ASYMPTOTIC STABILITY OF SOLITONS FOR MKDV

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Abstract. We prove a full asymptotic stability result for solitary wave solutions of the mKdV equation. We consider small perturbations of solitary waves with polynomial decay at infinity and prove that solutions of the Cauchy problem evolving from such data tend uniformly, on the real line, to another solitary wave as time goes to infinity. We describe precisely the asymptotics of the perturbation behind the solitary wave showing that it satisfies a nonlinearly modified scattering behavior. This latter part of our result relies on a precise study of the asymptotic behavior of small solutions of the mKdV equation.

Contents

1. Introduction

This paper is concerned with the Cauchy problem for the focusing modified Korteweg-de Vries (mKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^3) = 0 \\ u(t=0) = u_0 \end{cases} \tag{mKdV}$$

for $u = u(t,x) \in \mathbb{R}$ and $(t,x) \in \mathbb{R} \times \mathbb{R}$. This equation admits a family of solitary wave solutions of the form $u_c(t,x) = Q_c(x - ct)$ with

$$Q_c(\xi) = \sqrt{c}Q(\sqrt{c}\,\xi), \qquad Q(s) := \sqrt{2}/\cosh(s), \quad c > 0. \tag{1.1}$$

Our aim in this paper is to revisit the proof of global existence and modified scattering for (mKdV) for small and localized initial data, and then extend it in order to obtain new asymptotic stability results for solitary wave solutions.

Important conserved quantities[1] are the mass M, energy H, and momentum P

$$M = \int_{\mathbb{R}} u^2\,dx \qquad H = \int_{\mathbb{R}} \frac{1}{2}|\partial_x u|^2 - \frac{1}{4}|u|^4\,dx, \qquad P = \int_{\mathbb{R}} u\,dx. \tag{1.2}$$

Key words and phrases. mKdV, modified scattering, asymptotic stability, solitons.

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[1] As we will remark later, these are not needed to prove the small data result, which also applies to more general versions of (mKdV).

Moreover, we note that solutions of (mKdV) enjoy the scaling symmetry

$$u \longmapsto \lambda u(\lambda^3 t, \lambda x),$$

which is generated by the vector field $S = 1 + x\partial_x + 3t\partial_t$.

1.1. **Known results.**

Global well-posedness and asymptotic behavior. There is a vast body of literature dealing with the mKdV equation, and in particular with the local and global well-posedness of the Cauchy problem. Without trying to be exhaustive, we mention the early works on the local and global well-posedness by Kenig-Ponce-Vega [30] and Kato [28]. Global well-posedness in low regularity spaces, and in particular in the energy space H^1, was established in the seminal work of Kenig-Ponce-Vega [31]. In this latter paper the authors considered the wider class of generalized KdV (gKdV) equations $\partial_t u + \partial_x^3 u + \partial_x u^p = 0$, $p \geq 2$, which includes (mKdV) and the KdV equation ($p = 2$). Sharp, up to the end-point, global well-posedness in H^s for $s > 1/4$ was proved in the work of Colliander-Keel-Staffilani-Takaoka-Tao [7], for both the focusing and defocusing mKdV equation on the line (and for $s \geq 1/2$ in the periodic case). These results are complemented by several ill-posedness results; see for example Christ-Colliander-Tao [6] and references therein.[2]

Besides global regularity, another fundamental question for dispersive PDEs concerns the asymptotic behavior for large times. The first proof of global existence with a complete description of the asymptotic behavior of solutions of (mKdV) in the defocusing case, is due to Deift and Zhou [9], who used a steepest descent approach to oscillatory Riemann-Hilbert problems and the inverse scattering transform [53, 2]. In [9], thanks to the complete integrability of the defocusing mKdV equation, the authors were able to treat suitably localized initial data with arbitrary size. A proof of global existence and a (partial) derivation of the asymptotic behavior for small localized solutions, without making use of complete integrability, was later given by Hayashi and Naumkin [18, 19], following the ideas introduced in the context of the 1d nonlinear Schrödinger (NLS) equation in [17] - see also [20]. Recently, an alternative proof of the results in [19], with a precise derivation of asymptotics and a proof of asymptotic completeness, was given by Harrop-Griffiths [16], following the approach used for the 1d NLS equation in [22].

Our proof of global existence and asymptotic behavior - Theorem 1.1 - relies on the intuition developed in [29], where a very natural stationary phase argument is used to understand the large time behavior of small and localized solutions and derive asymptotic corrections. This approach was inspired by the space-time resonance method put forward in [13, 14, 15]. See section 1.3 below for a short explanation of these ideas in the present context. A similar approach was also successfully employed in the proofs of global regularity and modified scattering for 2d gravity [23, 24, 25] and capillary [26, 27] water waves, and in other higher dimensional dispersive models [29, 49].

Stability of solitons. The study of the stability of solitons also has a long history, but here we will only address results which are closer in spirit to the present paper. The asymptotic stability in front of the soliton[3] was first obtained by Pego-Weinstein [48] for initial perturbations of a soliton with exponential decay as $x \to +\infty$. This result was then refined

[2]For more on the local and global well-posedness and ill-posedness of KdV and generalized KdV equations we refer to the books of Tao [50] and Linares-Ponce [35].

[3]Similarly to how it is stated in Theorem 3.1.

by Mizumachi [44], who treated perturbations belonging to polynomially weighted spaces of sufficiently high order. For perturbations in the energy space H^1, definitive asymptotic stability results in front of the solitary wave have been obtained for the whole class of subcritical gKdV equations in a series of papers by Martel-Merle [36, 37, 38]. We also mention [42] on the L^2 stability of KdV solitons, [5] on the H^s $s \geq -1$ stability of KdV solitons, [40] on N-soliton solutions of subcritical gKdV equations, and [47, 45] for a different approach. For more on the asymptotic stability of solitons and multi-solitons for subcritical gKdV equations we refer the reader to the survey articles [52, 39] and references therein.

In [44] the author also obtained a full stability result for gKdV equations with a nonlinearity of degree $p \in (3, 5)$. More precisely, he showed that a solution that evolves from a small perturbation of a soliton will asymptotically resolve in a slightly differently modulated soliton, plus a radiation which behaves like a solution of the linear flow. Note that for the gKdV equation with quartic nonlinearity ($p = 4$), there are also scattering and asymptotic stability results in critical spaces rather than polynomially weighted ones, see [51, 34].

The results we present extend the above mentioned works by

i) proving the (modified) scattering result for the radiation in the (critical) case of (mKdV),
ii) allowing a wider class of small perturbations belonging to weighted Sobolev spaces with weak polynomial decay at infinity.

Because of the critical dispersive nature of the equation, in the case of (mKdV) the radiation does not behave linearly, but requires a nonlinear correction. See Theorem 1.5 and Remark 1.6 for more details. The proof that we give below combines the virial approach of Martel-Merle [37] and the weighted estimates of Pego-Weinstein [48], in the spirit of the recent work of Mizumachi-Tzvetkov [47] on the L^2 stability of solitons for KdV.

1.2. **Main results.** Our first main result concerns the stability of the zero solution under small perturbations.

Theorem 1.1 (Global Existence and Asymptotic Behavior)**.** *Let an initial data u_0 be given such that*

$$\|\langle x\rangle u_0\|_{H^1(\mathbb{R})} \leq \varepsilon_0. \tag{1.3}$$

There exists $\overline{\varepsilon_0} > 0$, such that for all $\varepsilon_0 \in (0, \overline{\varepsilon_0}]$ the Cauchy problem (mKdV) *admits a unique global solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$. This solution satisfies the decay estimates*

$$|u(t,x)| \lesssim \varepsilon_0 t^{-1/3} \langle x/t^{1/3}\rangle^{-1/4}, \qquad |\partial_x u(t,x)| \lesssim \varepsilon_0 t^{-2/3} \langle x/t^{1/3}\rangle^{1/4}. \tag{1.4}$$

Moreover, for $t \geq 1$ the solution u has the following asymptotics:

- *In the region $x \geq t^{1/3}$ we have the improved decay*

$$|u(t,x)| \lesssim \frac{\varepsilon_0}{t^{1/3}(x/t^{1/3})^{3/4}}; \tag{1.5}$$

- *In the region $|x| \leq t^{1/3+2\gamma}$, for some $\gamma > 0$ sufficiently small, the solution is approximately self-similar:*

$$\big|u(t,x) - \frac{1}{t^{1/3}}\varphi\big(\frac{x}{t^{1/3}}\big)\big| \lesssim \frac{\varepsilon_0}{t^{1/3+3\gamma/2}}, \tag{1.6}$$

where φ is a bounded solution of the Painlevé II equation[4]

$$\varphi'' - \frac{1}{3}\xi\varphi + \varphi^3 = 0, \qquad \int_{\mathbb{R}} \varphi(x)\,dx = \int_{\mathbb{R}} u_0(x)\,dx.$$

- *In the region $x \leq -t^{1/3+2\gamma}$, the solution has a nonlinearly modified asymptotic behavior: there exists $f_\infty \in L^\infty_\xi$ such that*

$$\Big| u(t,x) - \frac{1}{\sqrt{3t\xi_0}} \Re \exp\Big(-2it\xi_0^3 + \frac{i\pi}{4} + \frac{i}{6}|f_\infty(\xi_0)|^2 \log t\Big) f_\infty(\xi_0)\Big| \leq \frac{\varepsilon_0}{t^{1/3}(-x/t^{1/3})^{3/10}}, \quad (1.7)$$

where $\xi_0 := \sqrt{-x/(3t)}$, and $\Re$ denotes the real part.

Remark 1.2. In the proof of Theorem 1.1 above, the Hamiltonian structure of the equation, as well as the conservation of mass and energy, do not play any crucial role. For convenience we will work with (mKdV), but it will be clear that all our results also apply to the defocusing mKdV equation $\partial_t u + \partial_x^3 u - \partial_x(u^3) = 0$, and to other (not necessarily Hamiltonian) versions of the equation, such as

$$\partial_t u + \partial_x^3 u = a(t)\partial_x(u^3), \tag{1.8}$$

where $|a(t)| \leq 1, |a'(t)| \leq \langle t\rangle^{-7/6}$.

Remark 1.3. In Theorem 1.1 we have decided to state the global existence and scattering result for initial data satisfying (1.3), that is $xu_0 \in H^1$. However, in its proof we only make use of the assumption $xu_0 \in H^\alpha$, for some α close to, but less than, $1/2$. We can therefore treat a larger class of initial data with respect to [19, 16]. See Remark 2.6 for more details. We have, however, decided to state Theorem 1.1 assuming the stronger initial condition (1.3), in order to make its application in the proof of Theorem 1.5 below more convenient.

Remark 1.4. We chose to characterize the modified asymptotic behavior of u (1.7) in L^∞. Similar estimates can be obtained in L^2 topology: one obtains then in particular that u converges to zero to the right of $-t^{1/3}$, while to its left, it approaches

$$\frac{1}{\sqrt{3t\xi_0}} \Re \exp\Big(-2it\xi_0^3 + \frac{i\pi}{4} + \frac{i}{6}|f_\infty(\xi_0)|^2 \log t\Big) f_\infty(\xi_0),$$

which gathers asymptotically all the mass of the solution.

Our second main result is a strong asymptotic stability result for soliton solutions, under small perturbations belonging to a weak algebraically weighted space.

Theorem 1.5 (Full Asymptotic Stability of Solitons)**.** *Assume that*

$$u_0(x) = Q_{c_0}(x) + v_0(x), \tag{1.9}$$

for some $c_0 > 0$, with

$$\|\langle x\rangle v_0\|_{H^1(\mathbb{R})} + \|\langle x_+\rangle^m v_0\|_{H^1(\mathbb{R})} \leq \varepsilon_0, \tag{1.10}$$

[4]The smallness of $\int \varphi(x)\,dx$ guarantees the existence and uniqueness of a bounded solution to the Painlevé II equation. Its asymptotics are as follows: $\varphi(\xi) \sim 3^{1/9}\,\mathrm{Ai}(3^{1/3}\xi) \sim \frac{3^{7/36}}{2\sqrt{\pi}\xi^{1/4}} e^{-\frac{2}{3\sqrt{3}}\xi^{3/2}}$ as $\xi \to \infty$, while $\varphi(\xi) \sim \frac{3^{7/36}}{2\sqrt{\pi}|\xi|^{1/4}} d \cos\Big(-\frac{2}{3\sqrt{3}}|\xi|^{3/2} + \frac{\pi}{4} + \frac{3d^2}{4\pi}\log|y|^{3/2} + \theta\Big)$ as $\xi \to -\infty$, where d and θ are constants depending on $\int \varphi(x)\,dx$. We refer to [21] and [10] for this and much more on Painlevé II. Note that our proof will actually provide the existence of a bounded solution of the Painlevé equation.

for some $m > 3/2$. *Then, for* ε_0 *sufficiently small, there exists a unique solution* $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ *of* (mKdV) *and a continuous function* $C(\cdot)$ *with* $C(0) = 0$, *such that for some* $c_+ > 0$ *and* x_+ *with*

$$|c_+ - c_0| + |x_+| \lesssim C(\varepsilon_0), \tag{1.11}$$

we have

$$|u(t,x) - Q_{c_+}(x - c_+t - x_+) - R(t,x)| \lesssim C(\varepsilon_0)\langle t\rangle^{-1} \tag{1.12}$$

where:

- *The radiation* R *verifies the decay estimates*
$$\|R(t)\|_{L^\infty(\mathbb{R})} \lesssim C(\varepsilon_0)\langle t\rangle^{-1/3}. \tag{1.13}$$
- R *has the same asymptotics as a small solution to* (mKdV)*, and in particular possesses a modified scattering behavior as* $t \to \infty$ *as in* (1.7)*.*

Note that we prove a full asymptotic stability result by describing the behavior of the perturbation behind the solitary wave and that, because of the critical dispersive decay of the mKdV equation, the radiation has nonlinear asymptotic oscillation.

Remark 1.6. Note that the spatial decay (1.10) that we require in front of the solitary is only slightly more than $x^{3/2}u$ in L^2. This is at the same scale as the decay property which is used in the inverse scattering theory, where one requires xu_0 in L^1. Spatial decay conditions on the data are not explicitly stated in the work of Deift-Zhou on the defocusing mKdV equation [9], but the condition above is used in the application of direct and inverse scattering in [1, 41]. We also refer to [33] for a recent survey.

1.3. **Ideas of the proof.** We now briefly explain the main ideas and the intuition behind our results.

Global existence and modified scattering. In what follows we let

$$f(t) := e^{t\partial_x^3}u(t) \tag{1.14}$$

so that $\partial_t f = -e^{t\partial_x^3}\partial_x(u^3)$. Then we can write (mKdV) as

$$\partial_t \widehat{f}(t,\xi) = -\frac{1}{2\pi}\iint e^{-it\phi(\xi,\eta,\sigma)} i\xi\, \widehat{f}(t,\xi-\eta-\sigma)\widehat{f}(t,\eta)\widehat{f}(t,\sigma)\, d\eta\, d\sigma. \tag{1.15}$$

with

$$\phi(\xi,\eta,\sigma) = \xi^3 - (\xi-\eta-\sigma)^3 - \eta^3 - \sigma^3 = 3(\eta+\sigma)(\xi-\eta)(\xi-\sigma).$$

We follow the approach of the space-time resonances method [12] which is to view the above integral as an oscillatory integral, whose large-time behavior will thus be dictated by the stationary points (in η, and in t after time integration) of the phase ϕ. As observed in [29], this will give a very simple means of computing the large-time correction to scattering, due to the long-range effects of the critically dispersing nonlinear term.

