \otimes \Lambda_{\mathcal{O}_F}(\widetilde{\Gamma}_{c_o})).$$

Similarly as in §4.5 (see Remark 4.8), the Heegner classes $z_{f,\mathfrak{a}}^{\rho}$ lie in the Bloch–Kato Selmer group $\text{Sel}(K_c, T) \subset H^1(K_c, T)$; in particular, $\text{loc}_{\mathfrak{p}}(z_{f,\mathfrak{a}}^{\rho}) \in H_f^1(K_{c,\mathfrak{p}}, T)$. On the other hand, by [Nek06, Lem. 9.6.3] and [loc.cit., Prop. 12.5.9.2] the Bloch–Kato finite subspace $H_f^1(K_{c,\mathfrak{p}}, T)$ is identified with the image of the natural map $H^1(K_{c,\mathfrak{p}}, \mathcal{F}^+ T) \rightarrow H^1(K_{c,\mathfrak{p}}, T)$, and hence $\text{loc}_{\mathfrak{p}}(\mathbf{z}_{f,c_o,\alpha})$ naturally defines a class in $H_{\text{Iw}}^1(L_{\infty}, \mathcal{F}^+ T)$.

Definition 5.6 (Algebraic anticyclotomic p -adic L -functions). Let $\widehat{\psi} : \widetilde{\Gamma}_{c_o} \rightarrow \mathcal{O}_F^{\times}$ be as before. Set

$$\begin{aligned} \mathcal{L}_{\mathfrak{p}}^*(\mathbf{z}_f \otimes \widehat{\psi}^{-1}) &:= \text{cor}_{K_{c_o}/K}(\mathcal{L}_{\mathcal{F}^+ V}(\text{loc}_{\mathfrak{p}}(\mathbf{z}_{f,c_o,\alpha} \otimes \widehat{\psi}^{-1}))) \\ &= \sum_{\sigma \in \widetilde{\Gamma}_{c_o}/\Gamma_{c_o}} \mathcal{L}_{\mathcal{F}^+ V}(\text{loc}_{\mathfrak{p}}(\mathbf{z}_{f,c_o,\alpha}^{\sigma} \otimes \widehat{\psi}^{-1})) \widehat{\psi}(\sigma^{-1}) \in \mathbf{D}_{\text{cris}}(\mathcal{F}^+ V) \otimes \Lambda_{\widehat{F}^{\text{ur}}}(\widetilde{\Gamma}_{c_o}), \end{aligned}$$

and letting $\text{res}_{K_{p^{\infty}}} : \widetilde{\Gamma}_{c_o} \rightarrow \widetilde{\Gamma} = \text{Gal}(K_{p^{\infty}}/K)$ be the restriction map, define

$$\mathcal{L}_{\mathfrak{p},\psi}(\mathbf{z}_f) := \text{res}_{K_{p^{\infty}}}(\mathcal{L}_{\mathfrak{p}}^*(\mathbf{z}_f \otimes \widehat{\psi}^{-1})) \in \mathbf{D}_{\text{cris}}(\mathcal{F}^+ V) \otimes \Lambda_{\widehat{F}^{\text{ur}}}(\widetilde{\Gamma}).$$

Theorem 5.7. *Suppose $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K . Let $f \in S_{2r}^{\text{new}}(\Gamma_0(N))$ with $p \nmid N$ be a p -ordinary newform, and let ψ be an anticyclotomic Hecke character of infinity type $(r, -r)$ and conductor $c_o\mathcal{O}_K$ with $p \nmid c_o$. Then*

$$\langle \mathcal{L}_{\mathfrak{p},\psi}(\mathbf{z}_f), \omega_f \otimes t^{1-2r} \rangle = (-c_o^{r-1}) \cdot \mathcal{L}_{\mathfrak{p},\psi}(f) \cdot \sigma_{-1,\mathfrak{p}} \in \Lambda_{\widehat{F}_{\text{ur}}}(\widetilde{\Gamma}),$$

where $\sigma_{-1,\mathfrak{p}} := \text{rec}_{\mathfrak{p}}(-1)|_{K_{p^\infty}} \in \widetilde{\Gamma}$ is an element of order two.

PROOF. Let $\widehat{\phi} : \widetilde{\Gamma} \rightarrow \mathbf{C}_p^\times$ be the p -adic avatar of a Hecke character ϕ of infinity type $(r, -r)$ and conductor p^n , for any $n > 1$, and set $\chi := \widehat{\psi}^{-1}\widehat{\phi}$, which is a finite order character. Applying Lemma 5.4, we find that $\mathbf{z}_f^\chi = \alpha^{-n} \cdot z_{f,\chi}$, where \mathbf{z}_f^χ denotes the χ -specialization of \mathbf{z}_f . By Theorem 4.9 (with $j = 0$), we thus obtain

$$\begin{aligned} \mathcal{L}_{\mathfrak{p},\psi}(f)(\widehat{\phi}^{-1}) &= \frac{\mathfrak{g}(\phi_{\mathfrak{p}}^{-1})\phi_{\mathfrak{p}}(p^n)c_o^{1-r}\widehat{\psi}_{\mathfrak{p}}^{-1}(p^n)}{(r-1)!} \cdot \langle \log_{\mathfrak{p}}(z_{f,\chi}), \omega_f \otimes t^{1-r} \rangle \\ (5.6) \quad &= \alpha^n \cdot \frac{\mathfrak{g}(\phi_{\mathfrak{p}}^{-1})\phi_{\mathfrak{p}}(p^n)c_o^{1-r}\widehat{\psi}_{\mathfrak{p}}^{-1}(p^n)}{(r-1)!} \cdot \langle \log_{\mathfrak{p}}(\mathbf{z}_f^\chi) \otimes t^r, \omega_f \otimes t^{1-2r} \rangle. \end{aligned}$$

On the other hand, a straightforward calculation reveals that the ε -factor for the p -adic Galois character $\widehat{\phi}_{\mathfrak{p}}$ defined in [LZ14, §2.8] agrees with Tate's ε -factor for $\phi_{\mathfrak{p}}$, i.e. $\varepsilon(\widehat{\phi}_{\mathfrak{p}}) = \varepsilon(0, \phi_{\mathfrak{p}}) = \mathfrak{g}(\phi_{\mathfrak{p}}^{-1})\phi_{\mathfrak{p}}(-p^n)$. Therefore, by Theorem 5.1 combined with (5.3) and (5.4), we find that

$$\begin{aligned} &\langle \mathcal{L}_{\mathfrak{p},\psi}(\mathbf{z}_f), \omega_f \otimes t^{1-2r} \rangle (\widehat{\phi}^{-1}) \\ &= \langle \mathcal{L}_{\mathfrak{p},\psi}(\mathbf{z}_f), \omega_{f,\psi} \rangle (\widehat{\phi}^{-1}) \cdot \Omega_{\psi}^{-1} \\ (5.7) \quad &= \mathfrak{g}(\phi_{\mathfrak{p}}^{-1})\phi_{\mathfrak{p}}(-p^n) \cdot \alpha^n \widehat{\psi}_{\mathfrak{p}}^{-1}(p^n) \cdot \frac{(-1)^{r-1}}{(r-1)!} \cdot \langle \log_{\mathfrak{p}}(\mathbf{z}_f^\chi) \otimes t^r, \omega_{f,\psi} \rangle \Omega_{\psi}^{-1} \\ &= -\widehat{\phi}(\sigma_{-1,\mathfrak{p}})\alpha^n \cdot \frac{\mathfrak{g}(\phi_{\mathfrak{p}}^{-1})\phi_{\mathfrak{p}}(p^n)\widehat{\psi}_{\mathfrak{p}}^{-1}(p^n)}{(r-1)!} \cdot \langle \log_{\mathfrak{p}}(\mathbf{z}_f^\chi) \otimes t^r, \omega_f \otimes t^{1-2r} \rangle. \end{aligned}$$

Since ψ has conductor prime to p , we have $\mathfrak{g}(\phi_{\mathfrak{p}}^{-1}) = \mathfrak{g}(\chi_{\mathfrak{p}}^{-1})$ in formula (5.7). Comparing (5.6) and (5.7), we see that both sides of the desired equality agree when evaluated at $\widehat{\phi}^{-1}$. Since the set of all such characters $\widehat{\phi}$ (for varying $n > 1$) is Zariski-dense in the space of continuous p -adic characters of $\widetilde{\Gamma}$, and both sides of the desired equality are elements in the Iwasawa algebra $\Lambda_{\widehat{F}_{\text{ur}}}(\widetilde{\Gamma})$, the result follows from the p -adic Weierstrass preparation theorem. \square

We are now ready to prove the ‘‘explicit reciprocity law’’ relating the image of generalized Heegner classes under the dual exponential map to the central values of the Rankin L -series $L(f, \chi, s)$ associated with f and the theta series of an anticyclotomic locally algebraic Galois character χ of conductor $c\mathcal{O}_K$. Recall that $L(f, \chi, s)$ is defined by the analytic continuation of the Dirichlet series

$$L(f, \chi, s) = \zeta(2s + 1 - 2r) \sum_{\mathfrak{a}} \frac{\mathfrak{a}_{\mathbf{N}(\mathfrak{a})}(f)\chi(\sigma_{\mathfrak{a}})}{\mathbf{N}(\mathfrak{a})^s} \quad (\text{Re}(s) > r + \frac{1}{2}),$$

where \mathfrak{a} runs over ideals of \mathcal{O}_K with $(\mathfrak{a}, c\mathcal{O}_K) = 1$. In terms of automorphic L -functions, we have

$$L(f, \chi, s) = L(s + \frac{1}{2} - r, \pi_K \otimes \chi_{\mathbf{A}}),$$

where π_K is the base change of the automorphic representation π generated by f , and $\chi_{\mathbf{A}}$ is the Hecke character of K^\times associated to χ . Also, recall from (5.2) the relation $\omega_A \eta_A = t$.

