SPACE-TIME RESONANCES AND THE NULL CONDITION FOR WAVE EQUATIONS

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ABSTRACT. In this note we describe a recent result obtained by the author and Shatah [26], concerning global existence and scattering for small solutions of nonlinear wave equations. Based on the analysis of space-time resonances, we formulate a very natural non-resonance condition for quadratic nonlinearities that guarantees the existence of global solutions with linear asymptotic behavior. This non-resonance condition turns out to be a generalization of the null condition given by Klainerman in his seminal work [21].

1. INTRODUCTION

Global existence and asymptotic behavior of small solutions to nonlinear wave equations has been a subject under active investigation for over fifty years. One area of research, where much progress has been made, focuses on identifying nonlinearities that lead to global solutions for small initial data. In this note we consider first order systems on $\mathbb{R} \times \mathbb{R}^3$, of the form

$$\text{(W)} \qquad \begin{cases} \partial_t u = i\Lambda u + Q_1(u,v) + R_1(u,v) \\ \partial_t v = -i\Lambda v + Q_2(u,v) + R_2(u,v) \\ u(1,x) = u^0(x)\,, v(1,x) = v^0(x), \end{cases}$$

where $\Lambda := |\nabla|$, $Q_i(u,v)$ are bilinear in (u,v) and their complex conjugates, and R_i are of degree 3 or higher.

In this paper we focus on determining some general conditions, naturally arising from the *space time resonance analysis*, that guarantee global existence and scattering. Our non-resonant condition imposed on the Q_i, roughly states that time resonant wave interactions should be limited to waves with different group velocities (spatially non-resonant waves).

Since cubic and higher order terms do not require any condition to ensure global existence, we will drop the R_i's from any further consideration. Moreover by introducing the notation for bilinear pseudo-product operator

$$T_{m(\xi,\eta)}(f,g) := \mathcal{F}^{-1} \int m(\xi,\eta)\widehat{f}(\eta)\widehat{g}(\xi-\eta)\,d\eta,$$

where $\hat{g} = \mathcal{F}g$ is the Fourier transform of g, and without any loss of generality, we reduce the system to a single scalar equation

$$\text{(1.1)} \qquad \begin{cases} \partial_t u - i\Lambda u = T_{q_{+,+}(\xi,\eta)}(u,u) + T_{q_{-,+}(\xi,\eta)}(\bar{u},u) + T_{q_{-,-}(\xi,\eta)}(\bar{u},\bar{u}) \\ \\ u(1,x) = u^0(x)\,. \end{cases}$$

with quadratic nonlinearities. Here and throughout the paper, the signs $+,-$ correspond to the presence of u and $\bar{u}$ respectively.

Our work is motivated by some recent applications of the *space-time resonance method* to several problems in the field of nonlinear dispersive PDEs. This method was introduced in [7, 9] where non resonant nonlinearites were treated for Schrödinger equations, which corresponds to (W) with $\Lambda = |\nabla|^2 = -\Delta$. In these works, most of the existing results on global existence and scattering of small solutions were reproduced and explained by studying space time resonant frequencies. Subsequently the method was applied to gravity water waves [8], which corresponds to $\Lambda = |\nabla|^{1/2}$, and to capillary waves [10], which corresponds

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to $\Lambda = |\nabla|^{3/2}$. Thus it is natural to us to apply this method to system (W), where $\Lambda = |\nabla|$, which can be reduced to a system of nonlinear wave equations. Our main result is:

Theorem 1.1. *Assume that system* (1.1) *is non-resonant in the sense of definition 4.3, and that the initial datum satisfies*[1]

$$\|xu_0\|_{H^2} + \left\|\Lambda x^2 u_0\right\|_{H^1} + \|u_0\|_{H^N} \leq \epsilon \tag{1.2}$$

for some large enough integer N. Then, if ϵ is small enough, there exists a unique global solution to (1.1) *with*

$$\|u(t)\|_{L^\infty} \lesssim \frac{\epsilon}{t}\,.$$

Moreover, $u(t)$ scatters in H^2 to a linear solution as $t \to \infty$.

Our non resonant condition defined in 4.3 turns out to include the classical null condition for wave equations [21], wave equations which are not invariant under the full Lorentz group, as well as other systems where global existence and asymptotic behavior of small solutions was not known.

2. Background

Since our system can be reduced to nonlinear wave equations, we give a brief review of some of the main results about the long time existence of solutions for systems of quadratic nonlinear wave equations on $\mathbb{R}^{1+3}$:

$$\Box u_i = \sum a^{jk}_{i,\alpha\beta} \partial^\alpha u_j \partial^\beta u_k \;+\; \text{cubic terms} \tag{2.1}$$

where $i = 1, \ldots, N$ for some $N \in \mathbb{N}$, and the sum runs over $j,k = 1, \ldots, N$, and all multi-indices $\alpha, \beta \in \mathbb{N}^4$ with $|\alpha|, |\beta| \leq 2$, $|\alpha| + |\beta| \leq 3$, with the usual convention that $\partial_0 = -\partial^0 = \partial_t$. Let us first recall that in 3 space dimensions general quadratic nonlinearities have long range effects: the L^2 norm of the nonlinearity, computed on a linear solution, decays at the borderline non-integrable rate of t^{-1}. Thus, quadratic nonlinearties can contribute to the long time behavior of solutions. It is in fact known since the pioneering works of John [11, 12] that finite time blowup can occur even for solutions with small data. On the other hand, for some very general classes of quadratic nonlinearities solutions were shown to exist almost globally by John and Klainerman [13] and Klainerman [19].