Before explaining this argument, we need to describe precisely the stationary points of ϕ. A small computation gives that

$$\begin{cases} \partial_\eta \phi(\xi,\eta,\sigma) = 3(\xi-\sigma)(\xi-2\eta-\sigma) \\ \\ \partial_\sigma \phi(\xi,\eta,\sigma) = 3(\xi-\eta)(\xi-\eta-2\sigma) \end{cases} \tag{1.16}$$

and

$$\det \mathrm{Hess}_{\eta,\sigma}\,\phi = -36(\eta^2+\sigma^2+\eta\sigma-\xi\eta-\xi\sigma). \tag{1.17}$$

Notice that

$$\partial_\eta\phi = \partial_\sigma\phi = 0 \quad\iff\quad (\eta,\sigma)=(\eta_i,\sigma_i) \quad , \quad 1\le i\le 4 \tag{1.18}$$

with

$$\begin{cases} (\eta_1,\sigma_1)=(\xi,\xi)\\ (\eta_2,\sigma_2)=(\xi,-\xi)\\ (\eta_3,\sigma_3)=(-\xi,\xi)\\ (\eta_4,\sigma_4)=\left(\frac{\xi}{3},\frac{\xi}{3}\right), \end{cases} \tag{1.19}$$

and that, for $i\in\{1,2,3\}$,

$$\begin{aligned} &\phi(\xi,\eta_i,\sigma_i)=0\\ &\phi(\xi,\eta_4,\sigma_4)=(8/9)\xi^3\\ &\det \mathrm{Hess}_{\eta,\sigma}\,\phi(\xi,\eta_i,\sigma_i)=-36\xi^2\\ &\det \mathrm{Hess}_{\eta,\sigma}\,\phi(\xi,\eta_4,\sigma_4)=12\xi^2\\ &\mathrm{sign}\, \mathrm{Hess}_{\eta,\sigma}\,\phi(\xi,\eta_i,\sigma_i)=0\\ &\mathrm{sign}\, \mathrm{Hess}_{\eta,\sigma}\,\phi(\xi,\eta_4,\sigma_4)=1-\mathrm{sign}\,\xi, \end{aligned} \tag{1.20}$$

where $\mathrm{sign}\,M$ is the signature of the matrix M.

The above computations are the basis to derive - heuristically for the moment - the large time behavior of f. By the stationary phase lemma, assuming that $\widehat{f}$ is sufficiently smooth, (1.15) implies that

$$\partial_t\widehat{f}(t,\xi)=\frac{i\,\mathrm{sign}\,\xi}{6t}|\widehat{f}(\xi)|^2\widehat{f}(\xi)+\frac{ic}{t}e^{-it\frac{8}{9}\xi^3}\widehat{f}\left(\frac{\xi}{3}\right)^3+\{\text{integrable terms}\}. \tag{1.21}$$

The second summand on the right-hand side should not be asymptotically relevant, due to the time-oscillating term. Thus the above reduces to

$$\partial_t\widehat{f}(t,\xi)\sim\frac{i\,\mathrm{sign}\,\xi}{6t}|\widehat{f}(\xi)|^2\widehat{f}(\xi)+\{\text{integrable terms}\},$$

from which we infer that $|\widehat{f}|$ converges as $t\to\infty$ to an asymptotic profile F, while

$$\widehat{f}(t,\xi)\sim F(\xi)\exp\Big(i\frac{\mathrm{sign}\,\xi}{6t}|F(\xi)|^2\log t\Big),\quad \text{as}\quad t\to\infty.$$

Asymptotic stability of solitons. The key idea when proving asymptotic stability for the soliton will be the following decomposition

$$u(t,x)=Q_{c(t)}(y)+\underbrace{v_1(t,y)+v_2(t,y)}_{v(t,y)}.$$

We now explain precisely how the new coordinate y, the soliton parameter c, and the radiation part $v=v_1+v_2$ are determined. First, the new coordinates

$$y=x-\int_0^t c(s)\,ds+h(t)$$

correspond to adopting as a reference the moving frame of the soliton; the modulation parameters c and h will be chosen below to ensure a certain cancellation. The radiation part v satisfies the perturbed equation

$$\begin{cases} \partial_t v + \underbrace{\partial_y(-c + \partial_y^2 + 3Q_c^2)}_{\mathcal{L}_c} v = \partial_y((Q_c + v)^3 - Q_c^3 - 3Q_c^2 v) + \{\text{less important terms}\} \\ v(t = 0) = v_0. \end{cases}$$

The asymptotic stability of solitons follows from the decay of v; this in turn is given by decay estimates for the linear group $e^{t\mathcal{L}_c}$. However, the functions in the generalized kernel of $\mathcal{L}_c$ (of dimension 2) do not decay under this semi group. Thus one needs to make sure that, in the spectral decomposition associated to $\mathcal{L}_c$, the component of v in the generalized kernel of $\mathcal{L}_c$ is zero: this condition completely determines c and h.

Following the work of Mizumachi [43], see also [46, 47], the radiation part v is then split into $v = v_1 + v_2$, where v_1 simply solves (mKdV) in the y coordinates, with data v_0,

$$\begin{cases} \partial_t v_1 - (c + \dot{h})\partial_y v_1 + \partial_y(v_1^3) + \partial_y^3 v = 0 \\ v_1(t = 0) = v_0, \end{cases}$$

while v_2, the remainder term with zero initial data, solves

$$\begin{cases} \partial_t v_2 - \mathcal{L}_c v_2 = -\partial_y((Q_c + v_1 + v_2)^3 - Q_c^3 - v_1^3 - 3Q_c^2 v_2) + \{\text{less important terms}\} \\ v_2(t = 0) = 0. \end{cases}$$

The advantages of this decomposition become clear when one tries to obtain decay for the part of the wave which is to the right of the soliton, that is the region $\{y > 0\}$. In more technical terms, we want to obtain decay for $\|\langle y_+\rangle^m v_1\|_{L^2} + \|e^{ay} v_2\|_{L^2}$.

- The decay of $\|\langle y_+\rangle^m v_1\|_{L^2}$ is obtained by a virial-type argument, which one can apply since the equation for v_1 does not "see" the soliton, and the data are such that $\|\langle y_+\rangle^m v_0\|_{L^2} < \infty$.
- The decay of $\|e^{ay} v_2\|_{L^2}$ is obtained via decay estimates in exponentially weighted spaces (with norm of the type $\|e^{ay} \cdot \|_{L^2}$) for the linear group $e^{t\mathcal{L}_c}$. This requires the data, as well as the right-hand side of the v_2 equation, to belong to an exponentially weighted space. This is easily checked for the v_2 equation: the data is zero, and expanding the right-hand side, it appears that the slowly decaying v_1 factors are always coupled to v_2 or Q_c, thus ensuring exponential decay.

Organization of the paper. Section 2 contains the proof of Theorem 1.1 about the stability of the trivial solution. We begin by establishing some linear and simple multilinear decay estimates in 2.1. In 2.2 we prove energy estimates involving the scaling vector field. Sections 2.3 and 2.4 contain the heart of the proof of Theorem 1.1, that is the justification of the asymptotic expansion (1.21) and the control of the remainder terms. In section 2.5 we derive the complete asymptotic description of small solutions of (mKdV) relying on a refined linear estimate and the global bounds established before. Section 3 is devoted to the proof of Theorem 1.5 about the stability of soliton solutions. We first prove asymptotic stability à la Pego-Weinstein in section 3.1, and then give the proof of scattering for the radiation in section 3.2.

1.4. **Notations.** For $x \in \mathbb{R}$, we set

$$\langle x \rangle = \sqrt{1+x^2}.$$

We denote C for a constant whose value may change from one line to another. Given two quantities X and Y, we write

- $X \lesssim Y$ if $X \leq CY$ for a constant C.
- $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$.
- $X \ll Y$ if $X \leq cY$ for a small constant c.

We define the Fourier transform by

$$\mathcal{F}g(\xi) = \widehat{g}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} g(x)\, dx \qquad \Longrightarrow \qquad g(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \widehat{g}(\xi)\, d\xi \,.$$

The Fourier multiplier $m(\partial_x)$ with symbol m is given by

$$\mathcal{F}[m(\partial_x)f](\xi) = m(i\xi)\widehat{f}(\xi).$$

and the pseudoproduct operator T_m with symbol $m(\xi,\eta)$ by

$$\mathcal{F}[T_m(f,g)](\xi) = \int m(\xi,\eta)\widehat{f}(\xi-\eta)g(\eta)\, d\eta.$$

Let ψ be smooth, supported on $[-2,-\frac{1}{2}] \cup [\frac{1}{2},2]$, and satisfying

$$\sum_{j\in\mathbb{Z}} \psi\left(\frac{\xi}{2^j}\right) = 1, \qquad \text{for } \xi \neq 0.$$

Define

$$\chi = \sum_{j<0} \psi\left(\frac{\xi}{2^j}\right)$$

and the Littlewood-Paley operators

$$P_j = \psi\left(\frac{\partial_x}{i2^j}\right), \quad P_{<j} = \chi\left(\frac{\partial_x}{i2^j}\right), \quad P_{>j} = 1 - \chi\left(\frac{\partial_x}{i2^j}\right),$$
$$P_{\sim j} = \sum_{2^k \sim 2^j} P_k, \quad P_{\ll j} = \sum_{2^k \ll 2^j} P_k, \quad P_{\lesssim j} = \sum_{2^k \leq 2^{j+C}} P_k.$$

We will often denote $f_j := P_j f$, $f_{\sim j} := P_{\sim j} f$, and so on.

2. Stability of the zero solution

Recalling that $f(t) = e^{t\partial_x^3}u(t)$, we assume that the following X-norm is a priori small:

$$\|u\|_X = \sup_{t\geq 1} \left(t^{-\delta}\|u(t)\|_{H^1} + t^{-1/6}\big\|xf(t)\big\|_{H^1} + t^{\alpha/3-1/6}\big\||\partial_x|^\alpha xf(t)\big\|_{L^2} + \big\|\widehat{f}(t,\xi)\big\|_{L^\infty_\xi}\right) \leq \varepsilon_1 \,, \tag{2.1}$$

where $\alpha > 0$ is a suitable parameter less than, and close to, $1/2$ to be chosen later on in the course of our proof. We will then show

$$\|u\|_X \leq \varepsilon_0 + C\varepsilon_1^3 \tag{2.2}$$

for some absolute constant C. This a priori estimate, combined with a bootstrap argument, and the choice $\varepsilon_1 = \varepsilon_0^{2/3}$, gives global existence for sufficiently small ε_0. For simplicity, and without loss of generality, we only consider $t \geq 1$, assuming that a local solutions has been already constructed on the time interval $[0,1]$ by standard methods, such as those in [30]. Using also time reversibility we obtain a solution for all times.

2.1. Linear and multilinear estimates.

We begin by proving a refined linear estimate which also gives useful L^p bounds.

Lemma 2.1 (Linear Estimates). *For any $t \geq 1$, $x \in \mathbb{R}$, and $u(t,x) = e^{-t\partial_x^3} f(t,x)$ with f such that*

$$\|\widehat{f}(t)\|_{L^\infty} + t^{-1/6}\|xf(t)\|_{L^2} \leq 1, \tag{2.3}$$

the following estimate holds true:

$$\big||\partial_x|^\beta u(x,t)\big| \lesssim t^{-1/3-\beta/3}\big(1+|x/t^{1/3}|\big)^{-1/4+\beta/2}, \qquad 0 \leq \beta \leq 1, \tag{2.4}$$

$$\big|\partial_x u(x,t)\big| \lesssim t^{-2/3}\big(1+|x/t^{1/3}|\big)^{1/4}. \tag{2.5}$$

In particular, whenever $u = e^{-t\partial_x^3} f$ satisfies the a priori bounds (2.1), one has for any $\beta \in [0, \frac{1}{2})$, and all p with $p(1/4 - \beta/2) > 1$,

$$\||\partial_x|^\beta u(t)\|_{L^p} \lesssim t^{-1/3-\beta/3+1/(3p)}. \tag{2.6}$$

Remark 2.2. The refined linear estimate (2.4) in the case $\beta = 0$, and the estimate (2.5) coincide with the estimates obtained in [19]; see also [8] for related work. The improvement for $\beta > 0$ is needed to obtain (2.6), which will be in turn used to prove the trilinear estimate (2.12) below. (2.12) allows us to simplify our subsequent analysis (especially the key estimate of section 2.4) and give a sharper global existence result, see remark 1.3.

Proof. Denote $\Lambda(\xi) = \xi^3$, and write

$$u(t,x) = e^{-t\partial_x^3} f(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\phi(\xi)} \widehat{f}(\xi)\, d\xi, \qquad \phi(\xi) = \phi(\xi; x, t) := \xi(x/t) + \Lambda(\xi)$$

(we often drop for simplicity the dependence of ϕ on t and x). For $x \leq 0$, let

$$\xi_0^{\pm} := \pm\sqrt{-x/(3t)} \tag{2.7}$$

denote the stationary points of the phase $\phi(\xi)$, $\phi'(\xi_0^{\pm}) = 0$. In the case $x > 0$ there are no stationary points, and the estimate in this case is easier and follows from the same arguments that we present below.

We now restrict our attention to $x \leq 0$. Up to taking complex conjugates, we notice that in order to obtain (2.4)-(2.5) it suffices to show that

$$\Big|\int_0^\infty e^{it\phi(\xi)} |\xi|^\beta \widehat{f}(\xi,t)\, d\xi\Big| \lesssim t^{-1/3-\beta/3}\big(1+|x/t^{1/3}|\big)^{-1/4+\beta/2},$$

for all $\beta \in [0,1]$. Let us denote $\xi_0 := \sqrt{-x/(3t)}$ the only stationary point in the above integral. We see that since $|x/t|^{1/3} = 3(\xi_0 t^{1/3})^2$, it is then enough to show

$$\Big|\int_0^\infty e^{it\phi(\xi)} |\xi|^\beta \widehat{f}(\xi,t)\, d\xi\Big| \lesssim t^{-1/3-\beta/3} \max\big(t^{1/3}\xi_0, 1\big)^{-1/2+\beta}, \tag{2.8}$$

for any $t \geq 1$, $x \leq 0$, and any function f satisfying (2.3). We distinguish two cases depending on the size of ξ_0.

Case 1: $\xi_0 \leq t^{-1/3}$. In this case we only need to obtain a bound of $t^{-1/3-\beta/3}$ for the term in (2.8). We split the integral in (2.8) as follows:

$$\begin{aligned} &\int_0^\infty e^{it\phi(\xi)}|\xi|^\beta \widehat{f}(\xi)\,d\xi = A + B, \\ &A = \int_0^\infty e^{it\phi(\xi)}|\xi|^\beta \widehat{f}(\xi)\chi(2^{-10}t^{1/3}\xi)\,d\xi, \\ &B = \int_0^\infty e^{it\phi(\xi)}|\xi|^\beta \widehat{f}(\xi)(1-\chi(2^{-10}t^{1/3}\xi))\,d\xi. \end{aligned}$$

The first term can be very easily estimated using the first bound in (2.3). For the second term we notice that $|\xi| \gg \xi_0$ on the support of the integral, so that $|\partial_\xi \phi| = 3|\xi_0^2 - \xi^2| \gtrsim |\xi|^2 \gtrsim t^{-2/3}$. An integration by parts then gives:

$$\begin{aligned} &|B| \lesssim B_1 + B_2, \\ &B_1 = \frac{1}{t}\int_0^\infty \Big|\partial_\xi \Big(\frac{1}{\partial_\xi \phi}|\xi|^\beta (1-\chi(2^{-10}t^{1/3}\xi)\Big)\Big| |\widehat{f}(\xi)|\,d\xi, \\ &B_2 = \frac{1}{t}\int_0^\infty \Big|\frac{1}{\partial_\xi \phi}|\xi|^\beta (1-\chi(2^{-10}t^{1/3}\xi)\Big| |\partial_\xi \widehat{f}(\xi)|\,d\xi. \end{aligned}$$

We can then estimate

$$B_1 \lesssim \frac{1}{t}\|\widehat{f}\|_{L^\infty} \int_0^\infty \frac{1}{|\xi|^{3-\beta}}(1-\chi(2^{-10}t^{1/3}\xi)) + \frac{1}{|\xi|^{2-\beta}}|\chi'(2^{-10}t^{1/3}\xi)|t^{1/3}\,d\xi \lesssim t^{-1/3-\beta/3}.$$

Similarly, we can use the second bound provided by (2.3) to obtain:

$$B_2 \lesssim \frac{1}{t}\|\partial_\xi \widehat{f}\|_{L^2}\Big(\int_0^\infty \frac{1}{|\xi|^{4-2\beta}}(1-\chi(2^{-10}t^{1/3}\xi))\,d\xi\Big)^{1/2} \lesssim t^{-1/3-\beta/3}.$$

Case 2: $\xi_0 \geq t^{-1/3}$. In this case we aim to prove a bound of $t^{-1/2}\xi_0^{-1/2+\beta}$ for the left-hand side of (2.8). To separate the non-stationary and stationary cases we split the integral as follows (see 1.4):

$$\begin{aligned} &\int_0^\infty e^{it\phi(\xi)}|\xi|^\beta \widehat{f}(\xi)\,d\xi = C + D, \\ &C = \int_0^\infty e^{it\phi(\xi)}|\xi|^\beta \widehat{f}(\xi)(1-\psi(\xi/\xi_0))\,d\xi, \\ &D = \int_0^\infty e^{it\phi(\xi)}|\xi|^\beta \widehat{f}(\xi)\psi(\xi/\xi_0)\,d\xi. \end{aligned} \tag{2.9}$$

Integrating by parts we get

$$\begin{aligned} &|C| \lesssim C_1 + C_2, \\ &C_1 = \frac{1}{t}\int_0^\infty \Big|\partial_\xi \Big(\frac{1}{\partial_\xi \phi}|\xi|^\beta (1-\psi(\xi/\xi_0))\Big)\Big| |\widehat{f}(\xi)|\,d\xi, \\ &C_2 = \frac{1}{t}\int_0^\infty \frac{1}{|\partial_\xi \phi|}|\xi|^\beta (1-\psi(\xi/\xi_0))|\partial_\xi \widehat{f}(\xi)|\,d\xi. \end{aligned}$$

Using the fact that on the support of the above integrals $|\partial_\xi \phi| \gtrsim \max(\xi,\xi_0)^2$ we obtain

$$C_1 \lesssim \frac{1}{t}\|\widehat{f}\|_{L^\infty} \int_0^\infty \left(\frac{|\xi|^{\beta-1}}{\max(\xi,\xi_0)^2}(1-\psi(\xi/\xi_0)) + \frac{1}{|\xi|^{2-\beta}}|\psi'(\xi/\xi_0)|\xi_0^{-1}\right) d\xi \lesssim t^{-1}\xi_0^{-2+\beta},$$

which is stronger than the desired bound since $\xi_0 \geq t^{-1/3}$. Similarly

$$C_2 \lesssim \frac{1}{t}\|\partial_\xi \widehat{f}\|_{L^2}\Big(\int_0^\infty \frac{1}{\max(\xi,\xi_0)^{4-2\beta}}(1-\psi(\xi/\xi_0))\,d\xi\Big)^{1/2} \lesssim t^{-5/6}\xi_0^{-3/2+\beta}$$

which suffices since $\xi_0 \geq t^{-1/3}$.