Corollary 5.8. *With notations and assumptions as in Theorem 5.7, let $\chi : \text{Gal}(K_{c_o p^\infty}/K) \rightarrow \mathcal{O}_F^\times$ be a locally algebraic p -adic character of infinity type $(j, -j)$ with $j \geq r$ and conductor $c_o p^n \mathcal{O}_K$. Then*

$$\langle \exp^*(\text{loc}_{\mathfrak{p}}(\mathbf{z}_f^{\chi^{-1}})), \omega_f \otimes \omega_A^{-j-r+1} \eta_A^{j-r+1} \rangle^2 = c_{f,K} \cdot e'_{\mathfrak{p}}(f, \chi)^2 \cdot (p^{2r-1}/\alpha^2)^n \cdot \chi^{-1}\psi(\mathfrak{N}) \cdot \frac{L^{\text{alg}}(f, \chi, r)}{\Gamma(j-r+1)^2},$$

where $c_{f,K} = 8u_K^2 \sqrt{D_K} c_o^{2r-1} \varepsilon(f)$,

$$e'_p(f, \chi) = \begin{cases} (1 - \alpha^{-1} \chi(\sigma_p) p^{r-j-1}) (1 - \alpha^{-1} \chi(\sigma_{\bar{p}}) p^{r-j-1}) & \text{if } n = 0, \\ 1 & \text{if } n > 0; \end{cases}$$

$$L^{\text{alg}}(f, \chi, r) = \frac{\Gamma(j-r+1)\Gamma(j+r)}{(4\pi)^{2j+1}(\text{Im } \vartheta)^{2j}} \cdot L(f, \chi, r).$$

PROOF. Choose an anticyclotomic Hecke character ψ of infinity type $(r, -r)$ and conductor c_o such that the character $\widehat{\phi} = \chi \widehat{\psi}^{-1}$ is of infinity type $(j-r, r-j)$ and conductor p^n . Assume first that $n > 0$. By Theorem 5.7 and Theorem 5.1, we then see that

$$(5.8) \quad \begin{aligned} & \langle \mathcal{L}_{p,\psi}(\mathbf{z}_f), \omega_f \otimes t^{1-2r} \rangle (\widehat{\phi}) \\ &= \mathfrak{g}(\phi_p) \phi_p(-p^n) \cdot \alpha^n \widehat{\psi}_p^{-1}(p^n) \cdot (j-r)! \cdot \langle \exp^*(\text{loc}_p(\mathbf{z}_f^{\chi^{-1}})) \otimes t^{r-j}, \omega_f \otimes t^{1-2r} \rangle \\ &= \pm \alpha^n \cdot \varepsilon(0, \phi_p^{-1} \psi_p^{-1}) p^{-nr} \cdot (j-r)! \cdot \langle \exp^*(\text{loc}_p(\mathbf{z}_f^{\chi^{-1}})), \omega_f \otimes t^{1-r-j} \rangle \\ &= \pm \alpha^n \cdot \varepsilon(0, \phi_p^{-1} \psi_p^{-1}) p^{-nr} \cdot \Gamma(j-r+1) \cdot \langle \exp^*(\text{loc}_p(\mathbf{z}_f^{\chi^{-1}})), \omega_f \otimes t^{1-r} \otimes \omega_A^{-j} \eta_A^j \rangle \Omega_p^{2j}. \end{aligned}$$

On the other hand, by the interpolation formula in Proposition 3.8 (with $m = j-r$), we have

$$(5.9) \quad \left(\frac{\mathcal{L}_{p,\psi}(f)(\widehat{\phi})}{\Omega_p^{2j}} \right)^2 = L^{\text{alg}}(f, \chi, r) \cdot \varepsilon\left(\frac{1}{2}, \psi_p \phi_p\right)^{-2} \cdot \phi(\mathfrak{N}^{-1}) \cdot 2^3 u_K^2 \sqrt{D_K} c_o \varepsilon(f)$$

where

$$\varepsilon\left(\frac{1}{2}, \psi_p \phi_p\right)^{-2} = \varepsilon\left(\frac{1}{2}, \psi_p^{-1} \phi_p^{-1}\right)^2 = \varepsilon(0, \psi_p^{-1} \phi_p^{-1})^2 p^{-n}.$$

Combining (5.8) and (5.9) with the equality in Theorem 5.7, we find that

$$\langle \exp^*(\text{loc}_p(\mathbf{z}_f^{\chi^{-1}})), \omega_f \otimes t^{1-r} \otimes \omega_A^{-j} \eta_A^j \rangle^2 = \frac{L^{\text{alg}}(f, \chi, r)}{\Gamma(j-r+1)^2} \cdot (p^{2r-1}/\alpha^2)^n \cdot \phi(\mathfrak{N}^{-1}) \cdot 2^3 u_K^2 \sqrt{D_K} c_o^{2r-1} \varepsilon(f).$$

This proves the result when $n > 0$; the case $n = 0$ is similar, and is left to the reader. \square

6. THE ARITHMETIC APPLICATIONS

In this section, we state our main arithmetic applications in this paper, whose proof will be based on the results of the preceding sections combined with Kolyvagin's method of Euler systems. The details of the Euler system argument will be given in §7.

6.1. Setup and running hypotheses. Let $f \in S_{2r}^{\text{new}}(\Gamma_0(N))$ be a newform, and let F/\mathbf{Q}_p be a finite extension with the ring of integers $\mathcal{O} = \mathcal{O}_F$ containing the Fourier coefficients of f . Let

$$\rho_f : G_{\mathbf{Q}} \longrightarrow \text{GL}_F(V_f) \simeq \text{GL}_2(F)$$

be the p -adic Galois representation attached to f , and set $\rho_f^* := \rho \otimes \varepsilon_{\text{cyc}}^r$ and $V := V_f(r)$. Let $\chi : \text{Gal}(K_{c_o, p^\infty}/K) \rightarrow \mathcal{O}^\times$ be a locally algebraic character of infinity type $(j, -j)$ and conductor $c\mathcal{O}_K$ and set $V_{f,\chi} := V|_{G_K} \otimes \chi$. Recall that the *Bloch-Kato Selmer group* of $V_{f,\chi}$ is defined by

$$\text{Sel}(K, V_{f,\chi}) := \ker \left\{ H^1(K, V_{f,\chi}) \longrightarrow \prod_v \frac{H^1(K_v, V_{f,\chi})}{H_f^1(K_v, V_{f,\chi})} \right\},$$

where

$$H_f^1(K_v, V_{f,\chi}) = \begin{cases} \ker \left(H^1(K_v, V_{f,\chi}) \longrightarrow H^1(K_v^{\text{ur}}, V_{f,\chi}) \right) & \text{if } p \nmid v, \\ \ker \left(H^1(K_v, V_{f,\chi}) \longrightarrow H^1(K_v, V_{f,\chi} \otimes \mathbf{B}_{\text{cris}}) \right) & \text{if } p \mid v. \end{cases}$$

We summarize the running hypotheses in this section.

Hypothesis (H).

- (a) $p \nmid 2(2r-1)!N\varphi(N)$;
- (b) (Heeg) and (can) in §4.1;
- (c) $(c_o, N) = 1$;
- (d) $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ is split in K .

Let $\epsilon(V_{f,\chi}) = \pm 1$ be the sign in the functional equation for $L(f, \chi, s)$. To calculate the sign, we note that $\epsilon(V_{f,\chi}) = \prod_v \epsilon(\frac{1}{2}, \pi_{K_v} \otimes \chi_v)$ is a product of local root numbers over places v of \mathbf{Q} . By the formulae [Sch02, (9), (12)], we see that $\epsilon(\frac{1}{2}, \pi_{K_v} \otimes \chi_v) = 1$ for all finite place v under the hypothesis (Heeg). On the other hand, since π_∞ is the unitary discrete series of weight $2r - 1$, we have

$$\epsilon(\frac{1}{2}, \pi_{K_\infty} \otimes \chi_\infty) = \epsilon(\frac{1}{2}, \mu^{r-\frac{1}{2}+j}) \epsilon(\frac{1}{2}, \mu^{\frac{1}{2}-r+j}) = (\sqrt{-1})^{|2r-1+2j|+|1-2r+2j|},$$

where $\mu : \mathbf{C}^\times \rightarrow \mathbf{C}^\times$ is the character $z \mapsto z/\bar{z}$ ([Tat79, (3.2.5)]). Therefore, we find that

$$(6.1) \quad \epsilon(V_{f,\chi}) = (\sqrt{-1})^{|2r-1+2j|+|1-2r+2j|} = -1 \iff -r < j < r.$$

6.2. Nonvanishing of generalized Heegner cycles. Recall from §4.4 the construction of the generalized Heegner classes $z_{f,\chi} \in H^1(K, T \otimes \chi)$ in (4.7).

Theorem 6.1. *Suppose that $\epsilon(V_{f,\chi}) = -1$. The following two statements hold.*

- (1) *If $z_{f,\chi} \neq 0$, then $\text{Sel}(K, V_{f,\chi}) = F \cdot z_{f,\chi}$.*
- (2) *The classes $z_{f,\chi\phi}$ are nonzero in $H^1(K, V_{f,\chi\phi})$ for all but finitely many finite order characters $\phi : \text{Gal}(K_{p^\infty}/K) \rightarrow \mu_{p^\infty}$.*

PROOF. The first part is a restatement of Theorem 7.7. The second part follows immediately from Theorem 4.9 and the nonvanishing of the p -adic L -function in Theorem 3.9. \square

6.3. Vanishing of Selmer groups. Assume further that f is ordinary at p in this subsection.

Theorem 6.2. *If $L(f, \chi, r) \neq 0$, then $\text{Sel}(K, V_{f,\chi}) = \{0\}$.*

PROOF. The nonvanishing of the central value $L(f, \chi, r)$ implies that $\epsilon(V_{f,\chi}) = +1$, and hence χ has infinity type $(j, -j)$ with $j \geq r$ or $j \leq -r$ by (6.1). Let $\chi^\tau(g) := \chi(\tau g \tau)$, where τ is the complex conjugation. Then clearly $L(f, \chi^\tau, r) = L(f, \chi, r)$ and the action of τ induces an isomorphism $\text{Sel}(K, V_{f,\chi}) \simeq \text{Sel}(K, V_{f,\chi^\tau})$, so we may assume that $j \geq r$. One then immediately checks that $V_{f,\chi}|_{G_{K_{\bar{p}}}}$ has positive Hodge–Tate weights², while the Hodge–Tate weights of $V_{f,\chi}|_{G_{K_{\bar{p}}}}$ are all ≤ 0 . By [BK90, Thm. 4.1(ii)] we thus have

$$(6.2) \quad H_f^1(K_v, V_{f,\chi}) = \begin{cases} H^1(K_v, V_{f,\chi}) & \text{if } v = \mathfrak{p}, \\ \{0\} & \text{if } v = \bar{\mathfrak{p}}. \end{cases}$$

Let $\mathbf{z}_f^\chi \in H^1(K, T \otimes \chi)$ be the χ -specialization of the Iwasawa cohomology class \mathbf{z}_f defined in (5.5). By Corollary 5.8, the nonvanishing of $L(f, \chi, r)$ implies that $\text{loc}_{\mathfrak{p}}(\mathbf{z}_f^{\chi^{-1}}) \neq 0$ (note that the factor $e_{\mathfrak{p}}'(f, \chi)$ never vanishes). The result thus follows from Theorem 7.9. \square

Combined with the nonvanishing of the p -adic L -function in Theorem 3.9, the results of Theorem 6.1 and Theorem 6.2 allow us to immediately obtain the following analogue of the *growth number conjecture* in [Maz84] on the asymptotic behavior of the ranks of Selmer groups over ring class fields.

Theorem 6.3. *There exists a non-negative integer e such that the formula*

$$\dim_F \text{Sel}(K_{p^n}, V_{f,\chi}) = \frac{(1 - \epsilon(V_{f,\chi}))}{2} \cdot [K_{p^n} : K] + e$$

holds for all sufficiently large n .

6.4. The parity conjecture. In combination with Nekovář’s results on the parity of a p -adic family of Galois representations [Nek07], our results imply the following parity conjecture for $V_{f,\chi}$. We heartily thank Ben Howard for drawing this application to our attention.

Theorem 6.4. *Suppose that f is ordinary at p . Then we have*

$$\text{ord}_{s=r} L(f, \chi, s) \equiv \dim_F \text{Sel}(K, V_{f,\chi}) \pmod{2}.$$

²Here our convention is that p -adic cyclotomic character has Hodge–Tate weight $+1$.