The main breakthrough in identifying classes of nonlinear wave equations where solutions with small data exist globally and scatter was in the works of Klainerman [21], Choquet-Bruhat and Christodoulou [2], and Christodoulou [3]. The class of nonlinearities that satisfy the "null condition" was introduced by Klainerman [21], and for semlinear systems

$$\Box u_i = \sum_{|\alpha|,|\beta|=1} a^{jk}_{i,\alpha\beta} \partial^\alpha u_j \partial^\beta u_k + \text{ cubic terms} \tag{2.2}$$

is given by the condition

$$\sum a^{jk}_{i,\alpha\beta} \xi_\alpha \xi_\beta = 0 \qquad \text{for any } \xi \in \mathbb{R}^4 \text{ such that } -\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 0\,. \tag{2.3}$$

For such systems it was shown by Klainerman [21] that in $3+1$ dimension small data solutions exists globally. This seminal work of Klainerman is based on the invariance of Minkowski space under the Lorentz group and on energy estimates using the vector fields that generate the Lorentz group [19].

Later on, building on Klainerman's original ideas, the problem of bypassing the use of the full invariance under the Lorentz group was dealt with by other authors. In [22] Klainerman and Sideris proved almost global existence of solutions for quadratic systems (2.1) in divergence form, under the sole assumption of translation, rotation and scaling invariance. Further developments were made by Sideris in [27, 28], where global existence of nonlinear elastic waves is proven under the assumption of the null condition. Similar results include the almost global existence of solutions contained in the works of Keel, Smith and

[1] See the remark at the end of section 3 for some comments about these initial conditions.

Sogge [17, 18]. It also worth mentioning that several works have dealt with the question of identifying other conditions (weaker than the null condition) under which global existence of solutions of (2.1) can be proven; see for instance Lindblad [23], Alinhac [1], Lindblad and Rodnianski [24, 25], and Katayama [14].

Another approach that identifies the effects of nonlinearities on the long time behavior of solutions is based on time resonant computations. For ODE's this is the Poincare-Dulac normal form. For PDE's normal forms were introduced by Shatah [30] and Simon [32] who treated, respectively, the Cauchy problem and the final state problem for the Klein-Gordon equation in $3+1$ dimensions. Similar results were obtained by Klainerman using the vector fields method [20].

In the past several years a new algorithmic method, called the "space-time resonance method", was developed by Germain, Masmoudi, and Shatah, to study long time behavior of spatially localized small solutions to dispersive equations. By bringing together ideas from both vector fields and normal forms, this new method proved to be very effective in proving new results [8, 5, 6] as well as simplifying already existing ones [7, 9, 16]. A description of this method can be found in [7].

Notations. We use R to denote indistinctly any one of the components of the vector of Riesz transforms $R = \frac{\nabla}{\Lambda}$, where $\Lambda := |\nabla|$. L^p norms will be denoted either by $\|\cdot\|_{L^p}$ or $\|\cdot\|_p$. For $s \geq 0$, $p \geq 1$, we define the usual Sobolev norms

$$\|\varphi\|_{W^{s,p}} := \|\langle\nabla\rangle^s \varphi\|_{L^p}\,,$$
$$\|\varphi\|_{\dot{W}^{s,p}} := \|\Lambda^s \varphi\|_{L^p}\,,$$

where $\langle x\rangle := \left(1+|x|^2\right)^{1/2}$. We let $H^s := W^{s,2}$ and $\dot{H}^s := \dot{W}^{s,2}$.
Finally we write $A \lesssim B$ to mean $A \leq CB$ for some positive absolute constant C.

3. Resonance analysis and non-resonant bilinear forms

To compute resonances for an equation of the type

$$iu_t + P(\frac{1}{i}\nabla)u = T_{m(\xi,\eta)}(u,u),$$

we write Duhamel's formula in Fourier space for the "profile" of u, namely $f := e^{-itP(\frac{1}{i}\nabla)}u$, as follows:

$$\widehat{f}(t,\xi) = \widehat{u}_0(\xi) + \int_0^t\!\!\int e^{is\varphi(\xi,\eta)} m(\xi,\eta)\widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\,d\eta\,ds\,, \tag{3.1}$$

where $\varphi(\xi,\eta) := -P(\xi) + P(\eta) + P(\eta-\xi)$ (and obvious signs modifications occur if Q depends also on $\bar{u}$). We then define the *time resonant set*

$$\mathscr{T} := \{(\xi,\eta)\,:\,\varphi(\xi,\eta)=0\} \qquad \text{(no oscillations in } s)\,,$$

the *space resonant set*

$$\mathscr{S} := \{(\xi,\eta)\,:\,\nabla_\eta\varphi(\xi,\eta)=0\} \qquad \text{(no oscillations in } \eta)\,,$$

and the *space-time resonant set*

$$\mathscr{R} := \mathscr{T} \cap \mathscr{S}\,.$$

Since for system (1.1) both u and $\bar{u}$ are present in the bilinear terms, there are three types of interactions that we need to analyze.