To estimate the resonant contributions $\xi \approx \xi_0$ we let l_0 be the smallest integer such that $2^{l_0} \geq 1/\sqrt{t\xi_0}$ and bound the term D in (2.9) as follows:

$$\begin{aligned}
|D| &\leq \sum_{l=l_0}^{\log\xi_0+10} |D_l|,\\
D_{l_0} &:= \int_{\mathbb{R}} e^{it\phi(\xi)}|\xi|^\beta \widehat{f}(\xi)\psi(\xi/\xi_0)\chi\big(2^{-l_0}(\xi-\xi_0)\big)\,d\xi,\\
D_l &:= \int_{\mathbb{R}} e^{it\phi(\xi)}|\xi|^\beta \widehat{f}(\xi)\psi(\xi/\xi_0)\psi(2^{-l}(\xi-\xi_0))\,d\xi\,, \qquad l \geq l_0+1.
\end{aligned}$$

The choice of l_0 and the first bound in (2.3) immediately give us $|D_{l_0}| \lesssim \xi_0^\beta 2^{l_0} \lesssim t^{-1/2}\xi_0^{-1/2+\beta}$. To control the terms D_l we integrate by parts and see that:

$$\begin{aligned}
|D_l| &\lesssim D_{l,1} + D_{l,2},\\
D_{l,1} &= \frac{1}{t}\int_0^\infty \Big|\partial_\xi\Big(\frac{1}{\partial_\xi\phi}|\xi|^\beta\psi(\xi/\xi_0)\psi(2^{-l}(\xi-\xi_0))\Big)\Big||\widehat{f}(\xi)|\,d\xi,\\
D_{l,2} &= \frac{1}{t}\int_0^\infty \frac{1}{|\partial_\xi\phi|}|\xi|^\beta\psi(\xi/\xi_0)\psi(2^{-l}(\xi-\xi_0))|\partial_\xi\widehat{f}(\xi)|\,d\xi.
\end{aligned}$$

Using the fact that on the support of the integrals $|\partial_\xi\phi| = 3|\xi^2-\xi_0^2| \approx 2^l\xi_0$, we can estimate

$$D_{l,1} \lesssim t^{-1}\|\widehat{f}\|_{L^\infty}2^{-l}\xi_0^{-1+\beta},$$

which gives the desired bound upon summation in l. Similarly, we have

$$D_{l,2} \lesssim t^{-1}\|\partial_\xi\widehat{f}\|_{L^2}2^{-l/2}\xi_0^{-1+\beta}.$$

Using the second bound in (2.3) and summing in l we obtain a bound of $t^{-5/6}\xi_0^{-1+\beta}2^{-l_0/2}$, which is better than our desired bound. This concludes the proof of (2.4). The estimate (2.6) follows by integrating in L^p the inequality (2.4). $\square$

Lemma 2.3 (Multilinear Estimates)**.** *Let $u = e^{t\partial_x^3}f$ be a function satisfying the a priori assumptions*

$$\sup_{t\geq 1}\big(t^{-1/6}\big\|xf(t)\big\|_{L^2} + \big\|\widehat{f}(t)\big\|_{L^\infty}\big) \leq \varepsilon_1. \tag{2.10}$$

Then the following bilinear estimate holds:

$$\sup_{t\geq1} t\,\|u(t)u_x(t)\|_{L^\infty_x} \lesssim \varepsilon_1^2. \tag{2.11}$$

Moreover, for all $0 \le \alpha < \frac{1}{2}$ *we have*

$$\big\| |\partial_x|^\alpha u^3(t) \big\|_{L^2} \lesssim \varepsilon_1^3 |t|^{-5/6-\alpha/3}. \tag{2.12}$$

Proof. To obtain (2.11) it suffices to multiply the bounds provided by (2.4) in the case $\beta = 0$ and (2.6).

To show (2.12) we start by choosing $\beta \in (\alpha, 1/2)$, $p, q, p_1, p_2 \in (2, \infty)$, such that

$$1/p + 1/q = 1/2, \quad 1/p = \theta/p_1 + (1-\theta)/p_2 \quad \text{with} \quad \theta = \alpha/\beta, \tag{2.13}$$

and

$$p_1(1/4 - \beta/2) > 1, \qquad p_2 > 4$$

(to see that it is possible to choose parameters satisfying the above requirements, first fix $\beta \in (\alpha, 1/2)$; the other indexes are then fully determined by p and p_1; one checks that the above inequalities are satisfied if $p, p_1 \to \infty$). We use the fractional Leibniz rule (see for instance [30]), followed by the Gagliardo-Nirenberg inequality (Theorem 2.44 in[3]) to obtain

$$\big\| |\partial_x|^\alpha u^3 \big\|_{L^2} \lesssim \big\| |\partial_x|^\alpha u \big\|_{L^p} \|u^2\|_{L^q} \lesssim \big\| |\partial_x|^\beta u \big\|_{L^{p_1}}^{\theta} \|u\|_{L^{p_2}}^{1-\theta} \|u\|_{L^{2q}}^2$$

Using the linear estimate (2.6) we have

$$\big\| |\partial_x|^\beta u \big\|_{L^{p_1}} \lesssim t^{-1/3-\beta/3+1/(3p_1)}, \qquad \|u\|_{L^{p_2}} \lesssim t^{-1/3+1/(3p_2)}, \qquad \|u\|_{L^{2q}} \lesssim t^{-1/3+1/(6q)}.$$

Using these three inequalities we see that

$$\big\| |\partial_x|^\alpha u^3 \big\|_{L^2} \lesssim t^\gamma$$

where, using (2.13), we have

$$\begin{aligned} \gamma &= (-1/3 - \beta/3 + 1/(3p_1))\theta + (-1/3 + 1/(3p_2))(1-\theta) + 2(-1/3 + 1/(6q)) \\ &= -1 + (1/3)\big(\theta(-\beta + 1/p_1) + (1-\theta)/p_2\big) + 1/(3q) = -5/6 - \alpha/3 \end{aligned}$$

as desired. $\square$

2.2. **Energy estimates.** We now prove energy and weighted energy estimates.

Lemma 2.4. *For data* u_0 *satisfying* (1.3), *let* $u \in C([0,T]; H^1)$ *be a solution of* (mKdV) *satisfying the apriori bounds* (2.1). *Then*

$$\|u(t)\|_{H^1} \le \varepsilon_0 + C\varepsilon_1^2.$$

Moreover, if $u = e^{-t\partial_x^3} f$,

$$\|xf(t)\|_{H^1} \le C(\varepsilon_0 + \varepsilon_1^3)\langle t\rangle^{1/6}. \tag{2.14}$$

Finally, for $0 \le \alpha < \frac{1}{2}$,

$$\big\| |\partial_x|^\alpha x f(t) \big\|_{L^2} \le C(\varepsilon_0 + \varepsilon_1^3)\langle t\rangle^{1/6-\alpha/3}. \tag{2.15}$$

Proof. The first estimate follows from the conservation of Mass and Energy (1.2). The estimates (2.14) and (2.15) will be obtained by energy estimates performed on the (mKdV) equation itself, or on the equation obtained after commuting the scaling vector field $S := 1 + x\partial_x + 3t\partial_t$, i.e.

$$\partial_t Su + \partial_x^3 Su + 3\partial_x(u^2 Su) = 0. \tag{2.16}$$

Proof of (2.14). In the following, we denote, given a function a, Ia for the antiderivative of a vanishing at $-\infty$: $[Ia](x) = \int_{-\infty}^{x} a$. Applying I to (2.16) gives

$$\partial_t ISu + \partial_x^2 Su + 3u^2 Su = 0.$$

Multiplying by ISu and integrating in space yields

$$\frac{1}{2}\frac{d}{dt}\big\|ISu\big\|_{L^2}^2 + \int \partial_x^2 Su\, ISu\, dx + 3\int u^2 Su\, ISu\, dx = 0,$$

or, after integrating by parts, and taking (2.11) into account,

$$\frac{1}{2}\frac{d}{dt}\big\|ISu\big\|_{L^2}^2 = \frac{3}{2}\int \partial_x(u^2)\big(ISu\big)^2\, dx \lesssim \frac{\varepsilon_1^2}{t}\big\|ISu\big\|_{L^2}^2.$$

By Gronwall's lemma

$$\big\|ISu\big\|_{L^2}^2 \lesssim \varepsilon_0 t^{C\varepsilon_1^2}. \tag{2.17}$$

Observe now that $xf = ISf - 3t\partial_t If = e^{t\partial_x^3} ISu + 3e^{t\partial_x^3} tu^3$, from which the above inequality, combined with (2.12), gives the desired result:

$$\|xf\|_{L^2} \leq \|ISu\|_{L^2} + 3t\|u^3\|_{L^2} \lesssim \varepsilon_0 t^{C\varepsilon_1^2} + \varepsilon_1^3 t^{1/6}.$$

A similar argument applies to give a bound for $\|\partial_x xf\|_{L^2}$ and completes the proof of (2.14).

Proof of (2.15). As explained in Remark 1.3, for the proof here, we do not need to assume that xf is in H^1. We shall then give here a proof of (2.15) that does not require higher regularity. By using the H^1 regularity, a shorter proof is possible (we shall use such an argument in the proof of Lemma 3.10 below to handle the solitary wave stability). Starting from (2.16), a simple energy estimate leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\||\partial_x|^{\alpha-1}Su\|_{L^2}^2 &= -3\int |\partial_x|^{\alpha-1}\partial_x(u^2 Su)|\partial_x|^{\alpha-1}Su\, dx \\ &= -3\int |\partial_x|^{\alpha-1}\partial_x(wv)|\partial_x|^{\alpha-1}v\, dx\end{aligned}$$

where

$$w = u^2 \quad \text{and} \quad v = Su\,.$$

Recalling that $w_j = P_j w$ and $v_j = P_j v$, a paraproduct decomposition of the above right-hand side gives

$$\begin{aligned}\frac{d}{dt}\||\partial_x|^{\alpha-1}v\|_{L^2}^2 &= \int |\partial_x|^{\alpha-1}\partial_x\Big(\sum_{2^j\gg 2^k} w_j v_k + \sum_{2^j\sim 2^k} w_j v_k + \sum_{2^k\gg 2^j} w_j v_k\Big)|\partial_x|^{\alpha-1}v\, dx \\ &=: I + II + III.\end{aligned}$$

To estimate I, we use the dispersive estimate (2.11) and the standard properties of the Littlewood-Paley decomposition (see for example [3]) to obtain

$$
\begin{aligned}
I &\lesssim \Big\| |\partial_x|^{\alpha-1} \sum_{2^k \ll 2^j} \partial_x (w_j v_k) \Big\|_{L^2} \| |\partial_x|^{\alpha-1} v\|_{L^2} \\
&\lesssim \Bigg[\sum_j \Big(2^{\alpha j} \Big\| \sum_{2^k \ll 2^j} w_j v_k \Big\|_{L^2} \Big)^2 \Bigg]^{1/2} \| |\partial_x|^{\alpha-1} v\|_{L^2} \\
&\lesssim \Bigg[\sum_j \Big(\sum_{2^k \ll 2^j} 2^{(\alpha-1)j} 2^j \|w_j\|_{L^\infty} 2^{k(1-\alpha)} \| |\partial_x|^{\alpha-1} v_k\|_{L^2} \Big)^2 \Bigg]^{1/2} \| |\partial_x|^{\alpha-1} v\|_{L^2} \\
&\lesssim \frac{\varepsilon_1^2}{\langle t \rangle} \Bigg[\sum_j \Bigg(\sum_{2^k \ll 2^j} 2^{(j-k)(\alpha-1)} \| |\partial_x|^{\alpha-1} v_k\|_{L^2} \Bigg)^2 \Bigg]^{1/2} \| |\partial_x|^{\alpha-1} v\|_{L^2} \lesssim \frac{\varepsilon_1^2}{\langle t \rangle} \| |\partial_x|^{\alpha-1} v\|_{L^2}^2 .
\end{aligned}
$$

The estimate of II also relies on (2.11):

$$
\begin{aligned}
II &\lesssim \Big\| |\partial_x|^{\alpha-1} \partial_x \sum_{2^k \sim 2^j} (w_j v_k) \Big\|_{L^2} \| |\partial_x|^{\alpha-1} v\|_{L^2} \\
&\lesssim \Bigg[\sum_\ell \Big(2^{\alpha \ell} \Big\| P_\ell \Big(\sum_{2^k \sim 2^j} w_j v_k \Big) \Big\|_{L^2} \Big)^2 \Bigg]^{1/2} \| |\partial_x|^{\alpha-1} v\|_{L^2} \\
&\lesssim \Bigg[\sum_\ell \Big(\sum_{2^k \sim 2^j \gtrsim 2^\ell} 2^{\alpha \ell} 2^{-\alpha k} 2^j \|w_j\|_{L^\infty} 2^{k(\alpha-1)} \|v_k\|_{L^2} \Big)^2 \Bigg]^{1/2} \| |\partial_x|^{\alpha-1} v\|_{L^2} \\
&\lesssim \frac{\varepsilon_1^2}{\langle t \rangle} \Bigg[\sum_\ell \Big(\sum_{2^k \gtrsim 2^\ell} 2^{\alpha(\ell-k)} \| |\partial_x|^{\alpha-1} v_k\|_{L^2} \Big)^2 \Bigg]^{1/2} \| |\partial_x|^{\alpha-1} v\|_{L^2} \lesssim \frac{\varepsilon_1^2}{\langle t \rangle} \| |\partial_x|^{\alpha-1} v\|_{L^2}^2 .
\end{aligned}
$$

To estimate III, we need the classical commutator estimate

$$
\big\| \big[|\partial_x|^{\alpha-1} \partial_x, P_{\ll j} w \big] \, P_j f \big\|_{L^2} \lesssim 2^{(\alpha-1)j} \|\partial_x w\|_{L^\infty} \|P_j f\|_{L^2}, \tag{2.18}
$$

proved in the appendix in Lemma A.3. Commuting $|\partial_x|^{\alpha-1}\partial_x$ with $w_{\ll k}$ in III gives

$$
\begin{aligned}
III &= \int \sum_k \big[|\partial_x|^{\alpha-1} \partial_x, w_{\ll k} \big] \, v_k \, |\partial_x|^{\alpha-1} v \, dx + \int \sum_k w_{\ll k} \, \partial_x |\partial_x|^{\alpha-1} \, v_k \, |\partial_x|^{\alpha-1} v \, dx \\
&=: III_a + III_b
\end{aligned}
$$