PROOF. Let K_∞^-/K be the anticyclotomic \mathbf{Z}_p -extension and let $\Gamma_K^- := \text{Gal}(K_\infty^-/K)$. Let $\Lambda := \mathcal{O}[[\Gamma_K^-]]$ and let $\mathcal{X} : G_K \rightarrow \Lambda^\times$ be the universal deformation of χ defined by $g \mapsto \chi(g)g|_{K_\infty^-}$. Recall that $\tau \in G_{\mathbf{Q}} \setminus G_K$ is the complex conjugation. Let $\text{Ind}_K^{\mathbf{Q}} \mathcal{X} := \Lambda e_{\mathfrak{p}} \oplus \Lambda e_{\overline{\mathfrak{p}}}$ be the $G_{\mathbf{Q}}$ -module defined by

$$\begin{aligned} g(ae_{\mathfrak{p}} + be_{\overline{\mathfrak{p}}}) &= \mathcal{X}(g)ae_{\mathfrak{p}} + \mathcal{X}^\tau(g)be_{\overline{\mathfrak{p}}} \text{ for } g \in G_K, \\ \tau(ae_{\mathfrak{p}} + be_{\overline{\mathfrak{p}}}) &= be_{\mathfrak{p}} + ae_{\overline{\mathfrak{p}}}. \end{aligned}$$

Let $\mathcal{T} := T \otimes_{\mathcal{O}} \text{Ind}_K^{\mathbf{Q}} \mathcal{X}$, which is a self-dual left $\Lambda[G_{\mathbf{Q}}]$ -module equipped with a skew-symmetric paring defined in [Nek07, Example (5.3.4)], and define the $\Lambda[G_{\mathbf{Q}_p}]$ -submodule $\mathcal{T}_p^+ \subset \mathcal{T}$ by

$$\mathcal{T}_p^+ := \begin{cases} \mathcal{F}^+ T \otimes \Lambda & \text{if } -r < j < r, \\ T \otimes \Lambda e_{\mathfrak{p}} & \text{if } j \geq r \text{ or } j \leq -r. \end{cases}$$

Then $(\mathcal{T}, \mathcal{T}_p^+)$ satisfies [Nek07, (5.1.2) (1)–(4)]. Moreover, one verifies that for any finite order character $\phi : \Gamma_K^- \rightarrow \mu_{p^\infty}$, the specialization $\mathcal{T}_\phi = T \otimes \text{Ind}_K^{\mathbf{Q}} \chi\phi$ together with the corresponding subspace $\mathcal{T}_{p,\phi}^+$ also satisfy conditions (5)–(8) in *loc.cit.*³

Let $F(\phi)$ be the field generated over F by the values of ϕ , let $\mathcal{O}(\phi)$ be the ring of integers of $F(\phi)$, and put $\mathcal{V}_\phi := \mathcal{T}_\phi \otimes_{\mathcal{O}(\phi)} F(\phi)$. Let $\epsilon(\mathcal{V}_\phi) \in \{\pm 1\}$ be the sign of the Weil–Deligne representation attached to \mathcal{V}_ϕ . Under Hypothesis (H), it is well-known that $\epsilon(\mathcal{V}_\phi) = \epsilon(V_{f,\chi})$ is independent of ϕ , and as already noted we have

$$\epsilon(V_{f,\chi}) = \begin{cases} -1 & \text{if } -r < j < r, \\ +1 & \text{if } j \geq r \text{ or } j \leq -r. \end{cases}$$

Now choose a Hecke character ψ of infinity type $(r, -r)$ and conductor $c_o \mathcal{O}_K$ such that $\chi \widehat{\psi}^{-1}$ is of p -power conductor. By Theorem 3.9, we can choose ϕ sufficiently wildly ramified such that

$$(6.3) \quad \mathcal{L}_{\mathfrak{p},\psi}(f)(\chi \widehat{\psi}^{-1} \phi) \neq 0.$$

Thus Proposition 3.8 and Theorem 6.2 imply that $\dim_{F(\phi)} \text{Sel}(K, V_{f,\chi\phi}) = 0$ if $\epsilon(V_{f,\chi}) = +1$, while Theorems 5.7 and 6.1 imply that $\dim_{F(\phi)} \text{Sel}(K, V_{f,\chi\phi}) = 1$ if $\epsilon(V_{f,\chi}) = -1$. On the other hand, by Shapiro’s lemma we can verify that

$$\text{Sel}(K, V_{f,\chi\phi}) \simeq \text{Sel}(\mathbf{Q}, \mathcal{V}_\phi).$$

Therefore, by [Nek07, Cor. (5.3.2)] (see also [Nek09]), we conclude that

$$\dim_F \text{Sel}(K, V_{f,\chi}) \equiv \dim_{F(\phi)} \text{Sel}(K, V_{f,\chi\phi}) \equiv \epsilon(V_{f,\chi}) \pmod{2},$$

and the parity conjecture for $V_{f,\chi}$ follows. \square

7. KOLYVAGIN’S METHOD FOR GENERALIZED HEEGNER CYCLES

We keep the setup and Hypothesis (H) introduced in §6, except that we do *not* assume that p is split in K . In particular, $f \in S_{2r}^{\text{new}}(\Gamma_0(N))$ is a newform of level N prime to p , and $\chi : \text{Gal}(K_{c_o p^\infty}/K) \rightarrow \mathcal{O}^\times$ is a locally algebraic anticyclotomic Galois character of infinity type $(j, -j)$ and conductor $c\mathcal{O}_K$. Write $c = c_o p^s$ with $(c_o, pN) = 1$. The aim of this section is to develop a suitable extension of Kolyvagin’s method of Euler systems for the Galois representation $V \otimes \chi$. We largely follow Nekovář’s approach [Nek92].

³As explained in [Nek07, Example (5.3.4)(5)], this follows from properties [*loc.cit.*, (2)–(3)] for \mathcal{T}_ϕ , whose verification is immediate. Indeed, $(\mathcal{T}_\phi, \mathcal{T}_{p,\phi}^+)$ satisfies the Panchishkin condition of [Nek07, Def. (3.3.1)] by construction, and \mathcal{T}_ϕ is pure of weight 1 at all finite places, since Ramanujan’s conjecture holds for f ; and anticyclotomic Hecke characters are pure of weight 0.

7.1. Notation. For each positive integer n , let K_n be the ring class field of K of conductor n , and let $\Delta_n = \text{Gal}(K_n/K)$. If A is a G_{K_n} -module unramified outside pNc_o , let $H^1(K_n, A)$ denote the cohomology group $H^1(\text{Gal}(K^{\Sigma_n}/K_n), A)$, where K^{Σ_n} is the maximal extension of K unramified outside the prime factors of pNc_on .

Recall that T is the $G_{\mathbf{Q}}$ -stable \mathcal{O} -lattice of the self-dual Galois representation V as in [Nek92, §3]. By [Nek92, Prop. 3.1(2)], there is a $G_{\mathbf{Q}}$ -equivariant \mathcal{O} -linear perfect pairing

$$(7.1) \quad \langle \cdot, \cdot \rangle : T \times T \longrightarrow \mathcal{O}(1),$$

and for any local field L , let $\langle \cdot, \cdot \rangle_L : H^1(L, T) \times H^1(L, T) \rightarrow \mathcal{O}$ denote the local Tate pairing induced by $\langle \cdot, \cdot \rangle$. Let ϖ be a uniformizer of \mathcal{O} and let $\mathbb{F} = \mathcal{O}/(\varpi)$ be the residue field. If M is a positive integer, we abbreviate

$$\mathcal{O}_M := \mathcal{O}/\varpi^M \mathcal{O}, \quad T_M := T/\varpi^M \mathcal{O}.$$

We let ℓ always denote a rational prime *inert* in K , and let λ be the prime of \mathcal{O}_K above ℓ , K_λ be the completion of K at λ , and Frob_ℓ be the Frobenius element of λ in G_K . If A is a discrete $\mathcal{O}[G_K]$ -module, we denote by A^\vee the Pontryagin dual of A . Let $H_f^1(K_\lambda, A)$ and $H_s^1(K_\lambda, A) := H^1(K_\lambda, A)/H_f^1(K_\lambda, A)$ be the finite part and the singular quotient of $H^1(K_\lambda, A)$, respectively. Denote by $\text{loc}_\ell : H^1(K, A) \rightarrow H^1(K_\lambda, A)$ the localization map at ℓ and by

$$\partial_\ell : H^1(K, A) \longrightarrow H_s^1(K_\lambda, A)$$

the composition of loc_ℓ with the quotient map $H^1(K_\lambda, A) \rightarrow H_s^1(K_\lambda, A)$.

7.2. Kolyvagin's anticyclotomic Euler systems. Denote by \mathcal{K} the set of square-free products of primes ℓ inert in K with $\ell \nmid 2pNc_o$. Let τ denote the complex conjugation, and let $w_f \in \{\pm 1\}$ be the Atkin–Lehner eigenvalue of f .

Definition 7.1. An anticyclotomic Euler system attached to (T, χ) is a collection $\{\mathbf{c}_n\}_{n \in \mathcal{K}}$ of classes $\mathbf{c}_n \in H^1(K_{cn}, T \otimes \chi^{-1})$ such that for every $n = m\ell \in \mathcal{K}$ we have:

- (E1) $\text{cor}_{K_{nc}, K_{mc}}(\mathbf{c}_n) = \mathbf{a}_\ell(f) \cdot \mathbf{c}_m$;
- (E2) $\text{loc}_\ell(\mathbf{c}_n) = \text{res}_{K_{mc, \lambda}, K_{nc, \lambda}}(\text{loc}_\ell(\mathbf{c}_m))^{\text{Frob}_\ell}$;
- (E3) if $\chi^2 = 1$, then $\mathbf{c}_n^\tau = w_f \cdot \chi(\sigma_{\overline{\eta}}) \cdot \mathbf{c}_n^{\sigma_{\overline{\eta}}}$.

We briefly recall the construction of derivative classes attached to an anticyclotomic Euler system $\mathbf{c} = \{\mathbf{c}_n\}_{n \in \mathcal{K}}$. First we make an auxiliary choice of a positive integer ν such that p^ν annihilates:

- (i) the kernel and cokernel of the map $\text{res}_{K, K_n} : H^1(K, T_M \otimes \chi^{-1}) \rightarrow H^1(K_n, T_M \otimes \chi^{-1})^{\Delta_n}$ for all positive integers n and M ;
- (ii) the local cohomology groups $H^1(K_v, T_M \otimes \chi^{-1})$ for all $v \mid c_oN$.

The existence of such ν follows from [Nek92, Prop. 6.3, Cor. 6.4, Lem. 10.1]. Define the constant

$$(7.2) \quad B_1 = \min \{ \text{ord}_\varpi(x-1) \mid x \cdot I_2 \in \rho_f^* \otimes \chi^{-1}(G_K), x \in \mathbf{Z}_p^\times \}.$$

A rational prime ℓ is called an M -admissible Kolyvagin prime if

- $\ell \nmid 2cNp$ is inert in K ;
- $\mathbf{a}_\ell(f) \equiv \ell + 1 \equiv 0 \pmod{\varpi^M}$;
- $\varpi^{M+B_1+1} \nmid \ell + 1 \pm \mathbf{a}_\ell(f)\ell^{1-r}$.

