## 1. Projects: Propagating the Iwasawa main conjecture via congruences

1.1. **Goal of these projects.** Let $f, g \in S_k(\Gamma_0(N))$ be normalized eigenforms (not necessarily newforms) of weight $k \geqslant 2$, say with rational Fourier coefficients $a_n, b_n \in \mathbf{Q}$ for simplicity, and assume that

$$f \equiv g \pmod p$$

in the sense that $a_n \equiv b_n \pmod p$ for all $n > 0$. Roughly speaking, the goal of these projects is to study how knowledge of the Iwasawa main conjecture for $f$ can be "transferred" to $g$.

For $k = 2$ and primes $p \nmid N$ of ordinary reduction, such study was pioneered by Greenberg–Vatsal [GV00], and in these projects we will aim to extend some of their results to:

- non-ordinary primes;
- certain anticyclotomic settings;
- (more ambitiously) some of the "residually reducible" cases which eluded the methods of [GV00], with applications to the $p$-part of the BSD formula in ranks 0 and 1.

1.2. **The method of Greenberg–Vatsal.** Before jumping into the specifics of each of those settings, let us begin with a brief outline of the method of Greenberg–Vatsal (which is beautifully explained in [GV00, §1]). Let $F_\infty/F$ be a $\mathbf{Z}_p$-extension of a number field $F$, and identify the Iwasawa algebra $\mathbf{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$ with the one-variable power series ring $\Lambda = \mathbf{Z}_p[[T]]$ in the usual fashion.

Recall that Iwasawa's main conjecture for $f$ over $F_\infty/F$ posits the following equality between principal ideals of $\Lambda$:

$$(L_p^{\mathrm{alg}}(f)) \stackrel{?}{=} (L_p^{\mathrm{an}}(f)), \tag{1.1}$$

where

- $L_p^{\mathrm{alg}}(f) \in \Lambda$ is a characteristic power series of a Selmer group for $f$ over $F_\infty/F$.
- $L_p^{\mathrm{an}}(f) \in \Lambda$ is a $p$-adic $L$-function interpolating critical values for $L(f/F, s)$ twisted by certain characters of $\mathrm{Gal}(F_\infty/F)$.

By the Weierstrass preparation theorem, we may uniquely write

$$L_p^{\mathrm{alg}}(f) = p^{\mu^{\mathrm{alg}}(f)} \cdot Q^{\mathrm{alg}}(f) \cdot U,$$

with $\mu^{\mathrm{alg}}(f) \in \mathbf{Z}_{\geqslant 0}$, $Q^{\mathrm{alg}}(f) \in \mathbf{Z}_p[T]$ a distinguished polynomial, and $U \in \Lambda^\times$ an invertible power series. Letting

$$\lambda^{\mathrm{alg}}(f) := \deg Q^{\mathrm{alg}}(f),$$

and similarly defining $\mu^{\mathrm{an}}(f)$ and $\lambda^{\mathrm{an}}(f)$ in terms $L_p^{\mathrm{an}}(f)$, the strategy of [GV00] is based on the following three observations:

**O1**. The equality (1.1) amounts to having:
  (1) $(L_p^{\mathrm{alg}}(f)) \supseteq (L_p^{\mathrm{an}}(f))$,
  (2) $\mu^{\mathrm{alg}}(f) = \mu^{\mathrm{an}}(f)$,
  (3) $\lambda^{\mathrm{alg}}(f) = \lambda^{\mathrm{an}}(f)$.

We shall place ourselves in a situation where one expects that $\mu^{\mathrm{alg}}(f) = \mu^{\mathrm{an}}(f) = 0$.

**O2**. For $\Sigma$ any finite set of primes $\ell \neq p, \infty$, the equality (1.1) is *equivalent* to the equality

$$(L_{p,\mathrm{alg}}^{\Sigma}(f)) \stackrel{?}{=} (L_{p,\mathrm{an}}^{\Sigma}(f)), \tag{1.2}$$

where $L_{p,\mathrm{alg}}^{\Sigma}(f)$ and $L_{p,\mathrm{an}}^{\Sigma}(f)$ are the "imprimitive" counterparts of $L_p^{\mathrm{alg}}(f)$ and $L_p^{\mathrm{an}}(f)$ obtained (roughly speaking) by relaxing the local conditions/removing the Euler factors at the primes $\ell \in \Sigma$.