It follows easily from the commutator estimate (2.18) and (2.11) that

$$
|III_a| \lesssim \|\partial_x w\|_{L^\infty} \| |\partial_x|^{\alpha-1} v\|_{L^2}^2 \lesssim \frac{\varepsilon_1^2}{\langle t \rangle} \| |\partial_x|^{\alpha-1} v\|_{L^2}^2 .
$$

To estimate III_b, we integrate by parts to obtain

$$\begin{aligned} III_b &= \int \sum_{2^j \sim 2^k} w_{\ll j} \partial_x |\partial_x|^{\alpha-1} v_j |\partial_x|^{\alpha-1} v_k \, dx \\ &= \int \sum_{2^j \sim 2^k} \partial_x w_{\ll j} |\partial_x|^{\alpha-1} v_j |\partial_x|^{\alpha-1} v_k \, dx + \{ \text{ remainder } \}. \end{aligned}$$

The remainder is easy to treat, thus we skip it, and the main term is not much harder:

$$|III_b| \lesssim \|\partial_x w\|_{L^\infty} \| |\partial_x|^{\alpha-1} v\|_{L^2}^2 \lesssim \frac{\varepsilon_1^2}{t} \| |\partial_x|^{\alpha-1} v\|_{L^2}^2.$$

Gathering the estimates on I, II and III, we obtain

$$\frac{d}{dt} \| |\partial_x|^{\alpha-1} v\|_{L^2}^2 \lesssim \frac{\varepsilon_1^2}{t} \| |\partial_x|^{\alpha-1} v\|_{L^2}^2,$$

which implies, by Gronwall's lemma,

$$\| |\partial_x|^{\alpha-1} Su\|_{L^2} = \| |\partial_x|^{\alpha-1} v\|_{L^2} \lesssim \varepsilon_0 t^{C\varepsilon_1^2}.$$

Finally, observe that

$$\begin{aligned} \| |\partial_x|^{\alpha} x f\|_{L^2} = \| |\partial_x|^{\alpha-1} (x\partial_x + 1) f\|_{L^2} &\leq \| |\partial_x|^{\alpha-1} Sf\|_{L^2} + 3t \| |\partial_x|^{\alpha-1} \partial_t f\|_{L^2} \\ &= \| |\partial_x|^{\alpha-1} Su\|_{L^2} + 3t \| |\partial_x|^{\alpha} u^3\|_{L^2}. \end{aligned}$$

The desired estimate follows by combining the bound on $\| |\partial_x|^{\alpha-1} Su\|_{L^2}$ and (2.12). $\square$

2.3. **Control of $\sup_t \|\widehat{f}(t)\|_\infty$.** This section is dedicated to proving the following key proposition:

Proposition 2.5. *Under the a priori assumptions* (2.1)*, the following estimate holds for a solution u of* (mKdV)*:*

$$\sup_t \big\|\widehat{f}(t)\big\|_\infty \leq \big\|\widehat{u_0}\big\|_\infty + C\varepsilon_1^3\,. \tag{2.19}$$

Proof. We will show the following key identity: for $t > 1$,

$$\partial_t \widehat{f}(t,\xi) = \frac{i \operatorname{sign} \xi}{6t} |\widehat{f}(t,\xi)|^2 \widehat{f}(t,\xi) + \frac{c}{t} e^{it\frac{8}{9}\xi^3} \Big[\mathbf{1}_{|\xi| > t^{-1/3}} \widehat{f}(t, \xi/3) \Big]^3 + R(t,\xi)\,, \tag{2.20}$$

where $c \in \mathbb{C}$ is a constant, and R satisfies the bound

$$\int_{-\infty}^{\infty} |R(t,\xi)| \, dt \lesssim \varepsilon_1^3\,. \tag{2.21}$$

The proofs of (2.20) and (2.21) above will be given in Section 2.4 below; let us first show how (2.20)-(2.21) imply the desired conclusion (2.19). Define the modified profile $\widehat{w}$ as follows:

$$\widehat{w}(t,\xi) := e^{-iB(t,\xi)} \widehat{f}(t,\xi) \qquad\qquad B(t,\xi) := \frac{1}{6} \operatorname{sign} \xi \int_1^t \big|\widehat{f}(s,\xi)\big|^2 \frac{ds}{s}\,. \tag{2.22}$$

Then we have

$$\begin{aligned} \partial_t \widehat{w}(t,\xi) &= e^{-iB(t,\xi)} \Big[\partial_t \widehat{f}(t,\xi) - i \partial_t B(t,\xi) \widehat{f}(t,\xi) \Big] \\ &= e^{-iB(t,\xi)} \left\{ \frac{c}{t} e^{it\frac{8}{9}\xi^3} \Big[\mathbf{1}_{|\xi| > t^{-1/3}} \widehat{f}(t,\xi/3) \Big]^3 + R(t,\xi) \right\} \end{aligned}$$

Integrating in time the above identity, using the fact that B is real, $|\widehat{w}(t,\xi)| = |\widehat{f}(t,\xi)|$, and the remainder estimate (2.21), we obtain

$$\big|\widehat{f}(t,\xi)\big| \leq |\widehat{u_0}(\xi)| + c\left|\int_{|\xi|^{-3}}^{t} e^{is\frac{8}{9}\xi^3} e^{-iB(s,\xi)} \mathbf{1}_{|\xi|>s^{-1/3}} \widehat{f}(s,\xi/3)^3 \frac{ds}{s}\right| + \varepsilon_1^3 .$$

The desired conclusion will then follow once we show that, for $t > |\xi|^{-3}$,

$$\Big|\int_{|\xi|^{-3}}^{t} e^{is\frac{8}{9}\xi^3} e^{-iB(s,\xi)} \widehat{f}(s,\xi/3)^3 \frac{ds}{s}\Big| \lesssim \varepsilon_1^3 . \tag{2.23}$$

Proof of (2.23). Integrating by parts in s using the identity $e^{is\frac{8}{9}\xi^3} = \frac{9}{8i\xi^3}\partial_s e^{is\frac{8}{9}\xi^3}$, we see that

$$\begin{aligned}
&\Big|\int_{|\xi|^{-3}}^{t} e^{is\frac{8}{9}\xi^3} e^{-iB(s,\xi)} \widehat{f}(s,\xi/3)^3 \frac{ds}{s}\Big| \lesssim J+K+L+M \\
&J = \frac{1}{|\xi|^3}|\widehat{f}(s,\xi/3)|^3 \frac{1}{s}\ \Big|_{s=|\xi|^{-3}}^{s=t} , \\
&K = \int_{|\xi|^{-3}}^{t} \frac{1}{|\xi|^3} |\partial_s \widehat{f}(s,\xi/3)| |\widehat{f}(s,\xi/3)|^2 \frac{ds}{s} , \\
&L = \int_{|\xi|^{-3}}^{t} \frac{1}{|\xi|^3} |\partial_s B(s,\xi)| |\widehat{f}(s,\xi/3)|^3 \frac{ds}{s} , \\
&M = \int_{|\xi|^{-3}}^{t} \frac{1}{|\xi|^3} |\widehat{f}(s,\xi/3)|^3 \frac{ds}{s^2} .
\end{aligned} \tag{2.24}$$

Since $t \geq |\xi|^{-3}$, the a priori assumption $\|\widehat{f}(t)\|_{L^\infty} \leq \varepsilon_1$ gives immediately that $J \lesssim \varepsilon_1^3$. Using again $\|\widehat{f}(t)\|_{L^\infty} \leq \varepsilon_1$, and (2.20)-(2.21) we can estimate

$$K \lesssim \int_{|\xi|^{-3}}^{t} \frac{1}{|\xi|^3} \Big[\frac{\varepsilon_1^3}{s} + R(s,\xi)\Big] \varepsilon_1^2 \frac{ds}{s} \lesssim \varepsilon_1^5 .$$

¿From the definition of B in (2.22) we see that

$$L \lesssim \int_{|\xi|^{-3}}^{t} \frac{1}{|\xi|^3} \frac{\varepsilon_1^2}{s} \varepsilon_1^3 \frac{ds}{s} \lesssim \varepsilon_1^5 .$$

The last term, M, is easily bounded by $|\xi|^{-3}\varepsilon_1^3 \int_{|\xi|^{-3}}^{t} \frac{ds}{s^2} \lesssim \varepsilon_1^3$, which completes the proof of (2.23). $\square$

2.4. **Proof of** (2.20)-(2.21)**.** Recall that we assume $t > 1$.

<u>*Some estimates on f.*</u> Recall that $f_j = P_j f$; we start by stating a few estimates on f_j that follow immediately from the a priori assumption (2.1):

$$\begin{aligned}
&\|\widehat{f_j}\|_{L^\infty} \lesssim \varepsilon_1, \\
&\|xf_j\|_{L^2} \lesssim \left[2^{-\frac{j}{2}} + \min\left(t^{\frac{1}{6}}, 2^{-\alpha j} t^{\frac{1}{6}-\frac{\alpha}{3}}\right)\right]\varepsilon_1, \\
&\|f_j\|_{L^1} \lesssim \|f_j\|_{L^2}^{1/2}\|xf_j\|_{L^2}^{1/2} \lesssim \left(1 + 2^{\frac{j}{4}} t^{\frac{1}{12}}\right)\varepsilon_1 \\
&\|f_j\|_{L^1} \lesssim \left(1 + 2^{j(\frac{1}{4}-\frac{\alpha}{2})} t^{\frac{1}{12}-\frac{\alpha}{6}}\right)\varepsilon_1, \\
&\||x|^{2\rho} f_j\|_{L^1} \lesssim \|f_j\|_{L^2}^{\frac{1}{2}-2\rho}\|xf_j\|_{L^2}^{\frac{1}{2}+2\rho} \lesssim \left(2^{-2\rho j} + 2^{j(\frac{1}{4}-\rho)} t^{\frac{1}{12}+\frac{\rho}{3}}\right)\varepsilon_1, \quad \text{for} \quad 0 \le \rho < \frac{1}{4}.
\end{aligned} \tag{2.25}$$

Moreover, if $2^j \ge t^{-\frac{1}{3}}$, then $f_{<j}$ satisfies

$$\begin{aligned}
&\|f_{<j}\|_{L^1} \lesssim 2^{\frac{j}{4}} t^{\frac{1}{12}} \varepsilon_1 \\
&\|f_{<j}\|_{L^1} \lesssim 2^{j(\frac{1}{4}-\frac{\alpha}{2})} t^{\frac{1}{12}-\frac{\alpha}{6}} \varepsilon_1 \\
&\||x|^{2\rho} f_{<j}\|_{L^1} \lesssim 2^{j(\frac{1}{4}-\rho)} t^{\frac{1}{12}+\frac{\rho}{3}} \varepsilon_1, \quad \text{for} \quad 0 \le \rho < \frac{1}{4}.
\end{aligned} \tag{2.26}$$

Let us prove the first inequality, the proofs of the other ones being similar. Observe that, by the a priori assumption (2.1),

$$\begin{aligned}
&\|f_{<-\frac{1}{3}\log_2 t}\|_{L^2} \lesssim \varepsilon_1 t^{-1/6}, \\
&\|xf_{<-\frac{1}{3}\log_2 t}\|_{L^2} = \|\partial_\xi[\chi(t^{1/3}\xi)\widehat{f}(\xi)]\|_{L^2} \le t^{1/3}\|\chi'(t^{1/3}\xi)\widehat{f}(\xi)\|_{L^2} + \|\chi(t^{1/3}\xi)\partial_\xi \widehat{f}(\xi)\|_{L^2} \lesssim \varepsilon_1 t^{1/6},
\end{aligned}$$

and estimate

$$\begin{aligned}
\|f_{<j}\|_{L^1} &\le \|f_{<-\frac{1}{3}\log_2 t}\|_{L^1} + \sum_{t^{-1/3} \le 2^k \le 2^j} \|f_k\|_{L^1} \\
&\lesssim \|f_{<-\frac{1}{3}\log_2 t}\|_{L^2}^{1/2} \|xf_{<-\frac{1}{3}\log_2 t}\|_{L^2}^{1/2} + \sum_{t^{-1/3} \le 2^k \le 2^j} 2^{\frac{k}{4}} t^{\frac{1}{12}} \varepsilon_1 \\
&\lesssim \left(1 + 2^{\frac{j}{4}} t^{\frac{1}{12}}\right)\varepsilon_1 \lesssim 2^{\frac{j}{4}} t^{\frac{1}{12}} \varepsilon_1.
\end{aligned}$$

<u>*Decomposition of $\partial_t \widehat{f}$.*</u> Assume that $|\xi| \in (2^j, 2^{j+1})$ and split

$$\begin{aligned}
\partial_t \widehat{f}(t,\xi) &= -\frac{i}{2\pi} \iint e^{-it\phi(\xi,\eta,\sigma)} \xi \widehat{f}(\xi-\eta-\sigma)\widehat{f}(\eta)\widehat{f}(\sigma)\, d\eta\, d\sigma \\
&= -\sum_{k,l} \underbrace{\frac{i}{2\pi} \iint e^{-it\phi(\xi,\eta,\sigma)} \xi \widehat{f}(\xi-\eta-\sigma)\widehat{f}(\eta)\widehat{f}(\sigma)\psi\left(\frac{\eta}{2^k}\right)\psi\left(\frac{\sigma}{2^\ell}\right) d\eta\, d\sigma}_{A_{k\ell}} \\
&= -\Big[\underbrace{\sum_{\substack{2^k,2^\ell \lesssim 2^j \\ 2^j > t^{-1/3}}} A_{k\ell}}_{I} + \underbrace{\sum_{\substack{2^k \sim 2^\ell \gg 2^j \\ 2^k > t^{-1/3}}} A_{k\ell}}_{II} + \underbrace{\sum_{\substack{2^k \gg 2^\ell, 2^j \\ 2^k > t^{-1/3}}} A_{k\ell}}_{III} + \underbrace{\sum_{2^j,2^k,2^\ell \lesssim t^{-1/3}} A_{k\ell}}_{IV}\Big] + \left\{ \begin{matrix} \text{symmetric} \\ \text{terms} \end{matrix} \right\}.
\end{aligned} \tag{2.27}$$

Contribution of I. It can be written

$$\begin{aligned} I &= \frac{i}{2\pi} \iint e^{-it\phi(\xi,\eta,\sigma)} \xi \widehat{f_{\lesssim j}}(\xi-\eta-\sigma) \widehat{f_{\lesssim j}}(\eta) \widehat{f_{\lesssim j}}(\sigma) \chi\left(\frac{\eta}{C2^j}\right) \chi\left(\frac{\sigma}{C2^j}\right)\, d\eta\, d\sigma \\ &= \frac{2^{3j} i}{2\pi} \iint e^{-it2^{3j}\phi(\xi',\eta',\sigma')} \xi' \widehat{f_{\lesssim j}}(2^j(\xi'-\eta'-\sigma')) \widehat{f_{\lesssim j}}(2^j\eta') \widehat{f_{\lesssim j}}(2^j\sigma') \chi\left(\frac{\eta'}{C}\right) \chi\left(\frac{\sigma'}{C}\right)\, d\eta'\, d\sigma' \end{aligned}$$

where we changed variables by setting $(\xi',\eta',\sigma') = 2^{-j}(\xi,\eta,\sigma)$. This can also be written

$$I = \frac{2^{3j} i}{2\pi} \iint e^{-it2^{3j}\phi(\xi',\eta',\sigma')} \xi' F(\eta',\sigma') \chi\left(\frac{\eta'}{C}\right) \chi\left(\frac{\sigma'}{C}\right)\, d\eta'\, d\sigma'$$

with

$$F(\eta',\sigma') = \widehat{f_{\lesssim j}}(2^j(\xi'-\eta'-\sigma')) \widehat{f_{\lesssim j}}(2^j\eta') \widehat{f_{\lesssim j}}(2^j\sigma')\,,$$

and where $|\xi'| \sim 1$. Applying Lemma A.1, in light of (1.18)–(1.20), we get, for $|\xi| \geq t^{-1/3}$,

$$I = \frac{i \operatorname{sign} \xi}{6t} |\widehat{f}(\xi)|^2 \widehat{f}(\xi) + \frac{ic}{t} e^{-it\frac{8}{9}\xi^3} \widehat{f}\left(\frac{\xi}{3}\right)^3 + 2^{3j} O\left(\frac{\|\langle (x,y)\rangle^{2\rho} \widehat{F}\|_{L^1}}{(2^{3j}t)^{1+\rho}} \right), \tag{2.28}$$

where c is a constant whose exact value will not matter. Now observe that

$$\widehat{F}(x,y) = \frac{2^{-3j}}{2\pi} \int e^{-iz\xi} f_{\lesssim j}(2^{-j}(z-x)) f_{\lesssim j}(2^{-j}z) f_{\lesssim j}(2^{-j}(y-z))\, dz$$

so that

$$\|\widehat{F}\|_{L^1} \lesssim \|f_{\lesssim j}\|_{L^1}^3 \qquad \text{and} \qquad \||(x,y)|^{2\rho}\widehat{F}\|_{L^1} \lesssim 2^{2\rho j} \|f_{\lesssim j}\|_{L^1}^2 \||x|^{2\rho} f_{\lesssim j}\|_{L^1}. \tag{2.29}$$