Let \mathcal{K}_M be the set of square-free products of M -admissible primes, and for each $n \in \mathcal{K}_M$ let G_n denote the Galois group $\text{Gal}(K_{nc}/K_c) \subset \Delta_{cn}$. For each $\ell \mid n$, the group G_ℓ is cyclic of order $\ell + 1$, and we have a canonical decomposition $G_n = \prod_{\ell \mid n} G_\ell$. Fixing a generator σ_ℓ for each G_ℓ , Kolyvagin's derivative operators are defined by

$$D_\ell = \sum_{i=1}^{\ell} i \sigma_\ell^i \in \mathbf{Z}[G_\ell]$$

and

$$D_n := \prod_{\ell \mid n} D_\ell \in \mathbf{Z}[G_n] \subset \mathcal{O}[\Delta_{nc}].$$

Then for each $n \in \mathcal{K}_M$ there is a unique $\mathcal{D}_M(n) \in H^1(K_c, T_M \otimes \chi^{-1})$ such that

$$\text{res}_{K_c, K_{nc}}(\mathcal{D}_M(n)) = p^{3\nu} D_n \mathbf{c}_n,$$

and the *derivative class* $\kappa_{\chi^{-1}}(n)$ is defined by

$$\kappa_{\chi^{-1}}(n) := \text{cor}_{K_c/K}(\mathcal{D}_M(n)) \in H^1(K, T_M \otimes \chi^{-1}).$$

We next introduce Euler systems with local conditions at p . Let $\mathcal{F} \subset H^1(K \otimes \mathbf{Q}_p, V \otimes \chi^{-1})$ be an F -vector subspace and let $\mathcal{F}^* \subset H^1(K \otimes \mathbf{Q}_p, V \otimes \chi)$ be the orthogonal complement of \mathcal{F} under the local Tate pairing. We assume that $\mathcal{F}^* = \mathcal{F}$ if $\chi^2 = 1$. Let $\mathcal{F}_T \subset H^1(K \otimes \mathbf{Q}_p, T \otimes \chi^{-1})$ be the inverse image of \mathcal{F} under the natural map $H^1(K \otimes \mathbf{Q}_p, T \otimes \chi^{-1}) \rightarrow H^1(K \otimes \mathbf{Q}_p, V \otimes \chi^{-1})$ and let $\mathcal{F}_M \subset H^1(K \otimes \mathbf{Q}_p, T_M \otimes \chi^{-1})$ be the image of \mathcal{F}_T under the reduction map $H^1(K \otimes \mathbf{Q}_p, T \otimes \chi^{-1}) \rightarrow H^1(K \otimes \mathbf{Q}_p, T_M \otimes \chi^{-1})$. For each positive integer n , let $\text{Sel}_{\mathcal{F}}^{(n)}(K, T_M \otimes \chi^{-1})$ be the n -imprimitive Selmer group defined by

$$\text{Sel}_{\mathcal{F}}^{(n)}(K, T_M \otimes \chi^{-1}) := \left\{ s \in H^1(K, T_M \otimes \chi^{-1}) \mid \begin{array}{ll} \text{loc}_v(s) \in H_f^1(K_v, T_M \otimes \chi^{-1}) & \text{for } v \nmid pn, \\ \text{loc}_p(s) \in \mathcal{F}_M & \text{if } p \nmid n \end{array} \right\}.$$

Note that if $p \mid n$, then $\text{Sel}_{\mathcal{F}}^{(n)}(K, T_M \otimes \chi^{-1})$ does not depend on the choice of \mathcal{F} . When $n = 1$ we shall simply write $\text{Sel}_{\mathcal{F}}(K, T_M \otimes \chi^{-1})$ for $\text{Sel}_{\mathcal{F}}^{(n)}(K, T_M \otimes \chi^{-1})$. We let

$$\text{Sel}_{\mathcal{F}}(K, V/T \otimes \chi^{-1}) := \varinjlim_M \text{Sel}_{\mathcal{F}}(K, T_M \otimes \chi^{-1})$$

and define $\text{Sel}_{\mathcal{F}^*}(K, T_M \otimes \chi)$ in a similar way.

Let

$$\mathbf{c}_K := \text{cor}_{K_c/K}(\mathbf{c}_1) \in H^1(K, T \otimes \chi^{-1}).$$

By [Nek92, Prop. 10.2 (2)(3)], the derivative classes $\kappa_{\chi^{-1}}(n)$ satisfy

$$(K1) \quad \kappa_{\chi^{-1}}(n) \in \text{Sel}_{\mathcal{F}}^{(np)}(K, T_M \otimes \chi^{-1}),$$

and by definition we see that

$$\kappa_{\chi^{-1}}(1) = p^{3\nu} \mathbf{c}_K \pmod{\varpi^M}.$$

If ℓ is an M -admissible prime, then G_{K_λ} acts trivially on $T_M \otimes \chi^{-1}$, and there are isomorphisms

$$\begin{aligned} \alpha_\ell: H_f^1(K_\lambda, T_M \otimes \chi^{-1}) &= H^1(K_\lambda^{\text{ur}}/K_\lambda, T_M) \xrightarrow{\sim} T_M, \\ \beta_\ell: H_s^1(K_\lambda, T_M \otimes \chi^{-1}) &= H^1(K_\lambda^{\text{ur}}, T_M) \xrightarrow{\sim} T_M, \end{aligned}$$

given by evaluation of cocycles at Frob_ℓ and γ_ℓ , respectively, where γ_ℓ is a generator of the pro- p part of the tame inertia group of K_λ . Define the *finite-to-singular* map

$$\varphi_\ell := \beta_\ell^{-1} \circ \alpha_\ell : H_f^1(K_\lambda, T_M \otimes \chi^{-1}) \xrightarrow{\sim} H_s^1(K_\lambda, T_M \otimes \chi^{-1}).$$

Then it is proved in [Nek92, Prop. 10.2] that for every M -admissible prime $\ell \mid n$, we have the relations

$$(K2) \quad \left(\frac{(-1)^{r-1} \epsilon_n \mathbf{a}_\ell(f) \ell^{1-r}}{\varpi^M} - \frac{\ell+1}{\varpi^M} \right) \varphi_\ell(\text{loc}_\ell(\kappa_{\chi^{-1}}(n/\ell))) = \left(\frac{\ell+1}{\varpi^M} \epsilon_n - \frac{\mathbf{a}_\ell(f) \ell^{1-r}}{\varpi^M} \right) \partial_\ell(\kappa_{\chi^{-1}}(n));$$

$$(K3) \quad \kappa_{\chi^{-1}}(n)^\tau = \epsilon_n \cdot \kappa_{\chi^{-1}}(n) \text{ if } \chi^2 = 1,$$

where $\epsilon_n = \chi(\sigma_{\mathfrak{N}}) \cdot w_f \cdot (-1)^{\omega(n)} \in \{\pm 1\}$ with $\omega(n)$ the number of prime divisors of n .

Definition 7.2. Let $\text{ES}(T, \chi, \mathcal{F})$ be the space of anticyclotomic Euler systems with local condition \mathcal{F} , consisting of anticyclotomic Euler systems $\mathbf{c} = \{\mathbf{c}_n\}_{n \in \mathcal{N}}$ satisfying, in addition to (E1–3) in Definition 7.1, the conditions:

$$(E4) \quad \mathbf{c}_K \in \text{Sel}_{\mathcal{F}}(K, T \otimes \chi^{-1}) \text{ and } \mathbf{c}_K^\tau \in \text{Sel}_{\mathcal{F}^*}(K, T \otimes \chi) \ (\Leftrightarrow \text{loc}_p(\mathbf{c}_K) \in \mathcal{F}_T \text{ and } \text{loc}_p(\mathbf{c}_K^\tau) \in \mathcal{F}_T^*);$$

$$(E5) \quad \text{for every } M \text{ and } n \in \mathcal{N}_M, \text{ we have } \kappa_{\chi^{-1}}(n) \in \text{Sel}_{\mathcal{F}}^{(n)}(K, T_M \otimes \chi^{-1}) \ (\Leftrightarrow \text{loc}_p(\kappa_{\chi^{-1}}(n)) \in \mathcal{F}_M).$$

The following is one of the key technical results in this paper.

Theorem 7.3. *If $\mathbf{c} \in \text{ES}(T, \chi, \mathcal{F})$ is an Euler system with local condition \mathcal{F} with*

$$\mathbf{c}_K \neq 0 \text{ in } H^1(K, V \otimes \chi^{-1}),$$

then $\text{Sel}_{\mathcal{F}^}(K, V \otimes \chi) = F \cdot \mathbf{c}_K^\tau$.*

In the next two sections we shall give the applications of this result to the Euler system constructed in this paper, postponing the proof of Theorem 7.3 to §7.5.

7.3. Euler system for generalized Heegner cycles (I). Suppose $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K , and for this section assume that χ has infinity type $(j, -j)$ with

$$-r < j < r.$$

We consider the χ^{-1} -component $z_{f,\chi^{-1},n}$ of the generalized Heegner classes $z_{f,n}$, as defined in (4.6).

Proposition 7.4. *If $n = m\ell$ with ℓ inert in K and $c \mid m$, then:*

- (1) $\text{cor}_{K_n, K_m}(z_{f,\chi^{-1},n}) = \mathbf{a}_\ell(f) \cdot z_{f,\chi^{-1},m}$.
- (2) $\text{loc}_\ell(z_{f,\chi^{-1},n}) = \text{res}_{K_{m,\lambda}, K_{n,\lambda}}(\text{loc}_\ell((z_{f,\chi^{-1},m})^{\text{Frob}_\ell}))$.
- (3) $(z_{f,\chi^{-1},n})^\tau = w_f \cdot \chi(\sigma_{\bar{\mathfrak{n}}})(z_{f,\chi,n})^{\sigma_{\bar{\mathfrak{n}}}}$.

PROOF. These properties follow from Proposition 4.4, Lemma 4.6, and Lemma 4.7, respectively. \square

Lemma 7.5. *Suppose $p > 2r - 1$. Let w be a place of K_c above p , and let $K_{c,w}$ be the completion of K_c at w . If $L/K_{c,w}$ is a finite extension and L'/L is a finite unramified extension, then the corestriction map*

$$\text{cor}_{L'/L} : H_f^1(L', T_M \otimes \chi) \longrightarrow H_f^1(L, T_M \otimes \chi)$$

is surjective, and the restriction map

$$\text{res}_{L,L'} : H^1(L, T_M \otimes \chi^{-1})/H_f^1(L, T_M \otimes \chi^{-1}) \longrightarrow H^1(L', T_M \otimes \chi^{-1})/H_f^1(L', T_M \otimes \chi^{-1})$$

is injective.

PROOF. By local Tate duality, it suffices to establish the first claim. Since $p > 2r - 1$, the Bloch–Kato group $H_f^1(L, T \otimes \chi)$ for the crystalline representation $T \otimes \chi$ admits a description in terms of Fontaine–Laffaille modules (see [BK90, Lem. 4.5(c)]). Thus let D be the Fontaine–Laffaille \mathcal{O}_L -module attached to $T \otimes \chi$ as a G_L -module. Then $D \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$ is the Fontaine–Laffaille module of $T \otimes \chi$ regarded as a $G_{L'}$ -module, and by *loc.cit.* we have the commutative diagram

$$\begin{array}{ccccccc} D^0 \otimes_{\mathcal{O}_L} \mathcal{O}_{L'} & \xrightarrow{f_0^{-1}} & D^0 \otimes_{\mathcal{O}_L} \mathcal{O}_{L'} & \longrightarrow & H_f^1(L, T \otimes \chi) & \longrightarrow & 0 \\ \downarrow 1 \otimes \text{Tr}_{L'/L} & & \downarrow 1 \otimes \text{Tr}_{L'/L} & & \downarrow \text{cor}_{L'/L} & & \\ D^0 & \xrightarrow{f_0^{-1}} & D^0 & \longrightarrow & H_f^1(L, T \otimes \chi) & \longrightarrow & 0, \end{array}$$

where f_0 is the usual Frobenius map. The surjectivity of $\text{cor}_{L'/L}$ thus follows from the surjectivity of the trace map $\text{Tr}_{L'/L} : \mathcal{O}_{L'} \rightarrow \mathcal{O}_L$. \square

For each $n \in \mathcal{K}$ define

$$\mathbf{c}_n^{\text{heeg}} := z_{f,\chi^{-1},nc}.$$

Set $\mathbf{c}^{\text{heeg}} := \{\mathbf{c}_n^{\text{heeg}}\}_{n \in \mathcal{K}}$ and let $\mathcal{F}_{\text{BK}} := H_f^1(K \otimes \mathbf{Q}_p, V \otimes \chi^{-1})$ be given by the usual Bloch–Kato finite subspaces.

