PROJECTS: PROPAGATING THE IWASAWA MAIN CONJECTURE VIA CONGRUENCES

FRANCESC CASTELLA

ABSTRACT. We describe projects for the course by Christopher Skinner at AWS 2018.

1. GOAL OF THESE PROJECTS

Let $f, g \in S_k(\Gamma_0(N))$ be normalized eigenforms (not necessarily newforms) of weight $k \geqslant 2$, say with rational Fourier coefficients $a_n, b_n \in \mathbf{Q}$ for simplicity, and assume that

$$f \equiv g \pmod{p}$$

in the sense that $a_n \equiv b_n \pmod{p}$ for all $n > 0$. Roughly speaking, the goal of these projects is to study how knowledge of the Iwasawa main conjecture for f can be "transferred" to g.

For $k = 2$ and primes $p \nmid N$ of ordinary reduction, such study was pioneered by Greenberg–Vatsal [GV00], and in these projects we will aim to extend some of their results to:

- non-ordinary primes;
- certain anticyclotomic settings;
- (more ambitiously) some of the "residually reducible" cases which eluded the methods of [GV00], with applications to the p-part of the BSD formula in ranks 0 and 1.

2. THE METHOD OF GREENBERG–VATSAL

Before jumping into the specifics of each of those settings, let us begin with a brief outline of the method of Greenberg–Vatsal (which is beautifully explained in [GV00, §1]). Let F_∞/F be a $\mathbf{Z}_p$-extension of a number field F, and identify the Iwasawa algebra $\mathbf{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$ with the one-variable power series ring $\Lambda = \mathbf{Z}_p[[T]]$ in the usual fashion.

Recall that Iwasawa's main conjecture for f over F_∞/F posits the following equality between principal ideals of Λ:

$$(2.1) \qquad (L_p^{\mathrm{alg}}(f)) \overset{?}{=} (L_p^{\mathrm{an}}(f)),$$

where

- $L_p^{\mathrm{alg}}(f) \in \Lambda$ is a characteristic power series of a Selmer group for f over F_∞/F.
- $L_p^{\mathrm{an}}(f) \in \Lambda$ is a p-adic L-function interpolating critical values for $L(f/F, s)$ twisted by certain characters of $\mathrm{Gal}(F_\infty/F)$.

By the Weierstrass preparation theorem, assuming $L_p^{\mathrm{alg}}(f)$ is nonzero, we may uniquely write

$$L_p^{\mathrm{alg}}(f) = p^{\mu^{\mathrm{alg}}(f)} \cdot Q^{\mathrm{alg}}(f) \cdot U,$$

with $\mu^{\mathrm{alg}}(f) \in \mathbf{Z}_{\geqslant 0}$, $Q^{\mathrm{alg}}(f) \in \mathbf{Z}_p[T]$ a distinguished polynomial, and $U \in \Lambda^\times$ an invertible power series. Letting

$$\lambda^{\mathrm{alg}}(f) := \deg Q^{\mathrm{alg}}(f),$$

and similarly defining $\mu^{\mathrm{an}}(f)$ and $\lambda^{\mathrm{an}}(f)$ in terms $L_p^{\mathrm{an}}(f)$, the strategy of [GV00] is based on the following three observations:

O1. The equality (2.1) amounts to having:
    (1) $(L_p^{\mathrm{alg}}(f)) \supseteq (L_p^{\mathrm{an}}(f))$,

(2) $\mu^{\mathrm{alg}}(f) = \mu^{\mathrm{an}}(f)$,
(3) $\lambda^{\mathrm{alg}}(f) = \lambda^{\mathrm{an}}(f)$.

We shall place ourselves in a situation where one expects that $\mu^{\mathrm{alg}}(f) = \mu^{\mathrm{an}}(f) = 0$.

O2. For Σ any finite set of primes $\ell \neq p, \infty$, the equality (2.1) is *equivalent* to the equality

$$(2.2) \qquad (L^{\Sigma}_{p,\mathrm{alg}}(f)) \overset{?}{=} (L^{\Sigma}_{p,\mathrm{an}}(f)),$$

where $L^{\Sigma}_{p,\mathrm{alg}}(f)$ and $L^{\Sigma}_{p,\mathrm{an}}(f)$ are the "imprimitive" counterparts of $L^{\mathrm{alg}}_p(f)$ and $L^{\mathrm{an}}_p(f)$ obtained (roughly speaking) by relaxing the local conditions/removing the Euler factors at the primes $\ell \in \Sigma$.