Combining (2.28) and (2.29) above, and using (2.26) gives

$$\begin{aligned} &\left| I - \frac{i \operatorname{sign} \xi}{6t} |\widehat{f}(\xi)|^2 \widehat{f}(\xi) - \frac{ic}{t} e^{-it\frac{8}{9}\xi^3} \widehat{f}\left(\frac{\xi}{3}\right)^3 \mathbf{1}_{|\xi|>t^{-1/3}} \right| \\ &\qquad \lesssim 2^{-3\rho j} t^{-1-\rho} \big(\|f_{\lesssim j}\|_{L^1}^3 + 2^{2\rho j} \|f_{\lesssim j}\|_{L^1}^2 \||x|^{2\rho} f_{\lesssim j}\|_{L^1} \big) \\ &\qquad \lesssim 2^{j(\frac{3}{4}-2\rho-\alpha)} t^{-\frac{3}{4}-\frac{2}{3}\rho-\frac{\alpha}{3}} \varepsilon_1^3. \end{aligned} \tag{2.30}$$

Recall that α is close to, but less than, $\frac{1}{2}$. Choosing ρ close to, but less than, $\frac{1}{4}$, we get that $\frac{3}{4} - 2\rho - \alpha =: -\kappa < 0$, and $-\frac{3}{4} - \frac{2}{3}\rho - \frac{\alpha}{3} = -1 - \frac{\kappa}{3}$. It follows that

$$\begin{aligned} &\int_{2^{-3j}}^{\infty} \left| I - \frac{i \operatorname{sign} \xi}{6t} |\widehat{f}(\xi)|^2 \widehat{f}(\xi) - \frac{ic}{t} e^{-it\frac{8}{9}\xi^3} \widehat{f}\Big(\frac{\xi}{3}\Big)^3 \mathbf{1}_{|\xi|>t^{-1/3}} \right| ds \\ &\qquad\qquad \lesssim \varepsilon_1^3 \int_{2^{-3j}}^{\infty} 2^{-\kappa j} s^{-1-\frac{\kappa}{3}}\, ds \lesssim \varepsilon_1^3. \end{aligned}$$

Contribution of II. We essentially follow the same approach as for I, and keep in particular the same values for ρ and α. A change of variables gives

$$II = \sum_{\substack{2^k \sim 2^\ell \gg 2^j \\ 2^k > t^{-1/3}}} \frac{2^{2k} i}{2\pi} \iint e^{-it2^{3k}\phi((2^{-k}\xi,\eta,\sigma)} \xi \widehat{f_{\lesssim k}}(\xi - 2^k(\eta+\sigma)) \widehat{f_{\lesssim k}}(2^k\eta) \widehat{f_{\lesssim k}}(2^k\sigma) \psi(\eta)\, \psi(\sigma)\, d\eta\, d\sigma.$$

Due to the absence of stationary points, Lemma A.1(ii) implies

$$|II| \lesssim \sum_{\substack{2^k \gg 2^j \\ 2^k > t^{-1/3}}} 2^j 2^{2k} \frac{\|\langle (x,y)\rangle^\rho \widehat{F_k}\|_{L^1}}{(2^{3k}t)^{1+\rho}}$$

where

$$F_k(\eta,\sigma) = \widehat{f_{\lesssim k}}(\xi - 2^k(\eta+\sigma))\widehat{f_{\lesssim k}}(2^k\eta)\widehat{f_{\lesssim k}}(2^k\sigma)$$

and, as above,

$$\||(x,y)|^\rho \widehat{F_k}\|_{L^1} \lesssim 2^{\rho k}\|f_{\lesssim k}\|_{L^1}^2 \||x|^\rho f_{\gtrsim k}\|_{L^1}.$$

As before, using (2.26), this leads to

$$\begin{aligned} |II| &\lesssim \sum_{\substack{2^k \gg 2^j \\ 2^k > t^{-1/3}}} 2^j 2^{2k} \frac{1}{(2^{3k}t)^{1+\rho}} (2^{\rho k}\|f_{\lesssim k}\|_{L^1}^2 \||x|^\rho f_{\gtrsim k}\|_{L^1} + \|f_{\lesssim k}\|_{L^1}^3) \\ &\lesssim \sum_{\substack{2^k \gg 2^j \\ 2^k > t^{-1/3}}} 2^j 2^{k(-\frac{1}{4}-\frac{5}{2}\rho-\alpha)} t^{-\frac{3}{4}-\frac{5\rho}{6}-\frac{\alpha}{3}}\varepsilon_1^3, \end{aligned} \tag{2.31}$$

and thus since $\frac{3}{4}+\frac{5\rho}{6}+\frac{\alpha}{3} > 1$,

$$\int_0^\infty |II|\,ds \lesssim \varepsilon_1^3 \int_0^\infty 2^j s^{-\frac{3}{4}-\frac{5\rho}{6}-\frac{\alpha}{3}} \max(2^j, s^{-1/3})^{-\frac{1}{4}-\frac{5}{2}\rho-\alpha}\,ds \lesssim \varepsilon_1^3.$$

Contribution of III. For the summands in III, $2^k \gg 2^\ell, 2^j$, thus $|\eta|$ is the largest variable and we can write $III = \sum_k A_k$, with

$$A_k(\xi) = \frac{i}{2\pi}\iint e^{-it\phi(\xi,\eta,\sigma)}\xi\widehat{f_{\sim k}}(\xi-\eta-\sigma)\widehat{f_{\sim k}}(\eta)\widehat{f_{\ll k}}(\sigma)\psi\left(\frac{\eta}{2^k}\right)\chi\left(\frac{\sigma}{2^k}\right)\,d\eta\,d\sigma\,.$$

On the support of the integrand, $|\partial_\sigma\phi| \sim 2^{2k}$ and

$$\left|\partial_\eta^{m_1}\partial_\sigma^{m_2}\frac{1}{\partial_\sigma\phi(\xi,\eta,\sigma)}\right| \lesssim 2^{-2k}2^{-(m_1+m_2)k} \tag{2.32}$$

for all integers $m_1, m_2 \in \{0,\dots,10\}$. We then integrate by parts in σ to get

$$\begin{aligned} III &= \sum_{2^k \gg 2^j, t^{-1/3}} III_k^{(1)} + III_k^{(2)} + III_k^{(3)} \\ III_k^{(1)} &:= \frac{i}{2\pi}\iint e^{-it\phi(\xi,\eta,\sigma)}\frac{\xi}{it\partial_\sigma\phi}\,\partial_\sigma\widehat{f_{\sim k}}(\xi-\eta-\sigma)\widehat{f_{\sim k}}(\eta)\widehat{f_{\ll k}}(\sigma)\psi\left(\frac{\eta}{2^k}\right)\chi\left(\frac{\sigma}{2^k}\right)\,d\eta\,d\sigma\,, \\ III_k^{(2)} &:= \frac{i}{2\pi}\iint e^{-it\phi(\xi,\eta,\sigma)}\frac{\xi}{it\partial_\sigma\phi}\widehat{f_{\sim k}}(\xi-\eta-\sigma)\widehat{f_{\sim k}}(\eta)\,\partial_\sigma\widehat{f_{\ll k}}(\sigma)\psi\left(\frac{\eta}{2^k}\right)\chi\left(\frac{\sigma}{2^k}\right)\,d\eta\,d\sigma\,, \\ III_k^{(3)} &:= \frac{i}{2\pi}\iint e^{-it\phi(\xi,\eta,\sigma)}\xi\,\partial_\sigma\left[\frac{1}{it\partial_\sigma\phi}\psi\left(\frac{\eta}{2^k}\right)\chi\left(\frac{\sigma}{2^k}\right)\right]\widehat{f_{\sim k}}(\xi-\eta-\sigma)\widehat{f_{\sim k}}(\eta)\widehat{f_{\ll k}}(\sigma)\,d\eta\,d\sigma\,. \end{aligned}$$

¿From (2.32) and Lemma A.2 it follows that

$$\left|III_{k\ell}^{(1)}\right| + \left|III_{k\ell}^{(2)}\right| \lesssim \frac{2^j}{t2^{2k}}\left[\|\partial\widehat{f_{\sim k}}\|_{L^2}\|\widehat{f_{\ll k}}\|_{L^2} + \|\widehat{f_{\sim k}}\|_{L^2}\|\partial\widehat{f_{\ll k}}\|_{L^2}\right]\|u_{\sim k}\|_{L^\infty} \lesssim \frac{2^j}{t^{7/6}2^{3k/2}}\varepsilon_1^3. \tag{2.33}$$

This gives the desired estimate after summing and integrating in time:

$$\int_0^\infty \sum_{\substack{2^k \gg 2^j \\ 2^k > t^{-1/3}}} |III_k^{(1)}| + |III_k^{(2)}| \, ds \lesssim \varepsilon_1^3 \int_{2^{-3k}}^\infty \frac{2^j}{s^{7/6} \max(2^j, s^{-1/3})^{3/2}} \, ds \lesssim \varepsilon_1^3.$$

The remaining term can be estimated similarly using again (2.32) and Lemma A.2:

$$\left| III_k^{(3)} \right| \lesssim \frac{2^j}{t 2^{3k}} \|\widehat{f_{\sim k}}\|_{L^2} \|\widehat{f_{\ll k}}\|_{L^2} \|u_{\sim k}\|_{L^\infty} \lesssim \frac{2^j}{t^{4/3} 2^{2k}} \varepsilon_1^3, \tag{2.34}$$

which gives the desired bound upon summation and time integration since

$$\int_{2^{-3k}}^\infty \frac{2^j}{s^{4/3} \max(2^j, s^{-1/3})^2} \, ds \lesssim 1.$$

<u>*Contribution of IV.*</u> Using simply $\|\widehat{f}\|_\infty \leq \varepsilon_1$, the term IV can be estimated by

$$|IV| \lesssim \sum_{2^j, 2^k, 2^\ell < t^{-1/3}} 2^{j+k+l} \varepsilon_1^3 \lesssim 2^j t^{-2/3} \mathbf{1}_{t < 2^{-3j}} \varepsilon_1^3,$$

which gives the desired result after time integration. □

We can now give some more specific comment about the class of initial data that we can treat in Theorem 1.1, and that we mentioned in Remark 1.3

Remark 2.6 (On the class of initial data)**.** Let us consider an initial datum u_0 satisfying

$$\|\langle x \rangle u_0\|_{H^\alpha(\mathbb{R})} \leq \varepsilon_0, \tag{2.35}$$

instead of (1.3), for α smaller but close to 1/2, as chosen in the proof of Proposition 2.5 above. Define the space Y by the norm

$$\|u\|_Y = \sup_{t \geq 1} \left(t^{-\delta} \|u(t)\|_{H^1} + t^{-1/6} \big\|xf(t)\big\|_{L^2} + t^{\alpha/3 - 1/6} \big\||\partial_x|^\alpha x f(t)\big\|_{L^2} + \big\|\widehat{f}(t,\xi)\big\|_{L^\infty_\xi} \right). \tag{2.36}$$

Notice that this is similar to the space X defined by (2.1) but does not contain the information that $xf \in H^1$. We claim that the results in Theorem 1.1 can be obtained under the weaker assumption (2.35) by a bootstrap argument in the weaker space Y. Indeed, assume that $\|u\|_Y \leq \varepsilon_1$. Notice that the linear estimates in Lemma 2.1 are obtained under the assumption (2.3), which is consistent with the apriori bound $\|u\|_Y \leq \varepsilon_1$, and does not require boundedness of $\|xf\|_{H^1}$. The same holds true for the bilinear and trilinear estimates in Lemma 2.3. Furthermore, one can see from the proof in Lemma 2.4 that, to obtain the weighted bounds $\|xf\|_{L^2} \leq C(\varepsilon_0 + \varepsilon_1^3 t^{1/6})$ and $\||\partial_x|^\alpha xf\|_{L^2} \leq C(\varepsilon_0 + \varepsilon_1^3 t^{1/6-\alpha/3})$, a bound for xf in H^1 is not used; see also the observation at the beginning of the proof of (2.15). To conclude the bootstrap argument in the Y-norm it then suffices to obtain the key bound (2.19). The reader can check by inspection of the proof of Proposition 2.5 that we only use the estimates (2.25)-(2.26) and the consequent pointwise decay of $t^{-1/3}$ for u. Therefore, one can obtain also the unifrom bound on $\widehat{f}$ under the weaker assumption $\|u\|_Y \leq \varepsilon_1$.

2.5. **Asymptotics.** In this section we derive the asymptotic behavior of solutions of (mKdV) as time goes to infinity. We are going to show the following:

Proposition 2.7 (Asymptotics for small solutions)**.** *Let u be a solution of* (mKdV) *satisfying the global bounds* (2.1)-(2.2)*. Then, for any $t \geq 2$, the following holds:*

- *In the region $x \geq t^{1/3}$ we have the decay estimate*
$$|u(t,x)| \lesssim \frac{\varepsilon_0}{t^{1/3}(x/t^{1/3})^{3/4}}; \tag{2.37}$$
- *In the region $|x| \leq t^{1/3+2\gamma}$, with $\gamma = 1/3(1/6 - C\varepsilon_1^2)$, the solution is approximately self-similar:*
$$\big|u(t,x) - \frac{1}{t^{1/3}}\varphi\big(\frac{x}{t^{1/3}}\big)\big| \lesssim \frac{\varepsilon_0}{t^{1/3+3\gamma/2}}, \tag{2.38}$$
where φ is a bounded solution of the Painlevé II equation
$$\varphi'' - 3\xi\varphi + \varphi^3 = 0, \qquad \text{p.v.} \int_{\mathbb{R}} \varphi(x)\,dx = \int_{\mathbb{R}} u_0(x)\,dx. \tag{2.39}$$
- *In the region $x \leq -t^{1/3+2\gamma}$, the solution has a nonlinearly modified asymptotic behavior: there exists $f_\infty \in L^\infty_\xi$ such that*
$$\Big|u(t,x) - \frac{1}{\sqrt{3t\xi_0}}\Re\exp\Big(-2it\xi_0^3 + \frac{i\pi}{4} + \frac{i}{6}|f_\infty(\xi_0)|^2\log t\Big)f_\infty(\xi_0)\Big| \leq \frac{\varepsilon_0}{t^{1/3}(-x/t^{1/3})^{3/10}}, \tag{2.40}$$
where $\xi_0 := \sqrt{-x/(3t)}$, and $\Re$ denotes the real part.

The proof of Proposition 2.7 is given in the remaining of this section.

Decaying region: Proof of (2.37). The proof of (2.37) follows from similar argument to those used in the proof of Lemma 2.1. As before we denote $\Lambda(\xi) = \xi^3$ and write
$$u(t,x) = e^{-t\partial_x^3}f(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{it\phi(\xi)}\widehat{f}(\xi)\,d\xi, \qquad \phi(\xi) = \phi(\xi;x,t) := \xi(x/t) + \Lambda(\xi).$$
Since for any $x > 0$ we have $\partial_\xi\phi = x/t + 3\xi^2 \geq \max(x/t, \xi^2)$, we integrate by parts in the above formula and bound:
$$\begin{aligned}
&|u(t,x)| \lesssim I + II,\\
&I = \int_{\mathbb{R}} \Big|\frac{1}{t\partial_\xi\phi(\xi)}\partial_\xi\widehat{f}(\xi)\Big|\,d\xi,\\
&II = \int_{\mathbb{R}} \Big|\frac{1}{t[\partial_\xi\phi(\xi)]^2}\partial_\xi^2\phi(\xi)\widehat{f}(\xi)\Big|\,d\xi.
\end{aligned}$$
Using the weighted L^2 bound in (2.1)-(2.2) we can estimate
$$|I| \lesssim \frac{1}{t}\Big[\int_{\mathbb{R}} (x/t + 3\xi^2)^{-2}\,d\xi\Big]^{1/2}\|xf\|_{L^2} \lesssim \frac{1}{t}(x/t)^{-3/4}\varepsilon_0 t^{1/6},$$
which is the desired bound. Similarly, we can use the bound on $\widehat{f}$ to obtain
$$|II| \lesssim \frac{1}{t}\int_{\mathbb{R}} (x/t + 3\xi^2)^{-2}|\xi|\,d\xi\|\widehat{f}\|_{L^\infty} \lesssim \frac{1}{t}(x/t)^{-1}\varepsilon_0,$$

which is a stronger bound than what we need since $x \geq t^{1/3}$.