The - - case. The phase $\varphi_{--} := -|\xi| - |\eta| - |\xi - \eta|$ clearly vanishes only at $\xi = \eta = 0$:

$$\mathscr{T}_{--} = \{\eta = \xi = 0\}\,.$$

Since the time resonant set is reduced to a point, we can perform a *normal form transformation*. This allows us to obtain the L^∞ decay in a more direct fashion (without the need to resort to weighted estimates). For completeness we compute

$$\mathscr{S}_{--} = \{\eta = \lambda\xi, \quad 0 \leq \lambda \leq 1\},$$

and

$$\mathscr{R}_{--} = \{\xi = \eta = 0\}\,.$$

The + + case. The phase $\varphi_{++} := -|\xi| + |\eta| + |\xi - \eta|$ vanishes on

$$\mathscr{T}_{++} = \{\eta = \lambda\xi, \quad 0 \leq \lambda \leq 1\}\,.$$

A simple computation shows that

$$\mathscr{S}_{++} = \{\eta = \lambda\xi, \quad 0 \leq \lambda \leq 1\}\,,$$

whence

$$\mathscr{R}_{++} = \mathscr{S}_{++} = \mathscr{T}_{++}\,.$$

The space time resonant set is very large, thus some additional structures are needed to help controlling these resonances. The first structure will be imposed on the interaction by requiring the symbol q_{++} to vanish on $\mathscr{R}_{++}$. The second structure is present in the phase

$$|\xi|\nabla_\xi \varphi_{++} = \frac{\eta - \xi}{|\eta - \xi|}\varphi_{++} - |\eta|\nabla_\eta \varphi_{++}\,,$$

and can be interpreted by saying that all resonant waves have the same group velocity and thus are spatially localized in the same region. This fact together with $\mathscr{S}_{++} = \mathscr{T}_{++}$ allows to control these resonances by solely relying on weighted energy estimates.

The - + and + - cases. Up to the change of variables[2] $\eta \to \xi - \eta$, these two case are the same. Therefore, we will just focus on the $-+$ case. Since the phase is $\varphi_{-+} := -|\xi| - |\eta| + |\xi - \eta|$, then

$$\begin{aligned}
\mathscr{T}_{-+} &= \{\eta = \lambda\xi, \quad \lambda \leq 0\} \cup \{\xi = 0\}\,,\\
\mathscr{S}_{-+} &= \{\eta = \lambda\xi, \quad \lambda \leq 0 \quad \text{or} \quad \lambda \geq 1\} \cup \{\xi = 0\}\,,\\
\mathscr{R}_{-+} &= \{\eta = \lambda\xi, \quad \lambda \leq 0\} \cup \{\xi = 0\}\,.
\end{aligned}$$

Again the set $\mathscr{R}_{-+}$ is very big and additional conditions are needed to ensure global existence and linear asymptotic behavior of solutions. These conditions are similar to the $++$ interaction, i.e., q_{-+} vanishes on $\mathscr{R}_{-+}$ and the fact that

$$|\xi|\nabla_\xi \varphi_{-+} = \frac{\eta - \xi}{|\eta - \xi|}\varphi_{-+} + |\eta|\nabla_\eta \varphi_{-+}\,.$$

However this interaction presents an additional difficulty over the $++$ case since $\mathscr{T}_{-+} \subsetneq \mathscr{S}_{-+}$, which requires both normal forms transformation and weighted estimates. The fact that this is an added difficulty is explained below.

[2] notice that $m_{12}(\xi, \xi - \eta) = -m_{12}(\xi, \eta)$.

4. Non-resonant bilinear forms

From Duhamel's formula for equation (1.1) in Fourier space the quadratic terms are expressed as

$$\widehat{B}_{\pm\pm}(t,\xi) = \int_1^t \int e^{is\varphi_{\pm\pm}(\xi,\eta)} q_{\pm\pm}(\xi,\eta) \widehat{f}_\pm(s,\eta) \widehat{f}_\pm(s,\xi-\eta)\, d\eta ds$$

where $f_+ = f$ and $f_- = \bar{f}$ and

$$\varphi_{\epsilon_1,\epsilon_2}(\xi,\eta) = -|\xi| + \epsilon_1|\xi-\eta| + \epsilon_2|\eta|\,,$$

for $\epsilon_i = \pm$. The quadratic interaction is given in terms of its symbol $q_{\pm\pm}$. To define non-resonant bilinear forms we start by defining the class of symbols that we will be dealing with:

Definition 4.1. A symbol $m = m(\xi,\eta)$ belongs to the class $\mathcal{B}_s$ if

• It is homogeneous of degree s;

• It is smooth outside of $\{\xi = 0\} \cup \{\eta = 0\} \cup \{\xi - \eta = 0\}$.

• For any labeling (ξ_1, ξ_2, ξ_3) of the three Fourier variables $(\xi, \eta, \xi-\eta)$ the following holds:

$$\text{for} \quad |\xi_1| \ll |\xi_2|, |\xi_3| \sim 1 \qquad m = \mathcal{A}\left(|\xi_1|, \frac{\xi_1}{|\xi_1|}, \xi_2\right)$$

for some smooth function $\mathcal{A}$.