Proposition 7.6. *We have $\mathbf{c}_K^{\text{heeg}} = z_{f,\chi^{-1}}$, and $\mathbf{c}^{\text{heeg}} \in \text{ES}(T, \chi^{-1}, \mathcal{F}_{\text{BK}})$ is an Euler system with local condition \mathcal{F}_{BK} .*

PROOF. The first claim is clear. On the other hand, it follows from Proposition 7.4 that \mathbf{c}^{heeg} satisfies conditions (E1–3) in Definition 7.1. To see that \mathbf{c}^{heeg} also satisfies conditions (E4) and (E5) in Definition 7.2, we note that $\text{loc}_p(z_{f,\chi^{-1},nc}) \in H_f^1(K_{nc}, T \otimes \chi^{-1})$ by [Niz97]. Since the action of complex conjugation induces an isomorphism $H_f^1(K \otimes \mathbf{Q}_q, T \otimes \chi^{-1}) \simeq H_f^1(K \otimes \mathbf{Q}_q, T \otimes \chi)$ for every prime q , we see that $(\mathbf{c}^{\text{heeg}})^\tau$ satisfies (E4). Therefore, we have $\mathbf{c}_K^{\text{heeg}} = z_{f,\chi^{-1}} \in \text{Sel}_{\mathcal{F}_{\text{BK}}}(K, T \otimes \chi^{-1})$ and

$$\text{loc}_w(\text{res}_{K_c, K_{cn}}(\mathcal{D}_M(n))) = \text{loc}_w(p^{3\nu} D_n z_{f,cn,\chi^{-1}}) \in H_f^1(K_{cn,w}, T_M \otimes \chi^{-1})$$

for each place $w \mid p$. By Lemma 7.5, this implies that $\text{loc}_w(\mathcal{D}_M(n)) \in H_f^1(K_{c,w}, T_M \otimes \chi^{-1})$, and hence $\text{loc}_p(\kappa_{\chi^{-1}}(n)) \in \mathcal{F}_M$, as was to be shown. \square

Theorem 7.7. *If $z_{f,\chi} \neq 0 \in H^1(K, V \otimes \chi)$, then*

$$\text{Sel}(K, V \otimes \chi) = \text{Sel}_{\mathcal{F}_{\text{BK}}}(K, V \otimes \chi) = F \cdot z_{f,\chi}.$$

PROOF. Note that by Proposition 7.4(3) we have the equivalence

$$z_{f,\chi} = 0 \iff z_{f,\chi^{-1}} = 0.$$

Thus Proposition 7.6 combined with Theorem 7.3 yields the result. \square

7.4. Euler system for generalized Heegner cycles (II). As in the preceding section, we assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K , but suppose now that χ has infinity type $(j, -j)$ with

$$j \geq r.$$

In addition, in this section we assume that f is ordinary at p .

Let \mathbf{z}_f^χ be the χ -specialization of the Iwasawa cohomology class \mathbf{z}_f defined in (5.5). For every place v of K above p , let $\mathcal{L}_v \subset H^1(K_v, V \otimes \chi)$ be the subspace spanned by $\text{loc}_v(\mathbf{z}_f^\chi)$. Then

$$\mathcal{L}_{T,v} := \mathcal{L}_v \cap H^1(K_v, T \otimes \chi) = \mathcal{O}\varpi^{-a_v} \text{loc}_v(\mathbf{z}_f^\chi) + H^1(K_v, T)_{\text{tor}}$$

for some $a_v \in \mathbf{Z}_{\geq 0}$, where $H^1(-)_{\text{tor}}$ denotes the torsion subgroup of $H^1(-)$. Let $\mathcal{L}_v^* \subset H^1(K_v, V \otimes \chi^{-1})$ be the orthogonal complement of \mathcal{L}_v , and set $\mathcal{L}^* := \mathcal{L}_{\mathfrak{p}}^* \oplus \mathcal{L}_{\bar{\mathfrak{p}}}^*$. We will choose the integer ν in §7.2 large enough so that $p^\nu H^1(K_v, T)_{\text{tor}} = \{0\}$ for each $v \mid p$.

Consider the Iwasawa cohomology classes $\mathbf{z}_{f,n} := \mathbf{z}_{f,n,\alpha}$ from (5.1), and for each $n \in \mathcal{H}$ define

$$\mathbf{c}_n^{\text{heeg},\dagger} := \mathbf{z}_{f,cn}^{\chi^{-1}} \in H^1(K_{cn}, T \otimes \chi^{-1})$$

to be the specialization of $\mathbf{z}_{f,cn}$ at χ^{-1} . Set $\mathbf{c}^{\text{heeg},\dagger} := \{\mathbf{c}_n^{\text{heeg},\dagger}\}_{n \in \mathcal{H}}$.

Proposition 7.8. *The collection $\mathbf{c}^{\text{heeg},\dagger} \in \text{ES}(T, \chi, \mathcal{L}^*)$ is an Euler system for the local condition \mathcal{L}^* with $\mathbf{c}_K^{\text{heeg},\dagger} = \mathbf{z}_f^{\chi^{-1}}$.*

PROOF. We begin by noting that for inert primes ℓ with $n = m\ell \in \mathcal{H}$, we have

- (1) $\text{cor}_{K_{nc}, K_{mc}}(\mathbf{z}_{f,n}) = \mathbf{a}_\ell(f) \cdot \mathbf{z}_{f,m}$;
- (2) $\text{loc}_\ell(\mathbf{z}_{f,n}) = \text{res}_{K_{mc,\lambda}, K_{nc,\lambda}}(\text{loc}_\ell(\mathbf{z}_{f,m})^{\text{Frob}_\ell})$;
- (3) $\mathbf{z}_{f,n}^\tau = w_f \cdot \sigma_{\bar{\mathfrak{n}}} \cdot \mathbf{z}_{f,n}$,

since by Lemma 5.4 and Proposition 7.4 these relations hold after specialization at every finite order ramified character. Specializing the same relations to χ^{-1} , we thus find that conditions (E1–3) are satisfied by $\mathbf{c}^{\text{heeg},\dagger}$. The validity of (E4) for $\mathbf{c}^{\text{heeg},\dagger}$ and its image under τ follows from the fact that if v and \bar{v} are the two places of K above p , then $\text{loc}_v(\mathbf{c}_K^{\text{heeg},\dagger}) = \text{loc}_v(\mathbf{z}_f^{\chi^{-1}})$ belongs to $H^1(K_v, \mathcal{F}^+ T \otimes \chi^{-1})$ and the action of complex conjugation sends $H^1(K_v, \mathcal{F}^+ T \otimes \chi^{-1})$ to $H^1(K_{\bar{v}}, \mathcal{F}^+ T \otimes \chi)$. We now proceed to verify condition (E5) for $\mathbf{c}^{\text{heeg},\dagger}$. For any finite extension L/K_v , let

$$\langle \cdot, \cdot \rangle_L : H^1(L, T_M \otimes \chi^{-1}) \times H^1(L, T_M \otimes \chi) \longrightarrow \mathcal{O}/\varpi^M \mathcal{O}$$

be the canonical pairing. By [Rub00, Prop. 1.4.3], it suffices to show that $\langle \text{loc}_v(\kappa_{\chi^{-1}}(n)), \mathcal{L}_{T,v} \rangle_{K_v} = 0$, i.e.

$$(7.3) \quad \langle \text{loc}_v(\kappa_{\chi^{-1}}(n)), \varpi^{-a_v} \text{loc}_v(\mathbf{z}_f^\chi) + x \rangle_{K_v} \equiv 0 \pmod{\varpi^M}, \quad \text{for all } x \in H^1(K_v, T)_{\text{tor}}.$$

Let v be a place of K above p and let w/w_0 be places of K_{nc}/K_c above v . Let \mathcal{K} and \mathcal{N} be the completion of K_c and K_{nc} at w_0 and w , respectively, and note that \mathcal{N}/\mathcal{K} is an unramified extension. Set

$$\mathcal{K}_\infty := K_\infty \mathcal{K}, \quad \mathcal{N}_\infty := K_\infty \mathcal{N}.$$

Let Ψ_v be a set of representatives of $\Delta_c/\Delta_{c,w_0}$, where $\Delta_{c,w_0} := \text{Gal}(\mathcal{K}/K_v)$ is the decomposition group of v , and let $\Delta_c = \text{Gal}(K_c/K)$ as always. By Lemma 7.5, there exists $\mathbf{y}_{\mathcal{N},\sigma} \in H_{\text{Iw}}^1(\mathcal{N}_\infty, T_M)$ such that

$$\text{cor}_{\mathcal{N}/\mathcal{K}}(\mathbf{y}_{\mathcal{N},\sigma}) \equiv \text{loc}_{w_0}(\text{res}_{K_{c_o}, K_c}(\varpi^{-a_v} \sigma \mathbf{z}_{f,c_o})) \pmod{\varpi^M}.$$

It is easy to see that $\kappa_{\chi^{-1}}(n)$ is divisible by p^ν , and so $\langle \kappa_{\chi^{-1}}(n), H^1(K_v, T)_{\text{tor}} \rangle = 0$. On the other hand, we compute

$$\begin{aligned} \langle \kappa_{\chi^{-1}}(n), \text{loc}_v(\varpi^{-a_v} \mathbf{z}_f^\chi) \rangle_{K_v} &= \sum_{\sigma \in \Delta_{c_o}, \rho \in \Psi_v} \langle \text{loc}_{w_0}(\rho D_M(n)), \text{loc}_{w_0}(\text{res}_{K_{c_o}, K_c}(\varpi^{-a_v} \sigma \mathbf{z}_{f, c_o}^\chi)) \rangle_{\mathcal{K}} \\ &= \sum_{\sigma \in \Delta_{c_o}, \rho \in \Psi_v} \langle \text{loc}_{w_0}(\rho D_M(n)), \text{cor}_{\mathcal{N}/\mathcal{K}}(\mathbf{y}_{\mathcal{N}, \sigma}^\chi) \rangle_{\mathcal{K}} \\ &= \sum_{\sigma \in \Delta_{c_o}, \rho \in \Psi_v} p^{3\nu} \langle \text{loc}_w(\rho D_n \mathbf{z}_{f, cn}^{\chi^{-1}}), \mathbf{y}_{\mathcal{N}, \sigma}^\chi \rangle_{\mathcal{N}}. \end{aligned}$$