Self-similar region: Proof of (2.38). We now look at the self-similar region $|x| \leq t^{1/3+2\gamma}$. Define v through the identity

$$u(t,x) = \frac{1}{t^{1/3}} v\big(t, \frac{x}{t^{1/3}}\big), \quad v(t,x) = t^{1/3} u(t, t^{1/3}x). \tag{2.41}$$

Recall the definition of the scaling vectorfield $S = 1 + x\partial_x + 3t\partial_t$. A simple computation shows that

$$\partial_t v(t,x) = \frac{1}{3t^{2/3}} \big(Su\big)(t, t^{1/3}x). \tag{2.42}$$

Moreover, since u is a solution of (mKdV), one can verify that

$$\partial_t v(t,x) = \frac{1}{t} \partial_x \Big(\frac{1}{3} xv - v_{xx} - v^3 \Big)(t,x). \tag{2.43}$$

Our aim is to show that $v(t,x)$ is a Cauchy sequence in time with values in L^∞_x. For this we first show that, for all $|x| \leq t^{2\gamma}$, one has

$$\big|P_{\geq 2^{20}t^\gamma} v(t,x)\big| \leq \varepsilon_0 t^{-3\gamma/2}, \tag{2.44}$$

$$\big|\partial_t P_{\leq 2^{20}t^\gamma} v(t,x)\big| \leq \varepsilon_0 t^{-7/6+3\gamma/2+C\varepsilon_1^2}. \tag{2.45}$$

For (2.44), we recall that $f = e^{t\partial_x^3}u$, and write

$$P_{\geq 2^{20}t^\gamma} v(t,x) = t^{1/3} \int_{\mathbb{R}} e^{i\phi(\xi;x,t)} \chi(\xi t^{1/3-\gamma} 2^{-20}) \widehat{f}(t,\xi) d\xi, \qquad \phi(\xi;x,t) := x\xi t^{1/3} + t\xi^3.$$

Since for any $|x| \leq t^{2\gamma}$, we have $|\partial_\xi \phi| \gtrsim \xi^2 t \gtrsim t^{1/3+2\gamma}$ on the support of the above integral, an integration by parts argument similar to those in the proof of Lemma 2.1, shows the validity of (2.44). Notice that a similar bound also holds for $P_{\sim 2^{20}t^\gamma} v(t,x)$. Because of this, in order to obtain (2.45), it suffices to prove the estimate for $P_{\leq 2^{20}t^\gamma} \partial_t v(t,x)$. Observe that from (2.42) one has $\partial_x^{-1}\partial_t v = 1/(3t)(ISu)(t, t^{1/3}x)$. Therefore, using Bernstein's inequality, and the bound (2.17), we get

$$\big|P_{\leq 2^{20}t^\gamma} \partial_t v(t,x)\big| \lesssim t^{3\gamma/2} \|\partial_x^{-1}\partial_t v\|_{L^2} \lesssim t^{3\gamma/2-1} \|(ISu)\|_{L^2} t^{-1/6} \lesssim \varepsilon_0 t^{-7/6+3\gamma/2+C\varepsilon_1^2},$$

as desired.

We then write

$$v(t,x) = v(t,x)[1-\psi(x/t^{2\gamma})] + P_{\geq 2^{20}t^\gamma} v(t,x)\psi(x/t^{2\gamma}) + P_{\leq 2^{20}t^\gamma} v(t,x)\psi(x/t^{2\gamma}).$$

Combining the decay estimate (2.4) which gives $|v(t,x)[1-\psi(x/t^{2\gamma})]| \lesssim \varepsilon_0 t^{-\gamma/2}$, with (2.44)-(2.45), we see that there exists $\varphi := \lim_{t\to\infty} v(t)$ with $|v(t) - \varphi| \lesssim \varepsilon_0 t^{-\gamma/2}$. It also follows that, uniformly for $|x| \leq t^{2\gamma}$,

$$|v(t,x) - \varphi(x)| \lesssim \varepsilon_0 t^{-3\gamma/2} + \varepsilon_0 \int_t^\infty t^{-7/6+3\gamma/2+C_1\varepsilon_1^2} \lesssim \varepsilon_0 t^{-3\gamma/2}$$

where we recall our choice of $\gamma = 1/3(1/6 - C\varepsilon_1^2)$.

To verify that φ satisfies the first identity in (2.39) it suffices to notice that from (2.42) and (2.43) one has

$$\|xv - 3v_{xx} - 3v^3\|_{L^2} = \|ISu\|_{L^2} t^{-1/6} \lesssim \varepsilon_0 t^{-1/6+c\varepsilon_1^2}.$$

To prove the second identity in (2.39) we let $0 < a < \gamma/2$ and use $|v(t) - \varphi| \lesssim t^{-\gamma/2}$ to write

$$\int \varphi(x)\psi(x/t^a)\,dx = \int v(t,x)\psi(x/t^a)\,dx + O(t^{a-\gamma/2}).$$

Using Plancherel, and the moment conservation for u, we have

$$\int v(t,x)\psi(x/t^a)\,dx = \int (\widehat{u}(t,\xi/t^{1/3}) - \widehat{u}(t,0))\widehat{\psi}(\xi t^a)t^a\,dx + \int u_0(x)\,dx.$$

Using the bounds (2.1)-(2.2) we see that for all $|\xi| \leq 1$

$$|\widehat{u}(t,\xi/t^{1/3}) - \widehat{u}(t,0)| \leq |\widehat{f}(t,\xi/t^{1/3})(e^{i\xi^3} - 1)| + |\widehat{f}(t,\xi/t^{1/3}) - \widehat{f}(t,0)| \lesssim |\xi|^{1/2}.$$

This shows that

$$\Big| \int \varphi(x)\psi(x/t^a)\,dx - \int u_0(x)\,dx \Big| \lesssim t^{a-\gamma/2} + t^{-a/2},$$

which implies (2.39).

Modified scattering: Proof of (2.40). The next Lemma gives a refined version of the linear estimate (2.4).

Lemma 2.8 (Refined linear estimate). *Let $u = e^{-t\partial_x^3}f$, for $f \in L^2$ satisfying*

$$\sup_{t\geq 2}\big(t^{-1/6}\|\langle x\rangle f(t)\|_{L^2} + \|\widehat{f}(t)\|_{L^\infty_\xi}\big) \leq 1. \tag{2.46}$$

Then, for all $t \geq 2$ and $x \leq -t^{1/3}$,

$$\Big|u(t,x) - \frac{1}{\sqrt{3t\xi_0}}\,\Re\big(e^{-2it\xi_0^3 + i\frac{\pi}{4}}\widehat{f}(t,\xi_0)\big)\Big| \lesssim \frac{1}{t^{1/3}|x/t^{1/3}|^{3/10}}, \tag{2.47}$$

where $\xi_0 := \sqrt{-x/(3t)}$, and $\Re$ denotes the real part.

This result can be proven by similar arguments to those in the proof of Lemma 2.1, and those of Lemma 3.2 in [25]. For completeness we give the main ideas of proof below.

Proof of Lemma 2.8. We write

$$u(t,x) = \sqrt{\frac{2}{\pi}}\Re\int_0^\infty e^{it\phi(\xi)}\widehat{f}(t,\xi)\,d\xi, \qquad \phi(\xi) := \frac{x}{t}\xi + \xi^3. \tag{2.48}$$

As before we let $\xi_0 := \sqrt{-x/3t} \approx t^{-1/3}(-x/t^{1/3})^{1/2} \gtrsim t^{-1/3}$ be the only stationary point of the phase ϕ in (2.48).

We first look at the frequency region with $|\xi - \xi_0| \geq \xi_0/2$. There we have $|\partial_\xi\phi(\xi)| \gtrsim \max(\xi^2, \xi_0^2)$. Then, an integration by parts like the one in the proof of Lemma 2.1 (cfr. the terms C_1 and C_2 there) gives us a bound of the form $t^{-1}\xi_0^{-2} + t^{-5/6}\xi_0^{-3/2} \lesssim t^{-1/3}(-x/t^{1/3})^{-3/4}$, which is smaller than the right-hand side of (2.47).

We then analyze the case with $|\xi - \xi_0| \leq \xi_0/2$. If $|\xi - \xi_0| \approx 2^\ell$, for $\ell \geq \ell_0$ with

$$2^{\ell_0} \approx t^{-1/3}(-x/t^{1/3})^{-1/5},$$

we integrate by parts in frequency. Using $|\partial_\xi\phi(\xi)| \gtrsim 2^\ell\xi_0$, we bound these contributions by

$$t^{-1}\Big(\|\partial_\xi\widehat{f}\|_{L^2}\xi_0^{-1}2^{-\ell/2} + \|\widehat{f}\|_{L^\infty}\xi_0^{-1}2^{-\ell}\Big).$$

Using (2.46), and the definitions of ξ_0 and ℓ_0, we see that the contribution from the region $|\xi-\xi_0|\geq 2^{\ell_0}$ is of the order of $t^{-1/3}(-x/t^{1/3})^{-3/10}$, which is an acceptable remainder.

We are then left with estimating the contribution to the integral (2.48) coming from the region $|\xi-\xi_0|\leq 2^{\ell_0}$. We write this contribution as

$$\sqrt{\frac{2}{\pi}}\Re\int_0^\infty e^{it\phi(\xi)}\chi((\xi-\xi_0)2^{-\ell_0})\widehat{f}(t,\xi)\,d\xi = A+B+C$$
$$A=\sqrt{\frac{2}{\pi}}\Re\Big(e^{it\phi(\xi_0)}\widehat{f}(t,\xi_0)\int_0^\infty e^{it3\xi_0\xi^2/2}\chi(\xi/2^{\ell_0})\,d\xi\Big)$$
$$B=\sqrt{\frac{2}{\pi}}\Re\int_0^\infty\Big(e^{it\phi(\xi)}-e^{it\phi(\xi_0)+it\phi''(\xi_0)(\xi-\xi_0)^2/2}\Big)\chi((\xi-\xi_0)2^{-\ell_0})\widehat{f}(t,\xi)\,d\xi$$
$$C=\sqrt{\frac{2}{\pi}}\Re e^{it\phi(\xi_0)}\int_0^\infty e^{it\phi''(\xi_0)(\xi-\xi_0)^2/2}\chi((\xi-\xi_0)2^{-\ell_0})\big(\widehat{f}(t,\xi)-\widehat{f}(t,\xi_0)\big)\,d\xi.$$

Using the hypotheses we immediately see that

$$|B|\lesssim t2^{4\ell_0}\lesssim t^{-1/3}(-x/t^{1/3})^{-4/5},$$
$$|C|\lesssim t^{1/6}2^{3\ell_0/2}\lesssim t^{-1/3}(-x/t^{1/3})^{-3/10},$$

so that these terms are acceptable remainders.

Using the formula

$$\int_{\mathbb{R}} e^{-ax^2}\,dx=\sqrt{\frac{\pi}{a}},\qquad a\in\mathbb{C},\quad \Re a>0,$$

we see that

$$\int_0^\infty e^{it3\xi_0\xi^2/2}e^{-\xi^2/2^{\ell_0}}\,d\xi=\frac{1}{2}\sqrt{\frac{2\pi}{-i3t\xi_0}}+O\big(2^{2\ell_0}+2^{-\ell_0}(t\xi_0)^{-3/2}\big).$$

Finally, it follows that

$$A=\Re\sqrt{\frac{i}{3t\xi_0}}e^{it\phi(\xi_0)}\widehat{f}(t,\xi_0)+O\big(t^{-2/3}|x/t^{1/3}|^{-3/10}\big),$$

and this completes the proof of the Lemma. □

Notice that in the region $x\leq -t^{1/3+2\gamma}$ one has $\xi_0=\sqrt{x/(-3t)}\gtrsim t^{-1/3+\gamma}\gg t^{-1/3}$. Our next goal is then to identify an asymptotic profile for $\widehat{f}(\xi)$, where $f=e^{t\partial_x^3}u$ and u solves (mKdV), whenever $|\xi|\gg t^{-1/3+\gamma}$. This will then determine the leading order asymptotic term for u in this region via (2.47).

Lemma 2.9. *Let $f=e^{t\partial_x^3}u$ with u satisfying the bounds* (2.1)-(2.2)*, and let us define the modified profile as in* (2.22)*:*

$$\widehat{w}(t,\xi):=e^{-iB(t,\xi)}\widehat{f}(t,\xi),\quad B(t,\xi):=\frac{1}{6}\operatorname{sign}\xi\int_1^t\big|\widehat{f}(s,\xi)\big|^2\frac{ds}{s}. \tag{2.49}$$

Then there exists $w_\infty\in L^\infty$ such that, for all $t\geq 2$, and $|\xi|\geq t^{-1/3+\gamma}$

$$|\widehat{w}(t,\xi)-w_\infty(\xi)|\lesssim\varepsilon_1^3(|\xi|t^{1/3})^{-\kappa}, \tag{2.50}$$

for any $\kappa \in (0,1/4)$. Moreover, there exists $f_\infty \in L^\infty$ such that, for $|\xi| \geq t^{-1/3+\gamma}$, we have

$$\Big|\widehat{f}(t,\xi) - \exp\Big(\frac{i}{6}\operatorname{sign}\xi|f_\infty(\xi)|^2\log t\Big)f_\infty(\xi)\Big| \lesssim \varepsilon_0(|\xi|t^{1/3})^{-\kappa}. \tag{2.51}$$

Proof. To prove (2.50) it suffices to show that for all times $t_2 \geq t_1 \geq 2$, one has

$$|\widehat{w}(t_1,\xi) - \widehat{w}(t_2,\xi)| \leq \varepsilon_1^3\big(2^j t_1^{1/3}\big)^{-\kappa}. \tag{2.52}$$

for every $|\xi| \approx 2^j$, with $j \in \mathbb{Z}$ and $2^j \geq t_1^{-1/3+\gamma}$. The starting point to prove (2.52) is the formula (2.27) which, for $|\xi| \geq t^{-1/3+\gamma} \gg t^{-1/3}$, reads

$$\partial_t\widehat{f}(t,\xi) = I + II + III,$$

where all the terms on the right-hand side are defined in (2.27). ¿From (2.30) and the definition of the modified profile $\widehat{w}$ in (2.49), we see that, for $t_1 \leq t \leq t_2$,

$$\Big|\partial_t\widehat{w}(t,\xi) - e^{-iB(t,\xi)}\frac{ic}{t}e^{it\frac{8}{9}\xi^3}\widehat{f}(\xi/3)^3\Big| \lesssim 2^{-j\kappa}t^{-1-\kappa/3}\varepsilon_1^3 + |II(t,\xi)| + |III(t,\xi)|, \tag{2.53}$$

where we recall that we have previously defined $\kappa = -\frac{3}{4} + 2\rho + \alpha$, and we can choose $0 < \alpha < \frac{1}{2}$ and $0 < \rho < \frac{1}{4}$ so that $\kappa = 1/4 - \beta$, for any small $\beta > 0$. To prove (2.50) it will then suffice to show

$$\Big|\int_{t_1}^{t_2} e^{-iB(t,\xi)}\frac{ic}{t}e^{-it\frac{8}{9}\xi^3}\widehat{f}(\xi/3)^3\,dt\Big| \lesssim \varepsilon_1^3\big(2^j t_1^{1/3}\big)^{-\kappa}, \tag{2.54}$$

for all $|\xi| \geq t_1^{-1/3+\gamma}$, and

$$|II(t,\xi)| + |III(t,\xi)| \lesssim \varepsilon_1^3 t^{-1-\kappa/3}2^{-\kappa j}, \tag{2.55}$$

for $t_1 \leq t \leq t_2$, and $|\xi| \geq t^{-1/3+\gamma}$. Here we have used the fact that the first term on the right-hand side of (2.53) matches the right-hand side of (2.55), which, upon integration between t_1 and t_2, gives the desired bound.