O3. For appropriate Σ, the objects involved in (2.2) are well-behaved under congruences. Letting $\mu^{\Sigma}_{\mathrm{alg}}(f)$, $\lambda^{\Sigma}_{\mathrm{alg}}(f)$, etc. be the obvious invariants from the above discussion, this translates into:

Expectation 1. *Assume that $f \equiv g \pmod{p}$, and let $* \in \{\mathrm{alg}, \mathrm{an}\}$. If $\mu^{\Sigma}_*(f) = 0$, then $\mu^{\Sigma}_*(g) = 0$ and $\lambda^{\Sigma}_*(f) = \lambda^{\Sigma}_*(g)$.*

Now, if we are given $f \equiv g \pmod{p}$ and the divisibilities

$$(2.3) \qquad (L^{\mathrm{alg}}_p(f)) \supseteq (L^{\mathrm{an}}_p(f)) \quad \text{and} \quad (L^{\mathrm{alg}}_p(g)) \supseteq (L^{\mathrm{an}}_p(g)),$$

we see that the equivalence of **O2** combined with **Expectation 1** yields the implication

$$(2.4) \qquad (L^{\mathrm{alg}}_p(f)) = (L^{\mathrm{an}}_p(f)) \quad \Longrightarrow \quad (L^{\mathrm{alg}}_p(g)) = (L^{\mathrm{an}}_p(g)).$$

Note that this has interesting applications. Indeed, if for example the residual representation $\bar{\rho}_f$ is absolutely irreducible, then one can hope to establish (2.3) by an Euler/Kolyvagin system argument. Proving the opposite divisibility (either via Eisenstein congruences, or via a refined Euler/Kolyvagin system argument) often requires additional ramification hypotheses on $\bar{\rho}_f$ relative to the level of f (see below for specific examples), a restriction that could be ultimately removed thanks to (2.4).

3. On the cyclotomic main conjectures for non-ordinary primes

Here we let F_∞/F be the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, let $p \nmid N$ be a non-ordinary prime for $f \in S_k(\Gamma_0(N))$, and let α, β be the roots of the p-th Hecke polynomial of f. In this setting, Lei–Loeffler–Zerbes [LLZ10], [LLZ11], formulated[1] "signed" main conjectures:

$$(3.1) \qquad (L^{\sharp}_p(f)) \overset{?}{=} \mathrm{Char}_\Lambda(\mathrm{Sel}_\sharp(f)^\vee), \qquad (L^{\flat}_p(f)) \overset{?}{=} \mathrm{Char}_\Lambda(\mathrm{Sel}_\flat(f)^\vee),$$

where $\mathrm{Sel}_\sharp(f)$ and $\mathrm{Sel}_\flat(f)$ are Selmer groups cut out by local condition at p more stringent that the usual ones, and $L^{\sharp}_p(f), L^{\flat}_p(f) \in \Lambda$ are related to the p-adic L-functions $L^{\alpha}_p(f), L^{\beta}_p(f)$ of Amice–Vélu and Vishik in the following manner:

$$(3.2) \qquad \begin{pmatrix} L^{\alpha}_p(f) \\ L^{\beta}_p(f) \end{pmatrix} = Q^{-1}_{\alpha,\beta} M_{\log} \cdot \begin{pmatrix} L^{\sharp}_p(f) \\ L^{\flat}_p(f) \end{pmatrix},$$

where $Q_{\alpha,\beta} = \begin{pmatrix} \alpha & -\beta \\ -p & p \end{pmatrix}$ and $M_{\log}$ is a certain "logarithm matrix".

Project A. *Show **Expectation 1** for the signed p-adic L-functions. More precisely, for each $\bullet \in \{\sharp, \flat\}$, show that if $f \equiv g \pmod{p}$, then*

$$\mu(L^{\bullet}_p(f)) = 0 \quad \Longrightarrow \quad \mu(L^{\bullet}_p(g)) = 0$$

and the λ-invariants of Σ-imprimitive versions of $L^{\bullet}_p(f)$ and $L^{\bullet}_p(g)$ are equal.