**O3**. For appropriate $\Sigma$, the objects involved in (1.2) are well-behaved under congruences. Letting $\mu_{\mathrm{alg}}^{\Sigma}(f)$, $\lambda_{\mathrm{alg}}^{\Sigma}(f)$, etc. be the obvious invariants from the above discussion, this translates into:

**Expectation 1.** *Assume that* $f \equiv g \pmod{p}$*, and let* $* \in \{\mathrm{alg}, \mathrm{an}\}$*. If* $\mu_*^\Sigma(f) = 0$*, then* $\mu_*^\Sigma(g) = 0$ *and* $\lambda_*^\Sigma(f) = \lambda_*^\Sigma(g)$*.*

Now, if we are given $f \equiv g \pmod{p}$ and the divisibilities

$$(L_p^{\mathrm{alg}}(f)) \supseteq (L_p^{\mathrm{an}}(f)) \quad \text{and} \quad (L_p^{\mathrm{alg}}(g)) \supseteq (L_p^{\mathrm{an}}(g)), \tag{1.3}$$

we see that the equivalence of **O2** combined with **Expectation 1** yields the implication

$$(L_p^{\mathrm{alg}}(f)) = (L_p^{\mathrm{an}}(f)) \quad \Longrightarrow \quad (L_p^{\mathrm{alg}}(g)) = (L_p^{\mathrm{an}}(g)). \tag{1.4}$$

Note that this has interesting applications. Indeed, if for example the residual representation $\bar{\rho}_f$ is absolutely irreducible, then one can hope to establish (1.3) by an Euler/Kolyvagin system argument. Proving the opposite divisibility (either via Eisenstein congruences, or via a refined Euler/Kolyvagin system argument) often requires additional ramification hypotheses on $\bar{\rho}_f$ relative to the level of $f$ (see below for specific examples), a restriction that could be ultimately removed thanks to (1.4).

1.3. **On the cyclotomic main conjectures for non-ordinary primes.** Here we let $F_\infty/F$ be the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, let $p \nmid N$ be a non-ordinary prime for $f \in S_k(\Gamma_0(N))$, and let $\alpha, \beta$ be the roots of the $p$-th Hecke polynomial of $f$. In this setting, Lei–Loeffler–Zerbes [LLZ10], [LLZ11], formulated[1] "signed" main conjectures:

$$(L_p^\sharp(f)) \stackrel{?}{=} \mathrm{Char}_\Lambda(\mathrm{Sel}_\sharp(f)^\vee), \qquad (L_p^\flat(f)) \stackrel{?}{=} \mathrm{Char}_\Lambda(\mathrm{Sel}_\flat(f)^\vee), \tag{1.5}$$

where $\mathrm{Sel}_\sharp(f)$ and $\mathrm{Sel}_\flat(f)$ are Selmer groups cut out by local condition at $p$ more stringent that the usual ones, and $L_p^\sharp(f), L_p^\flat(f) \in \Lambda$ are related to the $p$-adic $L$-functions $L_p^\alpha(f), L_p^\beta(f)$ of Amice–Vélu and Vishik in the following manner:

$$\begin{pmatrix} L_p^\alpha(f) \\ L_p^\beta(f) \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log} \cdot \begin{pmatrix} L_p^\sharp(f) \\ L_p^\flat(f) \end{pmatrix}, \tag{1.6}$$

where $Q_{\alpha,\beta} = \left(\begin{smallmatrix} \alpha & -\beta \\ -p & p \end{smallmatrix}\right)$ and $M_{\log}$ is a certain "logarithm matrix".

**Project A.** *Show* **Expectation 1** *for the signed p-adic L-functions. More precisely, for each* $\bullet \in \{\sharp, \flat\}$*, show that if* $f \equiv g \pmod{p}$*, then*

$$\mu(L_p^\bullet(f)) = 0 \quad \Longrightarrow \quad \mu(L_p^\bullet(g)) = 0$$

*and the* $\lambda$*-invariants of* $\Sigma$*-imprimitive versions of* $L_p^\bullet(f)$ *and* $L_p^\bullet(g)$ *are equal.*