Thus to verify (7.3) it remains to show that $\langle \text{loc}_w(\rho D_n \mathbf{z}_{f, cn}^{\chi^{-1}}), \mathbf{y}_{\mathcal{N}, \sigma}^\chi \rangle_{\mathcal{N}} \equiv 0 \pmod{\varpi^M}$. Consider Perrin-Riou's Λ -adic local pairing ([PR94, 3.6.1]):

$$\langle \cdot, \cdot \rangle_{\mathcal{N}_\infty} : H_{\text{Iw}}^1(\mathcal{N}_\infty, T_M) \times H_{\text{Iw}}^1(\mathcal{N}_\infty, T_M) \longrightarrow \Lambda_{\mathcal{O}}(\Gamma) \otimes \mathcal{O}/\varpi^M.$$

Recall that for every $x = \varprojlim_m x_m$ and $y = \varprojlim_m y_m$ in $H_{\text{Iw}}^1(\mathcal{N}_\infty, T_M)$, the pairing is defined by

$$\langle x, y \rangle_{\mathcal{N}_\infty} = \varprojlim_m \sum_{\sigma \in \text{Gal}(\mathcal{N}_m/\mathcal{N})} \langle x_m, \sigma y_m \rangle_{\mathcal{N}_m} \sigma,$$

and it enjoys the interpolation property: if $\chi : \Gamma \rightarrow \mathcal{O}^\times$ is any p -adic character, then

$$\langle x, y \rangle_{\mathcal{N}_\infty}(\chi) = \langle x^\chi, y^{\chi^{-1}} \rangle_{\mathcal{N}}.$$

Since for any finite order character ϕ of Γ and any $\rho \in \Delta_{c_o}$, the classes $\mathbf{y}_{\mathcal{N}, \sigma}^\phi$ and $\text{loc}_w(\rho \mathbf{z}_{f, c_o}^\phi)$ belong to $H_f^1(\mathcal{N}, T \otimes \phi)$, we see that $\langle \text{loc}_w(\rho D_n \mathbf{z}_{f, cn}), \mathbf{y}_{\mathcal{N}, \sigma} \rangle_{\mathcal{N}} = 0$, and hence

$$\begin{aligned} \langle \text{loc}_w(\rho D_n \mathbf{z}_{f, cn}^{\chi^{-1}}), \mathbf{y}_{\mathcal{N}, \sigma}^\chi \rangle_{\mathcal{N}} &= \langle \text{loc}_w(\rho D_n \mathbf{z}_{f, cn})^{\chi^{-1}}, \mathbf{y}_{\mathcal{N}, \sigma}^\chi \rangle_{\mathcal{N}} \\ &= \langle \text{loc}_w(\rho D_n \mathbf{z}_{f, cn}), \mathbf{y}_{\mathcal{N}, \sigma} \rangle_{\mathcal{N}}(\chi) \equiv 0 \pmod{\varpi^M}. \end{aligned}$$

This completes the proof. \square

Theorem 7.9. *If $\text{loc}_p(\mathbf{z}_f^{\chi^{-1}}) \neq 0$, then $\text{Sel}(K, V \otimes \chi) = \{0\}$.*

PROOF. To every choice of subspaces $\mathcal{F}_v \subset H^1(K_v, V \otimes \chi)$ for every prime $v \mid p$, we associate the generalized Selmer group

$$H_{\mathcal{F}_p, \mathcal{F}_p}^1(K, V \otimes \chi) := \left\{ s \in H^1(K, V \otimes \chi) \mid \begin{array}{ll} \text{loc}_q(s) \in H_f^1(K_q, V \otimes \chi^{-1}) & \text{for } q \nmid p \\ \text{loc}_v(s) \in \mathcal{F}_v & \text{for } v \mid p \end{array} \right\}.$$

The nonvanishing hypothesis implies that $\text{loc}_p(\mathbf{z}_f^\chi) \neq 0$, and hence by Proposition 7.8 and Theorem 7.3, we have

$$(7.4) \quad H_{\mathcal{L}_p, \mathcal{L}_p}^1(K, V \otimes \chi) = F \cdot (\mathbf{z}_f^{\chi^{-1}})^\tau = F \cdot \mathbf{z}_f^\chi.$$

Note that $\text{loc}_p(\mathbf{z}_f^\chi)^\tau = \text{loc}_p(\mathbf{z}_f^{\chi^{-1}})$. The nonvanishing of $\text{loc}_p(\mathbf{z}_f^{\chi^{-1}})$ thus implies that $\text{loc}_p(\mathbf{z}_f^\chi) \neq 0$, and combined with (7.4) this shows that $H_{\mathcal{L}_p, 0}^1(K, V \otimes \chi) = \{0\}$. Finally, in light of the Poitou–Tate exact sequence

$$\begin{aligned} 0 \longrightarrow H_{0, \emptyset}^1(K, V \otimes \chi^{-1}) \longrightarrow H_{\mathcal{L}_p^*, \emptyset}^1(K, V \otimes \chi^{-1}) \xrightarrow{\text{loc}_p} \mathcal{L}_p^* \\ \longrightarrow H_{0, 0}^1(K, V \otimes \chi)^\vee \longrightarrow H_{\mathcal{L}_p, 0}^1(K, V \otimes \chi)^\vee \longrightarrow 0, \end{aligned}$$

we find that $H_{0, 0}^1(K, V \otimes \chi) = \text{Sel}(K, V \otimes \chi) = \{0\}$. \square

7.5. Kolyvagin's descent: Proof of Theorem 7.3. Let $\mathbf{c} \in \text{ES}(T, \chi, \mathcal{F})$ be an Euler system with $\mathbf{c}_K \neq 0 \in H^1(K, V \otimes \chi^{-1})$, or equivalently, with $\mathbf{c}_K \notin H^1(K, T \otimes \chi^{-1})_{\text{tor}}$.

Preliminaries. Let $R_\rho = \mathcal{O}[\rho_f^*(G_{\mathbf{Q}})] \subset M_2(\mathcal{O})$ and define

$$B_2 := \inf \{n \in \mathbf{Z}_{\geq 0} \mid \varpi^n M_2(\mathcal{O}) \subset R_\rho\}.$$

Since ρ_f^* is absolutely irreducible, we have $R_\rho \otimes F = M_2(F)$, and hence $B_2 < \infty$.

Lemma 7.10. *Let $E \subset K_{c_o p^\infty}$ be a p -ramified abelian extension of \mathbf{Q} . Then either $E = \mathbf{Q}$ or $\mathbf{Q}(\sqrt{p^*})$, where $p^* = (-1)^{\frac{p-1}{2}} p$.*

PROOF. Since $p \nmid D_K$, the fields E and K are linearly disjoint. It follows that EK is abelian and dihedral over \mathbf{Q} . Hence by class field theory we conclude that either $E = \mathbf{Q}$ or $\mathbf{Q}(\sqrt{p^*})$. \square

Let M be a positive integer. Then $\chi^{-1} \pmod{\varpi^M}$ factors through the Galois group $\text{Gal}(\mathcal{H}/K)$ for some ring class field $K_{c_o p^\infty}/\mathcal{H}/K_{c_o}$. Let \mathcal{H}^b be the maximal pro- p extension of K_{c_o} inside \mathcal{H} . Then $\text{Gal}(\mathcal{H}/\mathcal{H}^b)$ is a cyclic group of order dividing $p \pm 1$. Since $(c_o D_K, pN) = 1$, the intersection $\overline{\mathbf{Q}}^{\ker \rho_f^*} \cap \mathcal{H}^b$ is a solvable p -ramified extension over \mathbf{Q} , and by Lemma 7.10 this implies that $\overline{\mathbf{Q}}^{\ker \rho_f^*} \cap \mathcal{H}^b = \mathbf{Q}$. We thus conclude that $\rho_f^*(G_{\mathbf{Q}}) = \rho_f^*(G_{\mathcal{H}^b})$ and

$$R_\rho = \mathcal{O}[\rho_f^*(G_{\mathcal{H}^b})].$$

Lemma 7.11. *Let $\xi : G_{\mathcal{H}^b} \rightarrow \mathcal{O}^\times$ be a character.*

- (1) *If $T' \subset T_M \otimes \xi$ is an R_ρ -submodule with $T' \not\subset \varpi T_M$, then $\varpi^{B_2} T_M \subset T'$.*
- (2) *$\varpi^{B_2} \text{Hom}_{R_\rho}(T_M \otimes \xi, T_M \otimes \xi) = \varpi^{B_2} \mathcal{O} \cdot I_2$, where I_2 is the identity map.*

PROOF. This is essentially [Nek92, Lemma 12.3]. \square

Lemma 7.12 ([Nek92], proof of Prop. 12.2(b)). *Let $\xi_1, \dots, \xi_s : G_{\mathcal{H}^b} \rightarrow \mathcal{O}^\times$ be characters, and set*

$$S = \bigoplus_{i=1}^s \mathcal{O}_M / (\varpi^{n_i}), \quad \mathcal{V} = \bigoplus_{i=1}^s \varpi^{M-n_i} T_M \otimes \xi_i.$$

Let $\mathcal{W} \subset \mathcal{V}$ be an R_ρ -submodule. If the map $j : S \rightarrow \text{Hom}(\mathcal{W}, T_M)$ given by

$$a = (a_1, \dots, a_s) \longrightarrow j(a) : (w_1, \dots, w_s) \longmapsto a_1 w_1 + \dots + a_s w_s$$

is injective, then $\varpi^{(2^{s+1}-2)B_2} \mathcal{V} \subset \mathcal{W}$.

PROOF. We proceed by induction on s . For $s = 1$, the result follows from Lemma 7.11(1). Suppose $s > 1$, and let $\pi : \mathcal{V} \rightarrow \mathcal{V}' := \bigoplus_{i=2}^s \varpi^{M-n_i} T_M \otimes \xi_i$ be the map projecting onto the last $s - 1$ factors. Let

$$S' = \bigoplus_{i=2}^s \mathcal{O}_M / (\varpi^{n_i}), \quad \mathcal{W}' = \pi(\mathcal{W}) \subset \mathcal{V}'.$$

It is easy to see that $S' \rightarrow \text{Hom}(\mathcal{W}', T_M)$ is also injective given the injectivity of j , and hence by induction hypothesis we have $\varpi^\gamma \mathcal{V}' \subset \mathcal{W}'$ with $\gamma = (2^s - 2)B_2$. Let

$$\mathcal{V}_1 = \varpi^{M-n_1} T_M \otimes \xi_1 \hookrightarrow \mathcal{V}, \quad \mathcal{W}_1 = \mathcal{W} \cap \mathcal{V}_1 = \ker \pi,$$

and let $\mathcal{W}' \rightarrow \mathcal{V}_1/\mathcal{W}_1$ be the R_ρ -module map $w' \mapsto pr_1(w)$, where w is a lifting of w' in $\mathcal{W} \subset \mathcal{V}$, and $pr_1 : \mathcal{V} \rightarrow \mathcal{V}_1$ is the first projection. By Lemma 7.11(1), there exists $m \leq n_1$ such that

$$\varpi^{m+B_2} \mathcal{V}_1 \subset \mathcal{W}_1 \subset \varpi^m \mathcal{V}_1.$$

Let $j' : \mathcal{V}' \rightarrow \mathcal{V}_1/\varpi^m \mathcal{V}_1$ be the composition of R_ρ -module maps

$$j' : \mathcal{V}' \xrightarrow{\cdot \varpi^\gamma} \mathcal{W}' \longrightarrow \mathcal{V}_1/\mathcal{W}_1 \longrightarrow \mathcal{V}_1/\varpi^m \mathcal{V}_1 = \mathcal{O}_M / (\varpi^m)$$