To prove (2.54) we use an integration by parts argument similar to the one that gave us (2.23). Proceeding as in (2.24), we see that

$$\begin{aligned}
&\Big|\int_{t_1}^{t_2} e^{it\frac{8}{9}\xi^3}e^{-iB(t,\xi)}\widehat{f}(t,\xi/3)^3\frac{dt}{t}\Big| \lesssim J' + K' + L' + M' \\
&J' = \frac{1}{|\xi|^3}|\widehat{f}(t,\xi/3)|^3\frac{1}{t}\,\Big|_{t=t_1}^{t_2}, \\
&K' = \int_{t_1}^{t_2}\frac{1}{|\xi|^3}|\partial_t\widehat{f}(t,\xi/3)||\widehat{f}(t,\xi/3)|^2\frac{dt}{t}, \\
&L' = \int_{t_1}^{t_2}\frac{1}{|\xi|^3}|\partial_t B(t,\xi)||\widehat{f}(t,\xi/3)|^3\frac{dt}{t}, \\
&M' = \int_{t_1}^{t_2}\frac{1}{|\xi|^3}|\widehat{f}(t,\xi/3)|^3\frac{dt}{t^2}.
\end{aligned}$$

Using $\|\widehat{f}(t)\|_{L^\infty} \leq \varepsilon_1$ we immediately see that $J' \lesssim \varepsilon_1^3 2^{-3j}t_1^{-1}$, which is more than sufficient, since $2^j t_1^{1/3} \gg 1$. Using (2.20) and (2.21) we see that

$$K' \lesssim \int_{t_1}^{t_2}\frac{1}{|\xi|^3}\Big[\frac{\varepsilon_1^3}{t} + R(t,\xi)\Big]\varepsilon_1^2\frac{dt}{t} \lesssim \varepsilon_1^2 2^{-3j}t_1^{-1}\Big[\varepsilon_1^3 + \int_{t_1}^{t_2}R(t,\xi)\,dt\Big] \lesssim \varepsilon_1^5 2^{-3j}t_1^{-1}.$$

L' and M' can be bounded similarly, using also $|\partial_t B(t,\xi)| \leq \varepsilon_1^2 t^{-1}$.

We now prove (2.55). To bound II we look back at the estimate (2.31), recall that $\kappa = -\frac{3}{4} + 2\rho + \alpha$, and see that

$$|II(t,\xi)| \lesssim \varepsilon_1^3 2^j \sum_{2^k \gg 2^j} 2^{(-1-\kappa)k} t^{-1-\kappa/3} \lesssim \varepsilon_1^3 2^{-\kappa j} t^{-1-\kappa/3}.$$

To estimate III we recall (2.33) and (2.34), and, in the case $2^j \geq t^{-1/3+\gamma}$, deduce the following:

$$\begin{aligned}|III(t,\xi)| &\lesssim \varepsilon_1^3 2^j \sum_{2^k \gg 2^\ell \gtrsim 2^j} \big(t^{-7/6} 2^{-3k/2} + t^{-4/3} 2^{-2k}\big) \\ &\lesssim \varepsilon_1^3 (2^{-j/2} t^{-7/6} + 2^{-j} t^{-4/3}) \lesssim \varepsilon_1^3 t^{-1} (t^{1/3} 2^j)^{-1/2}.\end{aligned}$$

This completes the proof of (2.55) and gives us (2.52). We also deduce that $\widehat{w}(t)$ is a Cauchy sequence and obtain the existence of a limit profile w_∞ as in (2.50).

To prove (2.51) we begin by observing that (2.50) implies that for $t \geq 2$

$$\big| |\widehat{f}(t,\xi)| - |w_\infty(\xi)| \big| \lesssim \varepsilon_1^3 (|\xi| t^{1/3})^{-\kappa}. \tag{2.56}$$

Next, for B as in (2.49), we define

$$A(t,\xi) := B(t,\xi) - \frac{1}{6} \operatorname{sign} \xi \, \big|\widehat{f}(t,\xi)\big|^2 \log t. \tag{2.57}$$

Omitting the variable ξ, we calculate for $2 \leq t_1 \leq t_2$

$$A(t_2) - A(t_1) = \frac{1}{6} \operatorname{sign} \xi \int_{t_1}^{t_2} \big(\big|\widehat{f}(s)\big|^2 - \big|\widehat{f}(t_2)\big|^2\big) \frac{ds}{s} + \frac{1}{6} \operatorname{sign} \xi \big(\big|\widehat{f}(t_1)\big|^2 - \big|\widehat{f}(t_2)\big|^2\big) \log t_1.$$

¿From this and (2.56) we deduce that $A(t,\xi)$ is a Cauchy sequence in time, and there exists $A_\infty \in L^\infty_\xi$ such that

$$|A(t,\xi) - A_\infty(\xi)| \lesssim \varepsilon_1^3 (|\xi| t^{1/3})^{-\kappa} \log t.$$

Thanks to (2.56) and (2.57) we see that

$$\Big| B(t,\xi) - \big(A_\infty(\xi) + \frac{1}{6} \operatorname{sign} \xi \, |w_\infty(\xi)|^2 \log t\big) \Big| \lesssim \varepsilon_1^3 (|\xi| t^{1/3})^{-\kappa} \log t,$$

and, in view of (2.49) and (2.50), we obtain

$$\Big| \widehat{f}(t,\xi) - w_\infty(\xi) \exp\big(i A_\infty(\xi) + \frac{i}{6} \operatorname{sign} \xi \, |w_\infty(\xi)|^2 \log t\big) \Big| \lesssim \varepsilon_1^3 (|\xi| t^{1/3})^{-\kappa} \log t.$$

The desired conclusion (2.51) follows by defining $f_\infty(\xi) := w_\infty(\xi) \exp(i A_\infty(\xi))$. $\square$

Finally, we observe that in the space-time region $x/t^{1/3} \leq -t^{2\gamma}$ we have $\xi_0 = \sqrt{-x/3t} \approx (-x/t^{1/3})^{1/2} t^{-1/3} \geq t^{-1/3+\gamma}$, and we can then combine the refined linear estimate (2.47) in Lemma 2.8, and the modified asymptotic estimate (2.51) in Lemma 2.9 to obtain:

$$\begin{aligned}&\Big| u(t,x) - \frac{1}{\sqrt{3t\xi_0}} \Re\Big\{ \exp\Big(-2it\xi_0^3 + i\frac{\pi}{4} + \frac{i}{6} |f_\infty(\xi_0)|^2 \log t \Big) f_\infty(\xi_0) \Big\} \Big| \\ &\qquad \lesssim \varepsilon_0 (t\xi_0)^{-1/2} (t^{1/3}\xi_0)^{-\kappa} \log t + \varepsilon_0 t^{-1/3} |x/t^{1/3}|^{-3/10}\end{aligned}$$

for $f_\infty \in L^\infty$, and whenever $x/t^{1/3} \leq -t^{2\gamma}$. Since κ can be chosen arbitrarily close to $1/4$, this gives (2.40) and concludes the proof of Proposition 2.7. $\square$

3. Stability of solitons

In this section, shall study the the asymptotic stability of the solitons

$$Q_c(x-ct) = \sqrt{c}Q(\sqrt{c}(x-ct)), \qquad Q(s) := \sqrt{2}/\cosh(s), \quad c>0,$$

for the focusing mKdV equation

$$\partial_t u + \partial_x^3 u + \partial_x u^3 = 0. \tag{3.1}$$

The aim is to prove Theorem 1.5. This will be obtained by combining the modified scattering result of the previous section and an asymptotic stability result in a weighted space for the soliton (Theorem 3.1).

For a smooth non-negative weight w, we shall use the following notation for the weighted norms:

$$\|u\|_{L^2_w} = \|w\, u\|_{L^2}, \quad \|u\|^2_{H^s_w} = \sum_{k\le s} \|w\, \partial_x^k u\|^2_{L^2}.$$

In the following, we shall use as weights $w(x) = (1+\tanh(\delta x))^{1/2}$, with δ sufficiently small, and w'. We shall first prove:

Theorem 3.1. *For every $\epsilon_1 > 0$ there exists ϵ_0 such that the following holds true: if v_0 satisfies*

$$\|v_0\|_{H^1} + \|\langle x_+\rangle^m v_0\|_{H^1} \le \epsilon_0 \tag{3.2}$$

for some fixed $m > 1/2$, then there exists a shift $h(t)$ and a modulation speed $c(t)$ such that the solution of (3.1) with $u(t=0) = Q_{c_0} + v_0$ satisfies

$$u(t,x) = Q_{c(t)}(y) + v(t,y), \quad y = y(x,t) = x - \int_0^t c(s)\,ds + h(t), \tag{3.3}$$

with

$$\|\langle y_+\rangle^m v(t)\|_{H^1} + \langle t\rangle^m \|v(t)\|_{H^1_w} + \langle t\rangle^{2m}(|c'(t)| + |h'(t)|) \lesssim \epsilon_1, \quad \forall t\ge 0. \tag{3.4}$$

Moreover, one has the bound

$$\int_0^\infty \|v\|^2_{H^2_{w'}}\,dt \lesssim \epsilon_1^2. \tag{3.5}$$

Note that this Theorem gives in particular that perturbations of a solitary wave decay to its right. This kind of result was already obtained in [48, 44, 36]. Nevertheless, we establish here a form of the result which is appropriate for the proof of Theorem 1.5. In particular, we prove rates of decay that will be useful in order to describe the radiation behind the solitary wave, following the approach of the previous section in a second step.

3.1. **Proof of Theorem 3.1.**

We shall split the proof in several steps.

Step 1: Linear estimates in exponentially weighted spaces. In this first step, we shall recall the properties of the equation (3.1) linearized about the solitary wave Q_c. By changing variables from x to $y = x - ct$, we obtain the linearized equation

$$\partial_t v - c\partial_y v + \partial_y^3 v + 3\partial_y\left(Q_c^2 v\right) = 0. \tag{3.6}$$

Let us denote by $S_c(t)$ the linear group associated to this linear equation, so that the solution of (3.6) with initial value v_0 can be written as $v(t) = S_c(t)v_0$. We shall recall the decay results for S_c obtained by Pego-Weinstein [48] by using the weighted norms

$$\|f\|_{L^2_a} := \|e^{ay} f\|_{L^2}, \quad \|f\|^2_{H^1_a} := \|e^{ay} f\|^2_{L^2} + \|e^{ay}\partial_y f\|^2_{L^2}$$

where a is chosen so that

$$0 < a < \sqrt{c/3}. \tag{3.7}$$

Let us define

$$\mathcal{L}_c := \partial_y\big(- cv + \partial_y^2 v + 3Q_c^2 v\big). \tag{3.8}$$

and $\xi_c^1(y) = \partial_y Q_c$, $\xi_c^2(y) = \partial_c Q_c$, that describe the generalized kernel of $\mathcal{L}_c$:

$$\mathcal{L}_c \xi_c^1 = 0, \quad \mathcal{L}_c \xi_c^2 = \xi_c^1.$$

To define a projection on this generalized kernel, we use the generalized kernel of the adjoint (for the L^2 scalar product) $\mathcal{L}_c^*$. Let us set

$$\zeta_c^1(y) = -\alpha_1 \left(\int_{-\infty}^y \partial_c Q_c \right) + \alpha_2 Q_c(y), \quad \zeta_c^2(y) = \alpha_1 Q_c, \tag{3.9}$$

where the normalization factors α_1 and α_2 are chosen so that[5]

$$\int \xi_c^i \zeta_c^j = \delta_{ij}, \quad 1 \leq i,\, j \leq 2.$$

Note that

$$\mathcal{L}_c^* \zeta_1 = \zeta_2, \qquad \mathcal{L}_c^* \zeta_2 = 0.$$

Define the projections

$$\mathcal{P}_c v = (v, \zeta_c^1)_{L^2} \xi_c^1 + (v, \zeta_c^2)_{L^2} \xi_c^2, \qquad \mathcal{Q}_c = I - \mathcal{P}_c \tag{3.10}$$

Note that these projections are well defined on L_a^2 and commute with $\mathcal{L}_c$ as well as $S_c(t)$ for all t. ¿From the linear stability of the solitary wave, one has:

Theorem 3.2 (Pego-Weinstein [48], Theorem 4.2)**.** *We have the following decay and smoothing estimates:*

$$\begin{aligned}
&\|S_c(t)\mathcal{Q}_c v\|_{L^2_a} \lesssim e^{-bt}\|v\|_{L^2_a},\\
&\|S_c(t)\mathcal{Q}_c v\|_{H^1_a} + \|S_c(t)\mathcal{Q}_c \partial_y v\|_{L^2_a} \lesssim e^{-bt}\max\big(1, t^{-1/2}\big)\|v\|_{L^2_a}.
\end{aligned}$$

for some $b > 0$.

By induction, we can deduce from the above estimates and the Duhamel formula that

$$\begin{aligned}
&\|S_c(t)\mathcal{Q}_c v\|_{H^k_a} \lesssim e^{-bt}\|v\|_{H^k_a},\\
&\|S_c(t)\mathcal{Q}_c v\|_{H^{k+1}_a} + \|S_c(t)\mathcal{Q}_c \partial_y v\|_{H^k_a} \lesssim e^{-bt}\max\big(1, t^{-1/2}\big)\|v\|_{H^k_a}
\end{aligned}$$

for every $k \geq 0$.

[5]Note that these integrals are well defined thanks to the fast decay of the ξ_c^i.

Step 2: Decomposition of the perturbation. The perturbation of the solitary wave $v(t,y)$ defined in (3.3) evolves according to

$$\partial_t v - \tilde{c}\partial_y v + 3\partial_y(Q_{c(t)}^2 v) + \partial_y^3 v = \partial_y \mathcal{F}(v) + e_Q, \quad v_{/t=0} = v_0(x) \tag{3.11}$$

where

$$\begin{aligned} &\tilde{c}(t) = c(t) - \dot{h}(t) \\ &e_Q(t,y) = \dot{c}\partial_c Q_{c(t)}(y) + \dot{h}\partial_y Q_{c(t)}(y) = \dot{c}\xi^2_{c(t)}(y) + \dot{h}\xi^1_{c(t)}(y) \\ &\mathcal{F}(v) = -\left((Q_c + v)^3 - Q_c^3 - 3Q_c^2 v\right). \end{aligned} \tag{3.12}$$

The modulation parameters $(h(t), c(t))$ will be chosen to ensure the constraint

$$(v, \zeta_c^1)_{L^2} = (v, \zeta_c^2)_{L^2} = 0. \tag{3.13}$$

Note that these constraints are always well defined (even the first one) when v is such that $\langle y_+\rangle^m v \in L^2$ for $m > 1/2$.

We shall use Mizumachi's [45] approach that consists in splitting $v(t,y)$ defined in (3.3) into

$$v(t,y) = v_1(t,y) + v_2(t,y) \tag{3.14}$$

where v_1 will be estimated in H^1_w, and v_2 in H^1_a. We choose $v_1(t,y)$ as the solution of the free nonlinear equation

$$\partial_t v_1 - \tilde{c}\partial_y v_1 + \partial_y^3 v_1 + \partial_y v_1^3 = 0, \quad v_1(0) = v_0, \tag{3.15}$$

and v_2 as the solution of

$$\partial_t v_2 - \tilde{c}\partial_y v_2 + 3\partial_y(Q_{c(t)}^2 v_2) + \partial_y^3 v_2 = \partial_y \mathcal{N}(v) + e_Q, \quad v_2(0) = 0, \tag{3.16}$$

with

$$\mathcal{N}(v) = -(Q_{c(t)} + v_1 + v_2)^3 + Q_{c(t)}^3 + v_1^3 + 3Q_{c(t)}^2 v_2. \tag{3.17}$$

We shall solve this equation for v_2 in the weighted space H^1_a by using estimates for the linear semigroup S_c. Note that the choice of the equation for v_2 is made in order to ensure that the source term $\mathcal{N}(v)$ that involves v_1 lies in the weighted space L^2_a.

Let us define the norm:

$$\begin{aligned} N(t) := \langle t\rangle^m (\|v_1(t)\|_{H^1_w} + \|v_2(t)\|_{H^1_a}) + \|\langle y_+\rangle^m v_1(t)\|_{H^1} + \|v_2\|_{H^1} \\ + |c(t) - c_0| + |h(t) - h_0|, \end{aligned} \tag{3.18}$$

with the parameters δ in the definition of w, and a in the exponential weights, chosen so that the following relations hold:

$$Q^{1/3}_{c\pm 5\epsilon_0} e^{\kappa|x|} \lesssim w + w' \lesssim e^{ax}, \quad \forall x \in \mathbb{R}, \tag{3.19}$$

for a small constant $\kappa > 0$.