Loosely speaking, symbols in $\mathcal{B}_0$ are Coifman-Meyer [4] except, possibly, along the axes $\xi = 0, \eta = 0$ and $\xi - \eta = 0$, where they can have singularities like linear Mihlin-Hörmander multipliers. Symbols in $\mathcal{B}_s$ are essentially of the form $|\xi|^s m_0$ for some $m_0 \in \mathcal{B}_0$. The boundedness of these bilinear operators on L^p is given by the following Theorem:

Theorem 4.2 (Boundedness of bilinear multipliers [8],[26]). *Let p, q, r be given such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $1 < p, q, r < \infty$. The following hold*

(i) *If m belongs to the class $\mathcal{B}_0$*

$$\|T_m(f,g)\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}\,.$$

(ii) *If m belongs to the class $\mathcal{B}_s$ for $s \geq 0$ and k is an integer, then*

$$\left\|\Lambda^k T_m(f,g)\right\|_{L^r} \lesssim \|f\|_{W^{s+k,p}} \|g\|_{L^q} + \|f\|_{L^p} \|g\|_{W^{s+k,q}}\,.$$

(iii) *If m belongs to the class $\mathcal{B}_s$ and $M > 3$, then*

$$\left\|\Lambda^k T_m(f,g)\right\|_{L^2} \lesssim \|f\|_{H^{s+k}} \|g\|_{W^{1,M}} + \|f\|_{W^{1,M}} \|g\|_{H^{s+k}}\,.$$

With the class $\mathcal{B}_s$ defined above we can define a non-resonant system as follows:

Definition 4.3 (Non-resonant bilinear forms). System (1.1) is called non-resonant if

$$q_{\pm\pm}(\xi,\eta) = a(\xi,\eta)\varphi_{\pm\pm}(\xi,\eta) + b(\xi,\eta)\cdot\nabla_\eta\varphi_{\pm\pm}(\xi,\eta)\,, \tag{4.1}$$

with $a \in \mathcal{B}_{-1}$ and $b \in \mathcal{B}_0$. Additionally we require that

$$|\eta|\nabla_{(\xi,\eta)}a(\xi,\eta) \quad \text{or} \quad |\xi-\eta|\nabla_{(\xi,\eta)}a(\xi,\eta) = \frac{\mu_0^{(1)}(\xi,\eta)}{|\xi|} + \frac{\mu_0^{(2)}(\xi,\eta)}{|\eta|} + \frac{\mu_0^{(3)}(\xi,\eta)}{|\xi-\eta|} \tag{4.2}$$

for some $\mu_0^{(i)} \in \mathcal{B}_0$, and

$$\left\|T_{b(\xi,\eta)}(f,g)\right\|_{L^2} \lesssim \|f\|_{L^2} \sum_{j=0}^{k} \left\|R^j g\right\|_{L^\infty} + \sum_{j=0}^{k} \left\|R^j f\right\|_{L^\infty} \|g\|_{L^2} \tag{4.3}$$

for some $k \in \mathbb{N}$.

Some comments about this definition. Equation (4.1) asserts that bilinear interactions vanish on $\mathscr{R}_{\pm\pm}$, the space time resonant set. The presence of $\varphi_{\pm\pm}$ in equation (4.1) allows us to perform normal form transformation on one part of the bilinear terms (integration by parts in s), while the presence of $\nabla_\eta\varphi_{\pm\pm}$ allows us to treat the remaining part by weighted estimates (integration by parts in η). The *classical null condition* is equivalent to $a = 0$ (see below). Equation (4.2) essentially avoids having $a(\xi,\eta) \sim |\xi|^{-1}$ which would be too singular to handle. Equation (4.3) is needed due to failure of the $L^2 \times L^\infty \to L^2$ estimate for symbols in $\mathcal{B}_0$. It would be possible to avoid this last technical restriction by resorting to the use of Besov spaces, but for the sake of simplicity we do not pursue this matter here (see also remark 5.1 for more comments about this aspect).

Examples of non-resonant systems. Now we give examples of non-resonant systems and explain how they relate to existing definitions on "null systems", and how our definition is a natural extension of previous ones.

Classical null forms. Quadratic (semilinear) nonlinearities satisfying the null condition (2.3) are linear combinations of

$$Q_{ij}(u,v) = \partial_i u \partial_j v - \partial_j u \partial_i v\,, \tag{4.4a}$$

$$Q_{0i}(u,v) = \partial_t u \partial_i v - \partial_i u \partial_t v\,, \tag{4.4b}$$

$$Q_0(u,v) = \partial_t u \partial_t v - \nabla u \cdot \nabla v\,. \tag{4.4c}$$

By letting $(u_\pm, v_\pm) = (\partial_t u \mp i\Lambda u, \partial_t v \mp i\Lambda v)$, one can reduce systems (2.2) to first order systems in the unknowns $u_\pm$ and $v_\pm$. Then, one can check that the respective symbols of the above null forms (4.4), corresponding to interactions of u_{ϵ_1} and v_{ϵ_2}, are given (up to a constant factor) by

$$m_{ij}^{\epsilon_1,\epsilon_2}(\xi,\eta) = 2\frac{\eta_i\xi_j - \eta_j\xi_i}{|\eta||\xi-\eta|} = \partial_{\eta_i}\varphi_{++}(\xi,\eta)\partial_{\eta_j}\varphi_{+-}(\xi,\eta) - \partial_{\eta_j}\varphi_{++}(\xi,\eta)\partial_{\eta_i}\varphi_{+-}(\xi,\eta)$$

$$m_{0i}^{\epsilon_1,\epsilon_2}(\xi,\eta) = \left(\epsilon_1\frac{\eta_i}{|\eta|} + \epsilon_2\frac{\eta_i - \xi_i}{|\eta-\xi|}\right) = \partial_{\eta_i}\varphi_{\epsilon_1\epsilon_2}(\xi,\eta)$$

$$m_{0}^{\epsilon_1,\epsilon_2}(\xi,\eta) = 2\left(1 - \epsilon_1\epsilon_2\frac{\eta}{|\eta|}\cdot\frac{\xi-\eta}{|\xi-\eta|}\right) = |\nabla_\eta\varphi_{\epsilon_1\epsilon_2}(\xi,\eta)|^2\,.$$