Say $k = 2$ for simplicity. Similarly as in [GV00], the proof of this result would follow from the equality

$$L_p^{\Sigma,\bullet}(f) \equiv u L_p^{\Sigma,\bullet}(g) \pmod{p\Lambda},$$

for some unit $u \in \mathbf{Z}_p^\times$, which in turn would follow from establishing the congruence

$$L_p^{\Sigma,\bullet}(f, \zeta - 1) \equiv u L_p^{\Sigma,\bullet}(g, \zeta - 1) \pmod{p\mathbf{Z}_p[\zeta]}, \tag{1.7}$$

for all $\zeta \in \mu_{p^\infty}$ and some $u \in \mathbf{Z}_p^\times$ independent of $\zeta$. However, a point of departure here from the $p$-ordinary setting is that (unless $a_p = b_p = 0$) the signed $p$-adic $L$-functions $L_p^\bullet(f), L_p^\bullet(g)$ are not directly related to twisted $L$-values, and so the arguments of [GV00, §3] do not suffice to cover this case. Nonetheless, it should be possible to exploit the result of [Vat99, Prop. 1.7], which amounts to the congruence

$$L_p^{\Sigma,\star}(f, \zeta - 1) \equiv u L_p^{\Sigma,\star}(g, \zeta - 1) \pmod{p\mathbf{Z}_p[\zeta]}$$

for both $\star \in \{\alpha, \beta\}$, together with (1.6) to establish (1.7). This will involve a detailed analysis of the values of $M_{\log}$ at $p$-power roots of unity, for which some of the calculations in [LLZ17] (see esp. [*loc.cit.*, Lem. 3.7]) might be useful.

[1] Extending earlier work of Kobayashi, Pollack, Lei, and Sprung

*Remark* 1.1. The algebraic analogue of Project A has recently been established by Hatley–Lei (see [HL16, Thm. 4.6]). On the other hand, as shown in [LLZ11, Cor. 6.6], either of the main conjectures (1.5) is equivalent to Kato's main conjecture (see [LLZ11, Conj. 6.2]). Thus from the discussion of §1.2 and the main result of [KKS17], we see that a successful completion of Project A would yield[2] cases of the signed main conjectures beyond those covered by [Wan14] or [CÇSS17, Thm. B], where the following hypothesis is needed:

$$\text{there exists a prime } \ell \neq p \text{ with } \ell \| N \text{ such that } \bar{\rho}_f \text{ is ramified at } \ell.$$

(*cf.* [KKS17, §1.2.3]).

1.4. **On the anticyclotomic main conjecture of Bertolini–Darmon–Prasanna.** Here we let $F_\infty/F$ be the anticyclotomic $\mathbf{Z}_p$-extension of an imaginary quadratic field $K$ in which

$$p = \mathfrak{p}\overline{\mathfrak{p}} \text{ splits,}$$

let $f \in S_k(\Gamma_0(N))$, and let $p \nmid N$ be a prime. Assume also that every prime factor of $N$ splits in $K$; so $K$ satisfies the *Heegner hypothesis*, and $N^- = 1$ with the standard notation.

The Iwasawa–Greenberg main conjecture for the $p$-adic $L$-function $L_\mathfrak{p}(f) \in \overline{\mathbf{Z}}_p[[\mathrm{Gal}(F_\infty/F)]]$ introduced in [BDP13] predicts that

$$\mathrm{Char}_\Lambda(\mathrm{Sel}_\mathfrak{p}(f)^\vee)\Lambda_{\overline{\mathbf{Z}}_p} \stackrel{?}{=} (L_\mathfrak{p}(f)), \tag{1.8}$$

where $\Lambda_{\overline{\mathbf{Z}}_p} = \overline{\mathbf{Z}}_p[[T]]$ and $\mathrm{Sel}_\mathfrak{p}(f)$ is a Selmer group defined by imposing local triviality (resp. no condition) at the primes above $\mathfrak{p}$ (resp. $\overline{\mathfrak{p}}$).

**Project B.** *Show* **Expectation 1** *for the $p$-adic $L$-functions of* [BDP13]. *That is, if $f \equiv g \pmod p$, then $\mu(L_\mathfrak{p}(f)) = \mu(L_\mathfrak{p}(g)) = 0$*[3] *and the $\lambda$-invariants of $\Sigma$-imprimitive versions of $L_\mathfrak{p}(f)$ and $L_\mathfrak{p}(g)$ are equal.*

Similarly as for Project A, in weight $k = 2$ this problem can be reduced to establishing the congruence

$$L_\mathfrak{p}^\Sigma(f, \zeta - 1) \equiv u L_\mathfrak{p}^\Sigma(g, \zeta - 1) \pmod{p\overline{\mathbf{Z}}_p[\zeta]} \tag{1.9}$$

for all $\zeta \in \mu_{p^\infty}$ and some $u \in \overline{\mathbf{Z}}_p^\times$ independent of $\zeta$. Now, by the $p$-adic Waldspurger formula of [BDP13, Thm. 5.13], the congruence of [KL16, Thm. 2.9] amounts to (1.9) for $\zeta = 1$, and so a promising approach to Project B would be based on extending the result of [KL16, Thm. 2.9] to ramified characters.