---

[1] Extending earlier work of Kobayashi, Pollack, Lei, and Sprung

Say $k = 2$ for simplicity. Similarly as in [GV00], the proof of this result would follow from the equality

$$L_p^{\Sigma,\bullet}(f) \equiv u L_p^{\Sigma,\bullet}(g) \pmod{p\Lambda},$$

for some unit $u \in \mathbf{Z}_p^\times$, which in turn would follow from establishing the congruence

(3.3) $$L_p^{\Sigma,\bullet}(f, \zeta - 1) \equiv u L_p^{\Sigma,\bullet}(g, \zeta - 1) \pmod{p\mathbf{Z}_p[\zeta]},$$

for all $\zeta \in \mu_{p^\infty}$ and some $u \in \mathbf{Z}_p^\times$ independent of ζ. However, a point of departure here from the p-ordinary setting is that (unless $a_p = b_p = 0$) the signed p-adic L-functions $L_p^\bullet(f), L_p^\bullet(g)$ are not directly related to twisted L-values, and so the arguments of [GV00, §3] do not suffice to cover this case. Nonetheless, it should be possible to exploit the result of [Vat99, Prop. 1.7], which amounts to the congruence

$$L_p^{\Sigma,\star}(f, \zeta - 1) \equiv u L_p^{\Sigma,\star}(g, \zeta - 1) \pmod{p\mathbf{Z}_p[\zeta]}$$

for both $\star \in \{\alpha, \beta\}$, together with (3.2) to establish (3.3). This will involve a detailed analysis of the values of $M_{\log}$ at p-power roots of unity, for which some of the calculations in [LLZ17] (see esp. [*loc.cit.*, Lem. 3.7]) might be useful.

Remark 3.1. The algebraic analogue of Project A has recently been established by Hatley–Lei (see [HL16, Thm. 4.6]). On the other hand, as shown in [LLZ11, Cor. 6.6], either of the main conjectures (3.1) is equivalent to Kato's main conjecture (see [LLZ11, Conj. 6.2]). Thus from the discussion of §2 and the main result of [KKS17], we see that a successful completion of Project A would yield[2] cases of the signed main conjectures beyond those covered by [Wan14] or [CÇSS17, Thm. B], where the following hypothesis is needed:

there exists a prime $\ell \neq p$ with $\ell \| N$ such that $\bar{\rho}_f$ is ramified at ℓ.

(*cf.* [KKS17, §1.2.3]).

4. On the anticyclotomic main conjecture of Bertolini–Darmon–Prasanna

Here we let F_∞/F be the anticyclotomic $\mathbf{Z}_p$-extension of an imaginary quadratic field K in which

$$p = \mathfrak{p}\bar{\mathfrak{p}} \text{ splits},$$

let $f \in S_k(\Gamma_0(N))$, and let $p \nmid N$ be a prime. Assume also that every prime factor of N splits in K; so K satisfies the *Heegner hypothesis*, and $N^- = 1$ with the standard notation.

The Iwasawa–Greenberg main conjecture for the p-adic L-function $L_\mathfrak{p}(f) \in \overline{\mathbf{Z}}_p[[\mathrm{Gal}(F_\infty/F)]]$ introduced in [BDP13] predicts that

(4.1) $$\mathrm{Char}_\Lambda(\mathrm{Sel}_\mathfrak{p}(f)^\vee)\Lambda_{\overline{\mathbf{Z}}_p} \stackrel{?}{=} (L_\mathfrak{p}(f)),$$

where $\Lambda_{\overline{\mathbf{Z}}_p} = \overline{\mathbf{Z}}_p[[T]]$ and $\mathrm{Sel}_\mathfrak{p}(f)$ is a Selmer group defined by imposing local triviality (resp. no condition) at the primes above $\mathfrak{p}$ (resp. $\bar{\mathfrak{p}}$).

Project B. *Show* **Expectation 1** *for the p-adic L-functions of* [BDP13]. *That is, if $f \equiv g \pmod p$, then $\mu(L_\mathfrak{p}(f)) = \mu(L_\mathfrak{p}(g)) = 0$[3] and the λ-invariants of Σ-imprimitive versions of $L_\mathfrak{p}(f)$ and $L_\mathfrak{p}(g)$ are equal.*

Similarly as for Project A, in weight $k = 2$ this problem can be reduced to establishing the congruence

(4.2) $$L_\mathfrak{p}^\Sigma(f, \zeta - 1) \equiv u L_\mathfrak{p}^\Sigma(g, \zeta - 1) \pmod{p\overline{\mathbf{Z}}_p[\zeta]}$$

---

[2]Subject to the nonvanishing mod p of some "Kurihara number"

[3]Note that in this case the vanishing of μ-invariants is known under mild hypotheses by [Hsi14, Thm. B] and [Bur17, Thm. B]

for all $\zeta \in \mu_{p^\infty}$ and some $u \in \overline{\mathbf{Z}}_p^\times$ independent of ζ. Now, by the p-adic Waldspurger formula of [BDP13, Thm. 5.13], the congruence of [KL16, Thm. 2.9] amounts to (4.2) for $\zeta = 1$, and so a promising approach to Project B would be based on extending the result of [KL16, Thm. 2.9] to ramified characters.