These symbols are of the form (4.1) with $a = 0$ and vanish on the space resonant set. Thus classical null forms are spatially non-resonant, and therefore can be treated by weighted estimates and without normal forms transformation. Note that in our system the $+-$ interactions have $\mathscr{T}_{+-} \subsetneq \mathscr{S}_{+-}$, and our symbols (4.1) only vanish on the smaller set $\mathscr{T}_{+-}$. To treat these interactions a normal form transformation is needed, leading to terms which are not well spatially localized. This causes a difficulty that will be elaborated on below.

The classical quasilinear null forms are also non-resonant in that their symbols satisfy (4.1) where the homogeneities of a and b are increased by 1. However, for general first order systems of the type (1.1), quasilinear equations lose derivatives in the energy estimates unless there are cancellations present. In the special case where the first order quasilinear system comes from a second order system of wave equations with quasilinear null terms, cancellations are present in the energy estimates. Thus, our results will apply for such nonlinearities as well[3].

Systems with multiple speeds. For systems

$$\partial_t u_\ell - ic_\ell\Lambda u_\ell = \sum_{j,k} T_{q^\ell_{j,k}(\xi,\eta)}(u_j,u_k) \tag{4.5}$$

the phases are given by $-c_\ell|\xi| + c_j|\xi-\eta| + c_k|\eta|$. In the case $c_j \neq \pm c_k$ one has $\mathscr{S} = \{0,0\}$, so that our results will trivially apply. If $c_j = \pm c_k$, $\mathscr{R} = \{0,0\}$ unless $c_k = \pm c_\ell$. Therefore, in the case $c_k \neq \pm c_\ell$,

[3] More specifically, in the quasilinear case the most efficient proof of the analogue of Theorem 1.1 would consist of two steps: 1) establishing energy and weighted energy estimates directly on the second order wave equation so to obtain the weighted bounds in (5.7); 2) run our proof to show the decay of solutions.

global existence can be obtained provided a suitable null condition is imposed at $\{0,0\}$. This is similar to the work on quadratic NLS [9]. The full non-resonance condition is then needed only for interactions of the form $-c_\ell|\xi| + c_\ell|\xi-\eta| \pm c_\ell|\eta|$. This extension is similar to the result of Sideris and Tu [29]. We also refer the reader to the work of Katayama and Yokoyama [15] and references therein for more on systems with multiple speeds.

Some examples of interest that can be treated using our techniques are:

1. First order systems of the form

$$\left\{ \begin{array}{l} \partial_t u + ic\Lambda u = |v|^2 \\ \partial_t v + i\Lambda v = T_m(v,v) + u^3 \end{array} \right. \tag{4.6}$$

where $c > 1$ and m is non-resonant as in definition 4.1. Here no special null condition at the origin is assumed on the bilinear form in the first equation. This system does not satisfy any existing null condition criteria set by the vector fields method. In fact we believe that this system is not amenable to analysis by the vector fields method due to the simultaneous failure of the Lorentz invariance and the need of a normal form transform.

Our method works by first applying a normal form transformation on u (notice that the phase is bounded below by $(c-1)|\xi|$), and then handling the singularity introduced by such a transformation through a spread-tight splitting as explained in section 5.

2. Systems of wave equations of the form

$$\left\{ \begin{array}{l} \Box_1 u = (\partial v)^2 + \partial u (\partial w)^2 \\ \Box_1 v = \partial v \partial w + Q(v,v) \\ \Box_c w = \partial v \partial w + \partial u (\partial v)^2 \end{array} \right. \tag{4.7}$$

where $\Box_c := \partial_t^2 - c^2\Delta$, $c \neq 1$ and Q is any null form. This is an example of a nonrelativistic system satisfying the weak null condition. Global existence can be obtained with a weaker decay on u of the form $\|\partial u\|_{L^\infty} \lesssim t^{-1+\epsilon}$, for $\epsilon \ll 1$.

Non-locality and absence of Lorentz invariance. The class of systems (W), under the non-resonance condition given by definition 4.3, includes the class of second order wave equations

$$\Box u = T_1 Q(T_2 u, T_3 u) \tag{4.8}$$

where the T_i's are zero-th order operator and Q is any combination of the nonlinear terms (4.4a)-(4.4c) (or their quasilinear version). For systems as (4.8) the action of the Lorentz boosts $L_i = x_i\partial_t + t\partial_i$ on the nonlinearity produces some terms which are too singular to be estimated. This makes the classical [21] vector fields method difficult to be applied.

In [27, 28] Sideris considered quasilinear hyperbolic systems governing the motion of isotropic, homogeneous, nonlinear elastic waves. Like systems with multiple speeds, these systems are only classical invariant, i.e. they do not possess Lorentz invariance. By imposing a null condition on the nonlinear terms of the form $F(\nabla u)\nabla^2 u$, he was able to show global existence of solutions. As mentioned in the introduction, several other works have dealt with the problem of long time existence for classically invariant systems on $\mathbb{R}^{3+1}$, see for example [22, 17, 18]. Our methods are also applicable to the systems considered in these works.