*Remark* 1.2. When $p$ is a good *ordinary* prime, the algebraic analogue of Project B has recently been established by Hatley–Lei (see [HL17, Prop. 4.2 and Thm. 5.4]). On the other hand, one can show that Howard's divisibility towards Perrin-Riou's Heegner point main conjecture implies one of the divisibilities predicted by (1.8) (see [How04, Thm. B] and [Cas17b, App. A]). Similarly as in [KKS17], it should be possible to show (this is work in progress) that a suitable refinement of the Kolyvagin system arguments of [How04] combined with Wei Zhang's proof of Kolyvagin's conjecture [Zha14][4] yields the full equality (1.8). In particular, this would yield new cases of conjecture (1.8) with $N^- = 1$ (not currently available in the literature), and even more cases (under a somewhat weaker version of Hypothesis ♠ in [Zha14], still with $N^- = 1$) after a successful completion of Project B.

Finally, in line with the previous remark, we note that the following should be possible:

**Project C.** *Extend the results of* [HL17] *to the non-ordinary case.*

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[2]Subject to the nonvanishing mod $p$ of some "Kurihara number"

[3]Note that in this case the vanishing of $\mu$-invariants is known under mild hypotheses by [Hsi14, Thm. B] and [Bur17, Thm. B]

[4]Which can be seen as proving "primitivity" in the sense of [MR04] of the Heeger point Kolyvagin system

1.5. **On the $p$-part of the Birch–Swinnerton-Dyer formula for residually reducible primes.** Here we consider the primes $p > 2$ for which the associated residual representation $\bar{\rho}_f$ is *reducible*. For simplicity, assume that $f$ corresponds to an elliptic curve $E/\mathbf{Q}$ (admitting a rational $p$-isogeny with kernel $\Phi$). The combination of [GV00, Thm. 3.12] (with a key input from [Kat04, Thm. 17.4]) and [Gre99, Thm.4.1] yields the $p$-part of the BSD formula for $E$ in analytic rank 0, i.e., when $L(E,1) \neq 1$, provided the following holds:

$$\text{(GV)} \qquad \text{the } G_{\mathbf{Q}}\text{-action on } \Phi \subset E[p] \text{ is either } \begin{cases} \text{ramified at } p \text{ and even, or} \\ \text{unramified at } p \text{ and odd.} \end{cases}$$

Similarly as in the residually irreducible cases considered in [JSW17], the above result (applied to a suitable quadratic twist of $E$) would be an important ingredient in the following:

**Project D.** *Prove the $p$-part of the BSD formula in analytic rank* 1 *for elliptic curves $E$ and primes $p > 2$ for which* (GV) *does not hold.*

Following the strategy of [JSW17] and [Cas17a], a key ingredient toward this[5] would be the proof of the relevant cases of the anticyclotomic main conjecture (1.8). By the discussion in §1.2, this could be approached in the following steps:

(1) establish the divisibility "$\supseteq$" in (1.8) (possibly after inverting $p$), based on a suitable refinement of the Kolyvagin system argument in [How04].
(2) show that $\mu(L_{\mathfrak{p}}(f)) = 0$ based on the congruence of [Kri16, Thm. 3] between $L_{\mathfrak{p}}(f)$ and an anticyclotomic Katz $p$-adic $L$-function, and Hida's results on the vanishing of $\mu$ for the latter.
(3) letting $L_{\mathfrak{p}}^{\mathrm{alg}}(f)$ be a generator of the characteristic ideal in (1.8), show that $\mu(L_{\mathfrak{p}}^{\mathrm{alg}}(f)) = 0$ and $\lambda(L_{\mathfrak{p}}^{\mathrm{alg}}(f)) = \lambda(L_{\mathfrak{p}}(f))$ based on an algebraic counterpart of [Kri16, Thm. 3] and the known cases of the main conjecture for the anticyclotomic Katz $p$-adic $L$-function.

After this is carried out, we could try to study the missing cases:

**Project E.** *Prove the $p$-part of the BSD formula for elliptic curves $E/\mathbf{Q}$ at residually reducible primes $p > 2$ when:*

- $L(E,1) \neq 0$ *and* (GV) *doesn't hold (complementing the cases that follow from* [GV00]*).*
- $\mathrm{ord}_{s=1} L(E,s) = 1$ *and* (GV) *holds (complementing the cases covered by* Project D*).*

Finally, we should note that $p = 2$ has been neglected throughout the above discussion, but one would of course like to understand this case as well. (See e.g. [CLZ17] for recent results in this direction.)

## References


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