Remark 4.1. When p is a good *ordinary* prime, the algebraic analogue of Project B has recently been established by Hatley–Lei (see [HL17, Prop. 4.2 and Thm. 5.4]). On the other hand, one can show that Howard's divisibility towards Perrin-Riou's Heegner point main conjecture implies one of the divisibilities predicted by (4.1) (see [How04, Thm. B] and [Cas17b, App. A]). Similarly as in [KKS17], it should be possible to show (this is work in progress) that a suitable refinement of the Kolyvagin system arguments of [How04] combined with Wei Zhang's proof of Kolyvagin's conjecture [Zha14][4] yields the full equality (4.1). In particular, this would yield new cases of conjecture (4.1) with $N^- = 1$ (not currently available in the literature), and even more cases (under a somewhat weaker version of Hypothesis ♠ in [Zha14], still with $N^- = 1$) after a successful completion of Project B.

Finally, in line with the previous remark, we note that the following should be possible:

Project C. *Extend the results of* [HL17] *to the non-ordinary case.*

5. On the p-part of the Birch–Swinnerton-Dyer formula for residually reducible primes

Here we consider the primes $p > 2$ for which the associated residual representation $\bar{\rho}_f$ is *reducible*. For simplicity, assume that f corresponds to an elliptic curve $E/\mathbf{Q}$ (admitting a rational p-isogeny with kernel Φ). The combination of [GV00, Thm. 3.12] (with a key input from [Kat04, Thm. 17.4]) and [Gre99, Thm.4.1] yields the p-part of the BSD formula for E in analytic rank 0, i.e., when $L(E, 1) \neq 1$, provided the following holds:

(GV)   the $G_{\mathbf{Q}}$-action on $\Phi \subset E[p]$ is either $\begin{cases} \text{ramified at } p \text{ and even, or} \\ \text{unramified at } p \text{ and odd.} \end{cases}$

Similarly as in the residually irreducible cases considered in [JSW17], the above result (applied to a suitable quadratic twist of E) would be an important ingredient in the following:

Project D. *Prove the p-part of the BSD formula in analytic rank* 1 *for elliptic curves E and primes $p > 2$ for which* (GV) *does not hold.*

Following the strategy of [JSW17] and [Cas17a], a key ingredient toward this[5] would be the proof of the relevant cases of the anticyclotomic main conjecture (4.1). By the discussion in §2, this could be approached in the following steps:

(1) establish the divisibility "⊇" in (4.1) (possibly after inverting p), based on a suitable refinement of the Kolyvagin system argument in [How04].
(2) show that $\mu(L_{\mathfrak{p}}(f)) = 0$ based on the congruence of [Kri16, Thm. 3] between $L_{\mathfrak{p}}(f)$ and an anticyclotomic Katz p-adic L-function, and Hida's results on the vanishing of μ for the latter.
(3) letting $L_{\mathfrak{p}}^{\mathrm{alg}}(f)$ be a generator of the characteristic ideal in (4.1), show that $\mu(L_{\mathfrak{p}}^{\mathrm{alg}}(f)) = 0$ and $\lambda(L_{\mathfrak{p}}^{\mathrm{alg}}(f)) = \lambda(L_{\mathfrak{p}}(f))$ based on an algebraic counterpart of [Kri16, Thm. 3] and the known cases of the main conjecture for the anticyclotomic Katz p-adic L-function.

After this is carried out, we could try to study the missing cases:

---

[4]Which can be seen as proving "primitivity" in the sense of [MR04] of the Heeger point Kolyvagin system

[5]Note that there are other points where the residually irreducible hypothesis is used in [JSW17], e.g. in the "anticyclotomic control theorem" of [*loc.cit.*, §3.3], but handling these should be relatively easy.

Project E. *Prove the p-part of the BSD formula for elliptic curves $E/\mathbf{Q}$ at residually reducible primes $p > 2$ when:*

- $L(E, 1) \neq 0$ *and* (GV) *doesn't hold (complementing the cases that follow from* [GV00]*).*
- $\mathrm{ord}_{s=1} L(E, s) = 1$ *and* (GV) *holds (complementing the cases covered by* Project D*).*

Finally, we should note that $p = 2$ has been neglected throughout the above discussion, but one would of course like to understand this case as well. (See e.g. [CLZ17] for recent results in this direction.)

REFERENCES

[BDP13]   Massimo Bertolini, Henri Darmon, and Kartik Prasanna, *Generalized Heegner cycles and p-adic Rankin L-series*, Duke Math. J. **162** (2013), no. 6, 1033–1148.