By Lemma 7.11(2), there exists $(a_2, \dots, a_s) \in \mathcal{O}^{s-1}$ such that

$$\varpi^{B_2} j'(v_2, \dots, v_s) = a_2 v_2 + \dots + a_s v_s.$$

In particular, for every $(w_1, \dots, w_s) \in \mathcal{W}$, we have

$$-\varpi^{\gamma+B_2} w_1 + \varpi^\gamma a_2 w_2 + \dots + \varpi^\gamma a_s w_s \in \varpi^m \mathcal{O}_M.$$

This shows that $(-\varpi^{n_1-m+\gamma+B_2} a_2, \varpi^{n_1-m+\gamma} a_2, \dots, \varpi^{n_1-m+\gamma} a_s) \in S$ annihilates \mathcal{W} . By the injectivity of $j : S \hookrightarrow \text{Hom}(\mathcal{W}, T_M)$, the equality $\varpi^{n_1-m+\gamma+B_2} = 0 \in \mathcal{O}/(\varpi^{n_1})$ implies that $m \leq \gamma + B_2$. Thus

we have proved the inclusions $\varpi^{\gamma+2B_2}\mathcal{V}_1 \subset \mathcal{W}_1$ and $\varpi^\gamma\mathcal{V}' \subset \mathcal{W}'$, and it follows that $\varpi^{2\gamma+2B_2}\mathcal{V} \subset \mathcal{W}$, concluding the proof of the lemma. \square

Put $\rho_M := \rho_f^* \otimes \mathcal{O}_M : G_{\mathbf{Q}} \rightarrow \text{Aut}_{\mathcal{O}}(T_M)$, let $\mathbf{Q}(T_M) := \overline{\mathbf{Q}}^{\ker \rho_M}$ be the splitting field of T_M , and set

$$L = \mathcal{H}(T_M) := \mathcal{H} \cdot \mathbf{Q}(T_M).$$

Consider the $\text{Gal}(L/\mathbf{Q})$ -module $H^1(L, T_M) = \text{Hom}(\text{Gal}(\overline{\mathbf{Q}}/L), T_M)$, where $\sigma \in \text{Gal}(L/\mathbf{Q})$ acts via

$$(\sigma \cdot f)(s) = \sigma f(\sigma^{-1}s).$$

If $S \subset H^1(L, T_M)^{\text{Gal}(L/\mathcal{H})}$ is a $\mathcal{O}[\text{Gal}(\mathcal{H}/\mathbf{Q})]$ -submodule, we let $L_S := \bigcap_{s \in S} \overline{\mathbf{Q}}^{\ker s}$ be the splitting of S over L , and put $\mathcal{G}_S := \text{Gal}(L_S/L)$. We then have an inclusion

$$S \hookrightarrow H^1(\mathcal{G}_S, T_M)^{\text{Gal}(L/\mathcal{H})}$$

and a $\text{Gal}(L/\mathbf{Q})$ -equivariant map

$$\mathcal{G}_S \hookrightarrow V_S := \text{Hom}_{\mathcal{O}}(S, T_M).$$

Lemma 7.13. *Let $s = \dim_{\mathbb{F}} S \otimes \mathbb{F}$. Then $\varpi^{(2^{s+1}-2)B_2}V_S \subset \mathcal{O}[\mathcal{G}_S]$.*

PROOF. Since $\text{Gal}(\mathcal{H}/\mathcal{H}^b)$ has order dividing $p \pm 1$, the $\text{Gal}(\mathcal{H}/\mathcal{H}^b)$ -module S can be decomposed into a direct sum of cyclic \mathcal{O} -modules:

$$S = \bigoplus_{i=1}^s \mathcal{O}/(\varpi^{n_i}) \otimes \xi_i^{-1}$$

for some $\xi_i : \text{Gal}(\mathcal{H}/\mathcal{H}^b) \rightarrow \mathcal{O}^\times$, and so $V_S = \bigoplus_{i=1}^s \varpi^{M-n_i} T_M \otimes \xi_i$ as R_ρ -modules. Applying Lemma 7.12 with $\mathcal{W} := \mathcal{O}[\mathcal{G}_S]$, the result follows. \square

Let $\mathcal{G}_S^+ = \mathcal{G}_S^{\tau=1} = (1 + \tau)\mathcal{G}_S$, where τ is the complex conjugation.

Proposition 7.14.

- (1) $\varpi^{B_1} H^1(\text{Gal}(L/K), T_M \otimes \chi^{-1}) = \{0\}$.
- (2) $L_S \cap \mathcal{H}(T_{2M}) \subset \mathcal{H}(T_{M+B_1})$.
- (3) For each $g \in \mathcal{G}_S^+$, there exist infinitely many primes ℓ inert in K such that:
 - $\text{Frob}_\ell(L_S/K) (= \text{Frob}_\ell|_{L_S}) = g$,
 - $\varpi^M \mid \ell + 1 \pm \mathbf{a}_\ell(f)$
 - $\varpi^{M+B_1+1} \nmid \ell + 1 \pm \mathbf{a}_\ell(f)$.

PROOF. This can be proved by the same argument as in [Nek92, Prop. 12.2]. \square

The descent argument. Define the constants B_3, B_4 by

$$\begin{aligned} B_3 &:= \max \{n \in \mathbf{Z}_{\geq 0} \mid \mathbf{c}_K \in \varpi^n H^1(K, T \otimes \chi^{-1})\} \\ &= \max \{n \in \mathbf{Z}_{\geq 0} \mid \mathbf{c}_K^\tau \in \varpi^n H^1(K, T \otimes \chi)\}; \\ B_4 &:= \begin{cases} 0 & \text{if } \chi^2 = 1, \\ \min_{\sigma \in \text{Gal}(K_{c,p^\infty}/K)} \text{ord}_\varpi(\chi^2(\sigma) - 1) & \text{if } \chi^2 \neq 1. \end{cases} \end{aligned}$$

Put $C_1 := 6B_2 + B_1 + B_3 + B_4$, and choose a positive integer M with

$$M > 2C_1 + 2B_1.$$

Let $\kappa_\chi(1) = \mathbf{c}_K^\tau \pmod{\varpi^M}$, and for each $x \in T_M$ put

$$\text{ord}_\varpi(x) := \max \{n \in \mathbf{Z}_{\geq 0} \mid x \in \varpi^n T_M\}.$$

Lemma 7.15. *There is an M -admissible prime ℓ_1 such that*

$$\text{ord}_\varpi(\alpha_{\ell_1}(\kappa_{\chi^{-1}}(1))) = \text{ord}_\varpi(\alpha_{\ell_1}(\kappa_\chi(1))) \leq C_1.$$

PROOF. Let $\text{res}_{K,L} : H^1(K, T_M \otimes \chi^{-1}) \rightarrow H^1(L, T_M \otimes \chi^{-1}) = H^1(L, T_M)$ be the restriction map. Let $s_1 = \text{res}_{K,L}(\kappa_{\chi^{-1}}(1)) \in H^1(L, T_M)$, and consider the \mathcal{O} -submodule

$$S := \mathcal{O}s_1 + \mathcal{O}s_1^\tau \subset H^1(L, T_M)^{\text{Gal}(L/\mathcal{H})}.$$

Take an element $t \in \varpi^{B_1+B_3+B_4}T_M$ with $\text{ord}_\varpi(t) = B_1 + B_3 + B_4$, and define $f \in V_S$ by $f(s_1) = t$ and $f(s_1^\tau) = 0$ if $\chi \neq \chi^{-1}$. Using Proposition 7.14(1), it is easy to see that f is well-defined. Applying Lemma 7.13, we find that

$$\varpi^{6B_2}(1 + \tau)f = \sum_{g \in \mathcal{G}_S^+} a_g \cdot g, \quad (a_g \in \mathcal{O})$$

and evaluating at s_1 we obtain

$$\varpi^{6B_2}t = \sum_{g \in \mathcal{G}_S^+} a_g \cdot \kappa_{\chi^{-1}}(1)(g).$$

This shows that there is an element $g \in \mathcal{G}_S^+$ with $\text{ord}_\varpi(\kappa_{\chi^{-1}}(1)(g)) \leq C_1$, and the existence of a prime ℓ_1 as in the statement follows from Proposition 7.14. \square

Fix an M -admissible prime ℓ_1 as in Lemma 7.15, and let $S \subset H^1(L, T_M)$ be the image of the sum of $\text{Sel}_{\mathcal{F}}^{(\ell_1)}(K, T_M \otimes \chi^{-1})$ and its complex conjugate. Then $S \subset \text{Hom}(\mathcal{G}_S, T_M)^{\text{Gal}(L/\mathcal{H})}$ is an $\mathcal{O}[\text{Gal}(\mathcal{H}/\mathbf{Q})]$ -submodule. We will apply the discussion in the preceding paragraphs to this S .

Setting

$$d_0 := \dim_{\mathbb{F}}(V/T \otimes \chi^{-1})^{G_K}[\varpi] + \dim_{\mathbb{F}} \text{Sel}_{\mathcal{F}}(K, V/T \otimes \chi^{-1})[\varpi],$$

we have

$$\dim_{\mathbb{F}} S \otimes \mathbb{F} \leq 2 \dim_{\mathbb{F}} \text{Sel}_{\mathcal{F}}^{(\ell_1)}(K, T_M \otimes \chi^{-1})[\varpi] \leq 2d_0 + 4.$$

Let $B = 2C_1 + 2B_1 + 2B_4$, define

$$C_2 := B + (2^{2d_0+5} - 2)B_2,$$

and let $Y \subset V_S^+$ be the subset consisting of maps f such that $p^{2C_2}T_M$ is contained in the \mathcal{O} -submodule generated by $f(s_1)$ and $f(s_2)$, where

$$s_1 := \text{res}_{K,L}(\kappa_{\chi^{-1}}(1)), \quad s_2 := \text{res}_{H,L}(\kappa_{\chi^{-1}}(\ell_1)).$$

Lemma 7.16. *The set $\mathcal{G}_S^+ \cap Y$ is non-empty.*

PROOF. First suppose $\chi^2 \neq 1$. Define the \mathcal{O} -module map

$$\xi : V_S^+ \longrightarrow T_M \oplus T_M, \quad f \longmapsto \xi(f) := (f(s_1), f(s_2)) = (f(\kappa_{\chi^{-1}}(1)), f(\kappa_{\chi^{-1}}(\ell_1))).$$