A remark about the initial data. In contrast with the results mentioned previously, our initial data belong to a low weighted Sobolev space. In particular we only ask for $xu_0 \in H^2$ and $|x|^2\Lambda u_0 \in H^1$, see (1.2). In comparison, the spaces used in [22, 27, 17] would require $(|x|\Lambda)^i u_0 \in \Lambda(L^2)$, for $i = 0\dots k$ and some $k \geq 7$. This means that we can allow more oscillating data. For example, for data behaving at infinity like $\cos|x|/|x|^\alpha$, we can allow any $\alpha > \frac{7}{2}$, whereas in the other works one would need $\alpha > \frac{17}{2}$.

5. Outline of the proof

Before outlining the proof of Theorem 1.1 we would like to point out two difficulties in our problem:
$a)$ Although the space-time resonance method is algorithmic, its implementation on the space-time resonant set is very much problem dependent. This is due to the fact that the aforementioned set can be large with no clear criteria, set by the method, to address how large is large. Its application to nonlinear dispersive equations has been restricted so far to cases where the resonant set is very small. In particular, for problems such as the Schrödinger equation, the resonant set is a point; and for gravity water waves, there are no quadratic time resonances. However for hyperbolic systems this set is large. For the system we are considering here the space time resonant set is 4 dimensional in a 6 dimensional space. Treating such a big space-time resonant set required new ideas, which we present in this manuscript.
$b)$ When space resonant frequencies are different from time resonant frequencies, and when both types of resonances are present in the bilinear interactions, a normal form transformation is needed

$$u \to u + T_m(u,u) \Leftrightarrow \widehat{f} \to \widehat{f} + \int e^{is\varphi(\xi,\eta)} m(\xi,\eta) \widehat{f}(s,\eta) \widehat{f}(s,\xi-\eta)\, d\eta \,.$$

The bilinear term $T_m(u,u)$ need not be well localized in space since the outcome of the interaction may have a different group velocity, i.e., $\nabla_\xi \varphi \neq 0$, which is the case for (1.1). Thus weighted estimates on this bilinear interaction tend to grow at a fast rate with time. We refer to these bilinear interactions as *spread terms*. This is in contrast to non space resonant frequencies which are well spatially localized and for which weighted estimates tend to grow very slowly. We refer to such bilinear interactions as *tight terms*. The presence of tight and spread terms is problematic and requires a careful analysis when trying to establish the decay of solutions. This is the case here for the $+-$ interactions, as was the case for the 2D Schrödinger equation [9]. Our strategy in obtaining the pointwise decay of solutions will be explained below in more details.

Reduction of system (1.1)**.** By isolating the terms in equation (1.1) that are most difficult to estimate, we can considerably simplify our notations and the presentation of the main aspects of the proof of Theorem 1.1.

Reduction to the $-+$ case. As the analysis of resonances in section 3 showed, the $-+$ interactions lead to a more complicated resonant set than the $++$ and $--$ interactions. The $-+$ case actually contains the difficult aspects of both the $++$ and $--$ cases. More precisely, in the $-+$ case we will need to decompose the phase space in two sets: one containing $\mathscr{R}_{-+}$ but not $\mathscr{S}_{-+} \cap \mathscr{T}_{-+}^c$, and the other one containing $\mathscr{S}_{-+} \cap \mathscr{T}_{-+}^c$ but not $\mathscr{R}_{-+}$. The analysis on the set containing $\mathscr{R}_{-+}$, respectively on the set containing $\mathscr{S}_{-+} \cap \mathscr{T}_{-+}^c$, would be enough to take care of the $++$, respectively the $--$, interactions.

Therefore, we focus only on the $-+$ interactions, and we will drop the $-+$ indices for lighter notations.

Reduction to $a(\xi,\eta) = \frac{1}{|\eta|}$ and $b(\xi,\eta) = 1$. Recall that we are imposing the restriction (4.2) on a. This means that a can have singularities of the type $1/|\eta|$ or $1/|\xi-\eta|$, but not of the form $1/|\xi|$. By the symmetry between η and $\xi - \eta$, we can then assume that a is of the form $\mu_0(\xi,\eta)/|\eta|$ for some $\mu_0 \in \mathcal{B}_0$. Moreover, since the presence of symbols in the class $\mathcal{B}_0$ is irrelevant for our estimates on the terms corresponding to the symbol $a(\xi,\eta)\varphi(\xi,\eta)$, we can simply assume that a is given by $1/|\eta|$.

Finally, since we assume that b satisfies (4.3), and since we will show that $\|R^j u\|_{L^\infty}$ is controlled with a decay of t^{-1} (see remark 5.1 below) we can reduce matters to $b = 1$. It will be clear to the reader what minor modifications are necessary to perform the estimates for a general $b \in \mathcal{B}_0$ and satisfying (4.3).