[Bur17]   Ashay A. Burungale, *On the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p, II: Shimura curves*, J. Inst. Math. Jussieu **16** (2017), no. 1, 189–222. MR 3591965

[Cas17a]   Francesc Castella, *On the p-part of the Birch–Swinnerton-Dyer formula for multiplicative primes*, Camb. J. Math., to appear (2017).

[Cas17b]   ———, *p-adic heights of Heegner points and Beilinson-Flach classes*, J. Lond. Math. Soc. (2) **96** (2017), no. 1, 156–180. MR 3687944

[CÇSS17]   Francesc Castella, Mirela Çiperiani, Christopher Skinner, and Florian Sprung, *On two-variable main conjectures for modular forms at non-ordinary primes*, preprint (2017).

[CLZ17]   Li Cai, Chao Li, and Shuai Zhai, *On the 2-part of the Birch and Swinnerton-Dyer conjecture for quadratic twists of elliptic curves*, preprint, `arXiv:1712.01271` (2017).

[Gre99]   Ralph Greenberg, *Iwasawa theory for elliptic curves*, Arithmetic theory of elliptic curves (Cetraro, 1997), Lecture Notes in Math., vol. 1716, Springer, Berlin, 1999, pp. 51–144.

[GV00]   Ralph Greenberg and Vinayak Vatsal, *On the Iwasawa invariants of elliptic curves*, Invent. Math. **142** (2000), no. 1, 17–63. MR 1784796

[HL16]   Jeffrey Hatley and Antonio Lei, *Arithmetic properties of signed Selmer groups at non-ordinary primes*, preprint, `arXiv:1608.00257` (2016).

[HL17]   ———, *Comparing anticyclotomic Selmer groups of positive coranks for congruent modular forms*, preprint, `arXiv:1706.04531` (2017).

[How04]   Benjamin Howard, *The Heegner point Kolyvagin system*, Compos. Math. **140** (2004), no. 6, 1439–1472. MR 2098397 (2006a:11070)

[Hsi14]   Ming-Lun Hsieh, *Special values of anticyclotomic Rankin-Selberg L-functions*, Doc. Math. **19** (2014), 709–767. MR 3247801

[JSW17]   Dimitar Jetchev, Christopher Skinner, and Xin Wan, *The Birch and Swinnerton-Dyer formula for elliptic curves of analytic rank one*, Camb. J. Math. **5** (2017), no. 3, 369–434. MR 3684675

[Kat04]   Kazuya Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, Astérisque (2004), no. 295, ix, 117–290, Cohomologies p-adiques et applications arithmétiques. III. MR 2104361 (2006b:11051)

[KKS17]   Chan-Ho Kim, Myoungil Kim, and Hae-Sang Sun, *On the indivisibility of derived Kato's Euler systems and the main conjecture for modular forms*, preprint, `arXiv:1709.05780` (2017).

[KL16]   Daniel Kriz and Chao Li, *Congruences between Heegner points and quadratic twists of elliptic curves*, preprint, `arXiv:1606.03172` (2016).

[Kri16]   Daniel Kriz, *Generalized Heegner cycles at Eisenstein primes and the Katz p-adic L-function*, Algebra Number Theory **10** (2016), no. 2, 309–374. MR 3477744

[LLZ10]   Antonio Lei, David Loeffler, and Sarah Livia Zerbes, *Wach modules and Iwasawa theory for modular forms*, Asian J. Math. **14** (2010), no. 4, 475–528.

[LLZ11]   ———, *Coleman maps and the p-adic regulator*, Algebra Number Theory **5** (2011), no. 8, 1095–1131. MR 2948474

[LLZ17]   ———, *On the asymptotic growth of Bloch-Kato-Shafarevich-Tate groups of modular forms over cyclotomic extensions*, Canad. J. Math. **69** (2017), no. 4, 826–850. MR 3679697

[MR04]   Barry Mazur and Karl Rubin, *Kolyvagin systems*, Mem. Amer. Math. Soc. **168** (2004), no. 799, viii+96. MR 2031496 (2005b:11179)

[Vat99]   V. Vatsal, *Canonical periods and congruence formulae*, Duke Math. J. **98** (1999), no. 2, 397–419. MR 1695203

[Wan14]   Xin Wan, *Iwasawa main conjecture for supersingular elliptic curves*, preprint, `arXiv:1411.6352` (2014).

[Zha14]   Wei Zhang, *Selmer groups and the indivisibility of Heegner points*, Camb. J. Math. **2** (2014), no. 2, 191–253. MR 3295917

MATHEMATICS DEPARTMENT, PRINCETON UNIVERSITY, FINE HALL, PRINCETON, NJ 08544-1000, USA
E-mail address: `fcabello@princeton.edu`