Let $\mathcal{V}^+ := \xi(V_S^+) \subset T_M \oplus T_M$. We claim that

$$\varpi^B(T_M \oplus T_M) \subset \mathcal{V}^+.$$

Indeed, let $S_1 \subset S$ be the \mathcal{O} -submodule generated by $\{s_1, s_1^\tau, s_2, s_2^\tau\}$ where $s_i^\tau := \tau \cdot s_i$. For $(t_1, t_2) \in \varpi^B T_M \oplus \varpi^B T_M$, we define $g : S_1 \rightarrow T_M$ by

$$g(xs_1 + ys_2 + zs_1^\tau + ws_2^\tau) = xt_1 + yt_2.$$

Note that if $xs_1 + ys_2 + zs_1^\tau + ws_2^\tau = 0$, then $\varpi^{B_4}(xs_1 + ys_2) = \varpi^{B_4}(zs_1^\tau + ws_2^\tau) = 0$, and hence

$$\begin{aligned} & \varpi^{B_1+B_4}y\partial_{\ell_1}(\kappa_{\chi^{-1}}(\ell_1)) = \varpi^{B_1+B_4}z\partial_{\ell_1}(\tau \cdot \kappa_{\chi^{-1}}(\ell_1)) = 0 \\ \implies & \varpi^{2B_1+B_4}y\alpha_{\ell_1}(\kappa_{\chi^{-1}}(1)) = \varpi^{2B_1+B_4}z\alpha_{\ell_1}(\kappa_{\chi}(1)) = 0 \\ \implies & \text{ord}_\varpi(y), \text{ord}_\varpi(z) \geq M - C_1 - 2B_1 - B_4 \geq M - B, \end{aligned}$$

and similarly:

$$\begin{aligned} & \varpi^{C_1+B_4+2B_1}x\kappa_{\chi^{-1}}(1) + \varpi^{C_1+B_4+2B_1}z\kappa_{\chi}(1) = 0 \\ \implies & \varpi^{C_1+2B_4+2B_1}x\kappa_{\chi^{-1}}(1) = \varpi^{C_1+2B_4+2B_1}z\kappa_{\chi}(1) = 0 \\ \implies & \text{ord}_\varpi(x), \text{ord}_\varpi(z) \geq M - (2C_1 + 2B_4 + 2B_1) = M - B. \end{aligned}$$

We thus find that $xt_1 = yt_2 = 0$, and so g is well-defined. Extending g to a map $\tilde{g} : S \rightarrow T_M$, we put $f := \tilde{g} + \tilde{g}^\tau \in V_S^+$. Since we have

$$f(s_1) = g(s_1) + \tau g(s_1^\tau) = t_1, \quad f(s_2) = g(s_2) - \tau g(s_2^\tau) = t_2,$$

this verifies the claim.

Now let $q : T_M \oplus T_M \rightarrow \mathcal{O}_M$ be the quadratic form defined by $q(v) = v_1 \wedge v_2$ for all $v = (v_1, v_2)$, and let $I \subset \mathcal{O}_M$ be the ideal generated by $\{q(v)\}_{v \in \mathcal{V}^+}$. Note that $I \supset \varpi^{2B} \mathcal{O}_M$. By Lemma 7.13, $\varpi^{C_2-B} \mathcal{V}^+$ is contained in the \mathcal{O} -module generated by $\xi(\mathcal{G}_S^+)$. This implies that $\varpi^{2C_2} \mathcal{O}_M \subset \varpi^{2C_2-2B} I$ is contained in the ideal generated by $\{q(v)\}_{v \in \xi(\mathcal{G}_S^+)}$. We thus conclude that there exists $g \in \mathcal{G}_S^+$ such that $\xi(g) = (v_1, v_2)$ with $v_1 \wedge v_2 \in \varpi^r \mathcal{O}_M^\times$ and $r \leq 2C_2$. This shows that

$$\mathcal{O}v_1 + \mathcal{O}v_2 \supset \varpi^r T_M \supset \varpi^{2C_2} T_M,$$

and hence $g \in Y$.

Next we assume that $\chi^2 = 1$. Then we have

$$s_1^\tau = \epsilon s_1, \quad s_2^\tau = (-\epsilon) s_2$$

for some $\epsilon \in \{\pm 1\}$. Define the \mathcal{O} -module map

$$\xi : V_S^+ \longrightarrow T_M^\epsilon \oplus T_M^{-\epsilon} = T_M, \quad f \longmapsto \xi(f) = f(s_1) + f(s_2) = f(\kappa_{\chi^{-1}}(1)) + f(\kappa_{\chi^{-1}}(\ell_1)),$$

and let $\mathcal{V}^+ := \xi(V_S^+) \subset T_M$. We now claim that $\varpi^B T_M \subset \mathcal{V}^+$. Let $S_1 \subset H^1(L, T_M)$ be the submodule generated by $\{s_1, s_2\}$. For each $(t_1, t_2) \in \varpi^B T_M^\epsilon \oplus \varpi^B T_M^{-\epsilon} = \varpi^B T_M$, define $g : S_1 \rightarrow T_M$ by

$$g(xs_1 + ys_2) = xt_1 + yt_2 \quad (x, y \in \mathcal{O}).$$

One can verify that g is well-defined as before, and extending g to a map $\tilde{g} : S \rightarrow T_M$, we set $f := \tilde{g} + \tilde{g}^\tau$. Then $f(s_1) = 2t_1$ and $f(s_2) = 2t_2$, proving the claim. By Lemma 7.13, $\varpi^{C_2} T_M \subset \varpi^{C_2-B} \mathcal{V}^+$ is contained in the \mathcal{O} -module generated by $\xi(\mathcal{G}_S^+)$, and we find that

$$\xi(\mathcal{G}_S^+) \not\subset (\varpi^{C_2} T_M^+ \oplus \varpi^{C_2+1} T_M^-) \cup (\varpi^{C_2+1} T_M^+ \oplus \varpi^{C_2} T_M^-),$$

which implies that $\mathcal{G}_S^+ \cap Y$ is non-empty. \square

By Proposition 7.14 and Lemma 7.16, there is a finite set Σ_Y of M -admissible primes such that

$$\{\text{Frob}_\ell(L_S/K)\}_{\ell \in \Sigma_Y} = \mathcal{G}_S^+ \cap Y.$$

Define the Σ_Y -restricted Selmer group Sel_{Σ_Y} by

$$\text{Sel}_{\Sigma_Y} = \{s \in \text{Sel}_{\mathcal{F}}(K, T_M \otimes \chi) \mid s(\text{Frob}_\ell) = 0 \text{ for all } \ell \in \Sigma_Y\}.$$

Then we have the exact sequence:

$$(7.5) \quad \bigoplus_{\ell \in \Sigma_Y} H_s^1(K_\lambda, T_M \otimes \chi^{-1}) \longrightarrow \text{Sel}_{\mathcal{F}}(K, T_M \otimes \chi)^\vee \longrightarrow \text{Sel}_{\Sigma_Y}^\vee \longrightarrow 0$$

Lemma 7.17. $p^{B_1+2C_2+1} \text{Sel}_{\Sigma_Y} = \{0\}$.

PROOF. By definition, if $s \in \text{Sel}_{\Sigma_Y}$ then $s(\mathcal{G}_S^+ \cap Y) = 0$. Noting that $\mathcal{G}_S \cap Y + p^{2C_2+1} \mathcal{G}_S^+ \subset \mathcal{G}_S^+ \cap Y$, we thus find that

$$\begin{aligned} s(\mathcal{G}_S^+ \cap Y) = 0 &\implies s(p^{2C_2+1} \mathcal{G}_S^+) = 0 \\ &\implies p^{2C_2+1} \text{res}_{K,L}(s) = 0 \in H^1(L, T_M). \end{aligned}$$

By Proposition 7.14(1), it follows that $p^{B_1+2C_2+1} s = 0$. \square

Lemma 7.18. For each $\ell \in \Sigma_Y$, we have

$$p^{2B+B_1} H_s^1(K_\lambda, T_M \otimes \chi^{-1}) \subset \mathcal{O} \partial_\ell \kappa_{\chi^{-1}}(\ell) + \mathcal{O} \partial_\ell \kappa_{\chi^{-1}}(\ell \ell_1).$$

PROOF. By the choice of $\ell \in \Sigma_Y$, we have

$$\varpi^{2B} T_M \subset \mathcal{O}(\alpha_\ell(\kappa_{\chi^{-1}}(1)) + \mathcal{O}(\alpha_\ell(\kappa_{\chi^{-1}}(\ell_1))).$$

This is equivalent to $\varpi^{2B} H_f^1(K_\lambda, T_M \otimes \chi^{-1}) \subset \mathcal{O} \text{loc}_\ell(\kappa_{\chi^{-1}}(1)) + \mathcal{O} \text{loc}_\ell(\kappa_{\chi^{-1}}(\ell_1))$. The lemma thus follows from property (K2). \square

Now Theorem 7.3 is a consequence of the following result.

Theorem 7.19. There exists a positive integer C such that

$$p^C \cdot (\text{Sel}_{\mathcal{F}^*}(K, V/T \otimes \chi) / (F/\mathcal{O} \cdot \mathbf{c}_K^\tau)) = \{0\}.$$

PROOF. We denote by

$$\langle \cdot, \cdot \rangle_\lambda: H_f^1(K_\lambda, T_M \otimes \chi) \times H_s^1(K_\lambda, T_M \otimes \chi^{-1}) \longrightarrow \mathbf{Z}/M\mathbf{Z}$$

the Tate local pairing. By the exact sequence (7.5) combined with Lemma 7.17 and Lemma 7.18, for every $f \in \text{Sel}_{\mathcal{F}}(K, T_M \otimes \chi)^\vee$ we can write

$$p^{C_3} \cdot f = \sum_{\lambda \in \Sigma_Y} a_\ell \partial_\ell \kappa_{\chi^{-1}}(\ell) + b_\ell \partial_\ell \kappa_{\chi^{-1}}(\ell \ell_1), \quad C_3 := 2C_2 + 2B + 2B_1 + 1.$$

Thus for every $s \in \text{Sel}_{\mathcal{F}}(K, T_M \otimes \chi)$ we have

$$\begin{aligned} (p^{C_3} \cdot f)(s) &= f(p^{C_3} \cdot s) \\ &= \sum_{\ell \in \Sigma_Y} \langle \text{loc}_\lambda(s), b_\lambda \partial_\ell \kappa_{\chi^{-1}}(\ell \ell_1) \rangle_\lambda \\ &= \langle \text{loc}_{\lambda_1}(s), t_{\lambda_1} \rangle_{\lambda_1} \quad (t_{\lambda_1} := \sum_{\ell \in \Sigma_Y} -b_\lambda \partial_\ell \kappa_{\chi^{-1}}(\ell \ell_1)). \end{aligned}$$

This implies that p^{C_3} annihilates the kernel of the localization map

$$\text{loc}_{\lambda_1}: \text{Sel}_{\mathcal{F}}(K, T_M \otimes \chi) \longrightarrow H'_{\ell_1} := \{s \in H_f^1(K_{\lambda_1}, T_M \otimes \chi) \mid \langle s, \mathcal{O} \partial_{\ell_1}(\kappa_{\chi^{-1}}(\ell_1)) \rangle_{\lambda_1} = 0\}.$$

On the other hand, setting

$$a_1 := \text{ord}_\varpi(\alpha_{\ell_1}(\kappa_\chi(1))), \quad a_2 := \text{ord}_\varpi \beta_\ell(\partial_{\ell_1} \kappa_{\chi^{-1}}(\ell_1)),$$

by Lemma 7.15 and (K2) we have $a_1 \leq C_1$ and $a_2 \leq C_1 + B_1$. If $M > a_1 + a_2$, an elementary argument shows that

$$\varpi^{2C_1+B_1} H'_{\ell_1} \subset \varpi^{a_1+a_2} H'_{\ell_1} \subset \varpi^{a_2} \mathcal{O} \alpha_{\ell_1}(\kappa_\chi(1)).$$

Combining these together, we deduce that

$$p^{2C_1+B_1+C_3} \text{Sel}_{\mathcal{F}}(K, T_M \otimes \chi) \subset \mathcal{O} \kappa_\chi(1) = \mathcal{O}_M \mathbf{c}_K^\tau$$

for every $M > 2C_1 + 2B_1$, and the theorem follows. \square

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