In view of these reductions the non-resonant equation becomes

$$\widehat{f}(t,\xi) = \widehat{f_0}(\xi) + \int_1^t \int e^{is\varphi(\xi,\eta)} \left(\frac{\varphi(\xi,\eta)}{|\eta|} + \nabla_\eta \varphi(\xi,\eta) \right) \widehat{f}(s,\xi-\eta) \widehat{f}(s,\eta) d\eta ds \,. \tag{5.1}$$

Furthermore, we recall that $\nabla_\xi \varphi$ vanishes on the resonant set, and in particular the following identity holds:

$$|\xi|\nabla_\xi \varphi = \frac{\eta-\xi}{|\eta-\xi|}\varphi + |\eta|\nabla_\eta \varphi\,. \tag{5.2}$$

Splitting of the profile f. Integrating by parts in s in the terms containing the phase φ we get

$$\widehat{f}(t,\xi) \stackrel{def}{=} \widehat{f}_0(\xi) + \widehat{g}(t,\xi) + \widehat{h}(t,\xi) \stackrel{def}{=} \widehat{f}_0(\xi) + \widehat{g}(t,\xi) + \widehat{h}_0(t,\xi) + \widehat{h}_1(t,\xi)$$

where

$$\widehat{g}(t,\xi) \stackrel{def}{=} \int e^{it\varphi(\xi,\eta)} \frac{1}{|\eta|} \widehat{f}(t,\xi-\eta)\widehat{f}(t,\eta)\,d\eta\,, \tag{5.3a}$$

$$\widehat{h}_0(t,\xi) \stackrel{def}{=} \int_1^t \int e^{is\varphi(\xi,\eta)} \nabla_\eta \varphi(\xi,\eta) \widehat{f}(s,\xi-\eta)\widehat{f}(s,\eta)\,d\eta ds, \tag{5.3b}$$

$$\widehat{h}_1(t,\xi) \stackrel{def}{=} \int_1^t \int e^{is\varphi(\xi,\eta)} \frac{1}{|\eta|} \partial_s \left(\widehat{f}(s,\xi-\eta)\widehat{f}(s,\eta)\right)\,d\eta ds\,. \tag{5.3c}$$

This splitting can be understood in the following manner:
1) g comes from the normal form transformation, has very good time decay but is spatially spread;
2) h_0 is a spatially tight term due to the presence of $\nabla_\eta \varphi(\xi,\eta)$;
3) h_1 is cubic in f.

Regarding $h_0 = h_0(f,f)$ as a bilinear form of $f = f_0 + g + h$, we can decompose h_0 as

$$\begin{aligned} h_0(f,f) &= h_0(f_0, f_0+h) + h_0(f_0,g) + h_0(g,f) + h_0(h,h) + h_0(h,g) + h_0(h,f_0) \\ &= h_0(g,f) + h_0(h,h) + h_*\,, \end{aligned} \tag{5.4}$$

and thus decompose f further:

$$f = f_0 + g + h_0(h,h) + h_0(f,g) + h_1 + h_*. \tag{5.5}$$

A priori bounds. The proof of global existence and scattering of solutions is obtained from the following a priori bounds on $u = e^{it\Lambda}f$:

$$\left\{ \begin{array}{lll} \|u\|_{H^N} \lesssim t^\varepsilon, & \|u\|_{H^2} \lesssim 1, & \|u\|_{L^\infty}, \|Ru\|_{L^\infty} \lesssim \frac{1}{t}, \\ \|xf\|_{L^2} \lesssim t^\gamma, & \|\Lambda x f\|_{H^1} \lesssim 1, & \left\| |x|^2 \Lambda f \right\|_{H^1} \lesssim t\,, \end{array} \right. \tag{5.6}$$

and a continuation of the local-in-time solution. Here, N is a suitably large number possibly depending on ε and γ, which are arbitrarily small but fixed positive constants. This leads us to define the space X by the norm associated to these bounds:

$$\begin{aligned} \|u\|_X := \sup_{t\geq 1} \big[& t^{-\varepsilon}\|u\|_{H^N} + \|u\|_{H^2} + t(\|u\|_{L^\infty} + \|Ru\|_{L^\infty}) \\ & + t^{-\gamma}\|xf\|_{L^2} + \|\Lambda x f\|_{H^1} + t^{-1}\left\| |x|^2\Lambda f\right\|_{H^1} \Big]\,. \end{aligned} \tag{5.7}$$

Remark 5.1. The presence of $\|Ru\|_{L^\infty}$ is not surprising because the Riesz transform is already present in the interaction symbol $\nabla_\eta \varphi$. However we remark here that if b is any symbol in $\mathcal{B}_0$ satisfying (4.3), the same X-norm above would work. Indeed, as a byproduct of our estimates we have

$$\left\| R^k u \right\|_{L^\infty} \lesssim \frac{1}{t}\left[\epsilon + \|u\|_X^2\right] \tag{5.8}$$

for any k. This is because L^∞ estimates on $u = e^{-it\Lambda}u_0 + e^{-it\Lambda}g + e^{-it\Lambda}h$ are obtained by

a) using Sobolev's embedding on g: $\|e^{-it\Lambda}g\|_{L^\infty} \lesssim \|e^{-it\Lambda}g\|_{W^{1,p}}$ for $p \gg 1$, and then showing $t\|e^{-it\Lambda}g\|_{W^{1,p}} \lesssim \|u\|_X^2$;

b) estimating weighted L^2 norms of the main components of h by means of the following linear dispersive estimates:

$$\left\|e^{it\Lambda}h\right\|_{L^\infty} \lesssim \frac{1}{t}\|\langle x\rangle^{\frac{3}{2}+}\Lambda^2 h\|_{L^2} \tag{5.9}$$

$$\left\|e^{it\Lambda}h\right\|_{L^\infty} \lesssim \frac{1}{t}\left\||x|\Lambda^2 h\right\|_{L^2}^{\frac{1}{2}}\left\||x|^2\Lambda^2 h\right\|_{L^2}^{\frac{1}{2}}. \tag{5.10}$$

In both operations $a)$ and $b)$ the presence of Riesz transforms becomes irrelevant.

Bounds for g and h. A key aspect of our proof is the different treatment of the components g and h, and the different treatment of some components of h itself. In particular, the bound on $\|u\|_X$ will follow from the bounds on g

$$\begin{cases} \|g\|_{H^N} \lesssim \|u\|_X^2, \quad \|xg\|_{L^2} \lesssim t^\gamma\|u\|_X^2, \quad \|\Lambda xg\|_{H^1} \lesssim \|u\|_X^2, \\ \\ \left\|\Lambda|x|^2 g\right\|_{H^1} \lesssim t\|u\|_X^2, \quad \left\|e^{it\Lambda}g\right\|_{L^\infty} \lesssim \frac{1}{t}\|u\|_X^2. \end{cases} \tag{5.11}$$

and the bounds on h

$$\begin{cases} \|h\|_{H^N} \lesssim t^\varepsilon\|u\|_X^2, \quad \|h\|_{H^2} \lesssim \|u\|_X^2, \quad \|xh\|_{L^2} \lesssim t^\gamma\|u\|_X^2, \quad \|\Lambda xh\|_{H^1} \lesssim \|u\|_X^2, \\ \\ \left\|\Lambda|x|^2 h\right\|_{L^2} \lesssim t^a\|u\|_X^2, \quad \left\|\Lambda^2|x|^2 h\right\|_{L^2} \lesssim t^b\|u\|_X^2, \quad \left\|e^{it\Lambda}h\right\|_{L^\infty} \lesssim \frac{1}{t}\|u\|_X^2, \end{cases} \tag{5.12}$$

where a and b are (small) positive constants satisfying $0 < \gamma < b < \frac{a}{3}$ and $a < \frac{1}{8}$.

These a priori bounds will imply global existence provided the data is small:

$$\left\|e^{it\Lambda}(g+h)\right\|_X \lesssim \|u\|_X^2 \implies \|u\|_X \lesssim \left\|e^{it\Lambda}f_0\right\|_X + \left\|e^{it\Lambda}(g+h)\right\|_X \lesssim \epsilon + \|u\|_X^2, \tag{5.13}$$

which in turn gives $\|u\|_X \lesssim \epsilon$. Scattering is also a consequence of the bounds (5.11)-(5.12). To obtain Theorem 1.1 it is then enough to show (5.11) and (5.12). Below we briefly explain the main steps needed to obtain these bounds.

Main steps in the proof of (5.11) and (5.12). From (5.6) and (5.12) we see that h has the same energy and pointwise estimates as f, and better weighted estimates than f. Thus, the bilinear terms that need to be bounded are g, $h_0(h,h)$, $h_0(f,g)$ and h_1. All the remaining bilinear terms in (5.5), denoted by h_*, see (5.4), are easier to estimate because their arguments satisfy stronger bounds.

The first step in the proof of the bounds (5.11) and (5.12) consists of estimating Sobolev norms. Such estimates are pretty straightforward, since we are dealing with a semilinear equation.

We then obtain weighted and L^∞ estimates for the spread component g defined by (5.3a). This component is the one who is responsible for the fast growth in time of the weighted norms, see the first inequality in the second line of (5.11). On the other hand, since g consists of a bilinear term with no time integration, its decay in L^∞ can be obtained easily.

Subsequently, we prove a priori bounds on weighted L^2 norms of $h_0(h,h)$. Thanks to the presence of the symbol $\nabla_\eta\varphi$, and to the identity (5.2), we can always integrate by parts at least twice in time and/or frequency. As a consequence we can prove that $h_0(h,h)$ satisfies the stronger weighted bounds (5.12) that hold for the h component.

The next step consists of estimating the L^∞ norm of $h_0(h,h)$, and is one of the most delicate parts of the proof. In order to obtain the t^{-1} decay we perform an angular decomposition of the phase space into two regions. The first region contains the space resonant set $\mathscr{S}$ but is away from the time resonant set $\mathscr{T}$. In this region we can perform a normal form at the expense of introducing only a mild singularity when one of the Fourier variables vanishes. For the quadratic boundary terms arising in the integration by parts in time, the decay is obtained in a straightforward manner, as it is done for the g component. For the cubic terms the decay is obtained as a consequence of L^2-weighted estimates. The second complementary region is away from $\mathscr{S} \cap \mathscr{T}^c$ and contains $\mathscr{R}$. There we can combine the identity (5.2) and the fact that φ can be divided by $\nabla_\eta \varphi$, to conclude, roughly speaking, that $\nabla_\xi \varphi \sim \nabla_\eta \varphi$ in this region. This implies a good control on weighted norms, and decay is obtained by interpolating in an appropriate fashion these norms.

Eventually, the cubic terms $h_0(f,g)$ and h_1 are estimated using again the decomposition $f = g + h$. Also for these terms the pointwise decay is obtained as a consequence of the L^2-weighted bounds.

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