The bootstrap argument. We assume that

$$N(t) \lesssim \tilde{\varepsilon}_1, \quad \forall\, t \in [0,T] \tag{3.20}$$

and we will prove that, for all $t \in [0,T]$

$$N(t) \lesssim \varepsilon_0^{\frac{1}{2}}.$$

It will be convenient to use also the quantity

$$M(t) = \sup_{s\in[0,t]} \Big(\langle s\rangle^m (\|v_1(s)\|_{H^1_w} + \|v_2(s)\|_{H^1_a} \Big).$$

Note that by the bootstrap assumption, we also have that $M(t) \leq \tilde{\epsilon}_1$ on $[0,T]$.

Step 3: H^1 estimate. In this step we shall prove that

Proposition 3.3. *For $t \in [0,T]$ we have the estimate*

$$\|v_1(t)\|_{H^1} \lesssim \epsilon_0, \quad \|v_2(t)\|^2_{H^1} \lesssim \epsilon_0 + (1+\tilde{\epsilon}_1)(\|v_1(t)\|_{H^1_w} + \|v_2(t)\|_{H^1_a} + |c(t)-c_0|).$$

Note that the last estimate does not seem appropriate for the bootstrap. Nevertheless, we shall prove below that the estimates for $\|v_1(t)\|_{H^1_w}$, $\|v_2(t)\|_{H^1_a}$ and $|c(t)-c_0|$ are much better behaved in the sense that these quantities can be estimated in terms of ϵ_0 if $\tilde{\epsilon}_1$ is sufficiently small. We could use the orbital stability of the solitary wave to get better estimates at this stage.

Proof of Proposition 3.3. For the KdV type equation (3.15) we have the conservation of the quantities

$$\int_{\mathbb{R}} |v_1|^2\,dx, \quad \int_{\mathbb{R}} \left(\frac{1}{2}|\partial_x v_1|^2 - \frac{v_1^4}{4}\right)\,dx.$$

Using these and Sobolev inequalities we easily get

$$\|v_1(t)\|_{H^1} \lesssim \epsilon_0, \quad \forall\ t \in [0,T]. \tag{3.21}$$

To estimate v_2 we use the conserved quantities for (3.1). The mass conservation

$$\int_{\mathbb{R}} |u(t,x)|^2\,dx = \int_{\mathbb{R}} |Q_{c_0}(x) + v_0(x)|^2\,dx$$

implies, after expanding u as in (3.3) and (3.14), that

$$\int_{\mathbb{R}} |v_1 + v_2 + Q_{c(t)}|^2\,dx = \int Q^2_{c_0}\,dx + O(\varepsilon_0),$$

and thus

$$\int_{\mathbb{R}} |v_2|^2\,dy = \int_{\mathbb{R}} (Q^2_{c_0} - Q^2_{c(t)})\,dy - \int_{\mathbb{R}} v_1^2\,dy - 2\int_{\mathbb{R}} Q_{c(t)} v_1\,dy - 2\int_{\mathbb{R}} Q_{c(t)} v_2\,dy - 2\int_{\mathbb{R}} v_1 v_2\,dy + O(\epsilon_0).$$

This yields

$$\|v_2(t)\|^2_{L^2} \lesssim \epsilon_0 + |c(t)-c_0| + \|v_1(t)\|_{H^1_w} + \|v_2\|_{H^1_a},$$

if $\tilde{\epsilon}_1$ is chosen small enough.

To estimate $\|\partial_x v_2(t)\|^2_{L^2}$ one can proceed in a similar way, by using the conservation of the Hamiltonian for (3.1). $\square$

Step 4: Estimates of the modulation parameters. The existence of the modulation parameters is based on the following:

Lemma 3.4. *Let $c_0 > 0$, $h_0 \geq 0$. There exists $\delta > 0$ such that for every U satisfying*

$$U(t) - Q_{c_0}(\cdot - c_0 t + h_0) \in \mathcal{C}^1([0, T_0], H^1_{\langle x \rangle_+^m})$$

for some $m > 1/2$, with

$$\sup_{[0,T_0]} \| \langle (\cdot + h_0)_+^m \rangle \left(U(t) - Q_{c_0}(\cdot - c_0 t + h_0) \right) \|_{H^1} < \delta,$$

there exists $(h(t), c(t)) \in \mathcal{C}^1([0, T_0])$ such that

$$\int_{\mathbb{R}} \big(U(t,x) - Q_{c(t)}(y) \big) \, \zeta^k_{c(t)}(y) \, dx = 0, \quad k = 1,\, 2$$

where $y = x - \int_0^t c(s)\, ds + h(t)$.

The proof of this lemma is now very classical and relies on the use of the implicit function Theorem. We refer to [48, Proposition 5.1] or [44, Proposition 3.1] for the proof.

By using Lemma 3.4 for $U = u$, we get the existence of $c(t)$ and $h(t)$ such that the decompositions (3.3), and (3.14) with (3.13) hold.

Proposition 3.5. *On $[0, T]$ we have the following estimates for the modulation parameters:*

$$|\dot{h}(t)| + |\dot{c}(t)| \lesssim \langle t \rangle^{-2m} M(t)^2.$$

Note that by integrating in time the above estimate, we get that

$$|c(t) - c_0| + |h(t) - h_0| \leq M(t)^2, \quad \forall\, t \in [0, T]. \tag{3.22}$$

Proof of Proposition 3.5. By using the equation (3.11), we get by taking the time derivatives of the constraints (3.13) that the vector $\Gamma(t) = (h(t), c(t))^t$ verifies the ODE

$$A(t)\dot{\Gamma}(t) = - \begin{pmatrix} (\mathcal{F}(v), \partial_y \zeta_c^1) \\ (\mathcal{F}(v), \partial_y \zeta_c^2) \end{pmatrix}, \tag{3.23}$$

(using once again $(v, \zeta_c^2) = 0$) with

$$A(t) = Id - \begin{pmatrix} (v, \partial_y \zeta_c^1) & (v, \partial_c \zeta_c^1) \\ (v, \partial_y \zeta_c^2) & (v, \partial_c \zeta_c^2) \end{pmatrix} := Id - B(t).$$

Since $|B(t)| \lesssim \|v_2(t)\|_{L^2_w}$, we have that $A(t)$ is invertible for $\tilde{\epsilon}_1$ sufficiently small, with the norm of its inverse smaller than 2. Moreover, we can estimate the right hand side of (3.23) by using the localization provided by $\partial_y \zeta_c^i$. In particular, we obtain that

$$|(\mathcal{F}(v), \partial_y \zeta_c^i)| \lesssim (1 + \|v\|_{H^1}) \big(\|v_1\|^2_{L^2_w} + \|v_2\|^2_{L^2_a} \big) \lesssim \langle t \rangle^{-2m} M(t)^2,$$

for $t \in [0, T]$, which gives the desired estimate. $\square$

Step 5: Estimates of v_1. We shall now use localized virial type estimates in order to estimate the weighted norms of v_1.

Proposition 3.6. *For every* $t \in [0,T]$*, we have the estimates:*

$$\|v_1(t)\|_{H^1_w} \lesssim \epsilon_0 \langle t\rangle^{-m}, \qquad \|\langle y_+\rangle^m v_1(t)\|_{H^1} \lesssim \epsilon_0. \tag{3.24}$$

Moreover, we also have

$$\int_0^t \|v_1(s)\|^2_{H^2_{w'}}\, ds \lesssim \epsilon_0^2. \tag{3.25}$$

Proof. We first notice that on $[0,T]$, we have by assumption that $|c(t)-c_0| \leq \tilde\epsilon_1$ and also by using Proposition 3.5 that $|\dot h| \lesssim \tilde\epsilon_1$, consequently, by assuming that $\tilde\epsilon_1$ is sufficiently small, we can always ensure that

$$c_0/2 \leq \tilde c(t) \leq 2c_0, \quad \forall t \in [0,T]. \tag{3.26}$$

We shall use weights

$$\phi_k(t,y) := \chi_{k,\delta}(y+\sigma t + x_0) \tag{3.27}$$

with σ, $0 \leq \sigma < c_0/2$, $x_0 \in \mathbb{R}$, δ sufficiently small, and $\chi_{k,\delta}$ is given by

$$\chi_{k,\delta}(y) = \big(A_k + (\delta y)^2\big)^k \big(1+\tanh(\delta y)\big)$$

We choose A_k sufficiently big, so that the following inequalities hold:

$$\chi_{k,\delta} \sim w^2 \langle y\rangle^{2k}, \quad \chi'_{k,\delta} \geq 0, \quad \chi''_{k,\delta} \lesssim \delta \chi'_{k,\delta}, \quad \chi'''_{k,\delta} \lesssim \delta^2 \chi'_{k,\delta}.$$

¿From (3.15), we first obtain

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} \phi_k |v_1|^2 + \frac{1}{2}(\tilde c - \sigma)\int \phi_k' |v_1|^2 + \frac{3}{2}\int \phi_k' |\partial_y v_1|^2 = \frac{1}{2}\int \phi_k''' |v_1|^2 + \frac{1}{4}\int \phi_k' |v_1|^4.$$

Next, we observe that $|\phi_k'''| \lesssim \delta^2 \phi_k'$ and that $\|v_1\|_{L^\infty} \lesssim \|v_1\|_{H^1} \lesssim \epsilon_0$ thanks to Proposition 3.3. We thus obtain that

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} \phi_k |v_1|^2 + \Big(\frac{1}{2}(\tilde c - \sigma) - C\epsilon_0^2 - C\delta^2\Big)\int \phi_k' |v_1|^2 + \frac{3}{2}\int \phi_k' |\partial_y v_1|^2 \leq 0. \tag{3.28}$$

By setting

$$e_1 = \frac{1}{2}|\partial_y v_1|^2 - \frac{1}{4}|v_1|^4, \quad d_1 = \partial_y^2 v_1 + v_1^3,$$

we also get from (3.15) that

$$\partial_t e_1 - \tilde c \partial_y e_1 - d_1 \partial_y d_1 = -\partial_y(\partial_y d_1 \partial_y v_1).$$

Note that this is the infinitesimal conservation law corresponding to the conservation of the Hamiltonian. By integrating this identity against the weight ϕ_k, we obtain after some integration by parts that

$$\frac{d}{dt}\int \phi_k e_1 + (\tilde c - \sigma)\int \phi_k' e_1 + \frac{1}{2}\int \phi_k' |d_1|^2 = \int \phi_k' \partial_y d_1 \partial_y v_1.$$

To control the last integral we can integrate by parts and use Proposition 3.3 and $|\phi_k''| \lesssim \delta\phi_k'$ to get

$$\int \phi_k' \partial_y d_1 \partial_y v_1 + \int \phi_k' |d_1|^2 \lesssim \delta \int \phi_k'(|d_1|^2 + |\partial_y v_1|^2) + \epsilon_0 \int \phi_k'(|d_1|^2 + |v_1|^2).$$

We thus get that

$$\frac{d}{dt}\int \phi_k e_1 + (\tilde{c}-\sigma)\int \phi_k' e_1 + (\frac{3}{2} - C\delta - C\epsilon_0)\int \phi_k' |d_1|^2 \lesssim \delta \int \phi_k' |\partial_y v_1|^2 + \epsilon_0 \int \phi_k' |v_1|^2.$$

By combining the last identity and (3.28), we thus obtain that

$$\begin{aligned}\frac{d}{dt}\int \phi_k (e_1 + \frac{1}{2}|v_1|^2) + (\tilde{c} - \sigma - C\epsilon_0 - C\delta)\int \phi_k'(e_1 + \frac{1}{2}|v_1|^2) \\ + (\frac{3}{2} - C\delta - C\epsilon_0)\int \phi_k'(|d_1|^2 + |\partial_y v_1|^2) \leq 0.\end{aligned} \tag{3.29}$$

Note that for ϵ_0 sufficiently small, $e_1 + \frac{1}{2}|v_1|^2$ and $|d_1|^2 + |\partial_y v_1|^2 + |v_1|^2$ are positive quantities that control pointwise $|\partial_y v_1|^2 + |v_1|^2$ and $|\partial_y^2 v_1|^2 + |\partial_y v_1|^2 + |v_1|^2$ respectively.

By using this identity with $k = 0$, $\sigma = 0$, $x_0 = 0$, we obtain after integration in time that

$$\int_0^t \int_{\mathbb{R}} (w')^2 (|\partial_{yy} v_1|^2 + |\partial_y v_1|^2 + |v_1|^2)\, dydt \lesssim \|v_0\|_{H^1}^2. \tag{3.30}$$

By taking $k = 0$, $\sigma > 0$, small and $x_0 = -\sigma\tau$, we also get by integrating between 0 and τ that for every $\tau > 0$,

$$\|v_1(\tau)\|_{H^1_w}^2 \lesssim \int_{\mathbb{R}} \phi_0(0, y - \sigma\tau)(|v_0|^2 + |\partial_y v_0|^2)\, dy.$$

Since $\phi_0(y - \sigma\tau)/\langle y_+\rangle^{2m} \lesssim 1/\langle \tau\rangle^{2m}$, we also obtain that

$$\|v_1(\tau)\|_{H^1_w} \lesssim \frac{1}{\langle\tau\rangle^m}\|\langle y_+\rangle^m v_0\|_{H^1}, \quad \forall\, \tau \in [0, T]. \tag{3.31}$$

Finally, by using (3.29) with $\sigma = 0$ and $x_0 = 0$ but for $k = m$, we get that

$$\int_{\mathbb{R}} \phi_m \left(|\partial_y v_1|^2 + |v_1|^2\right)(t) \lesssim \|\langle y_+\rangle^m v_0\|_{H^1}^2, \quad \forall\, t \in [0, T].$$

Since ϕ_m behaves like y^{2m} for $y \geq 0$, we get, using also Proposition 3.3, that

$$\|\langle y_+\rangle^m v_1(t)\|_{H^1} \lesssim \epsilon_0.$$

This ends the proof of the proposition. □

Step 6: Estimate of v_2. We now estimate v_2 mainly using the semi-group estimates of Theorem 3.2.

Proposition 3.7. *For all $t \in [0, T]$ we have the estimates*

$$\langle t\rangle^m \|v_2(t)\|_{H^1_a} \lesssim \epsilon_0, \qquad \int_0^t \|v_2(\tau)\|_{H^2_a}^2 \lesssim \epsilon_0^2. \tag{3.32}$$

Proof. We can first write the equation (3.16) for v_2 as

$$\partial_t v_2 + \mathcal{L}_{c_0} v_2 = \partial_y \mathcal{N}(v) + e_Q + \partial_y \mathfrak{e}_Q,$$

where e_Q and $\mathcal{N}$ are defined in (3.12), (3.17) and $\mathfrak{e}_Q$ is given by

$$\mathfrak{e}_Q = -3(Q_{c(t)}^2 - Q_{c_0}^2)v_2 + (\tilde{c} - c_0)v_2. \tag{3.33}$$

By using the semi-group S_{c_0} we get that v_2 is given by the following Duhamel formula

$$v_2(t) = \int_0^t S_{c_0}(t-\tau)\left(\partial_y \mathcal{N}(v) + e_Q + \partial_y \mathfrak{e}_Q\right)(\tau)\, d\tau. \tag{3.34}$$

We shall first estimate $\mathcal{P}_{c_0}v(t)$. By using the definition of $\mathcal{P}_c$ and the fact that v_2 satisfies the constraint (3.13), we get that

$$\|\mathcal{P}_{c_0}v(t)\|_{H^2_a} \lesssim \|v_1\|_{L^2_w} + |c(t)-c_0|\|v_2\|_{L^2_a}.$$

Since

$$\|\mathcal{P}_{c_0}v_2(t)\|_{H^2_a} \leq \|\mathcal{P}_{c_0}v_1(t)\|_{H^2_a} + \|\mathcal{P}_{c_0}v(t)\|_{H^2_a} \lesssim \|v_1\|_{L^2_w} + \|\mathcal{P}_{c_0}v(t)\|_{H^2_a},$$

by using Proposition 3.6 and (3.22), we get

$$\langle t\rangle^m \|\mathcal{P}_{c_0}v_2(t)\|_{H^2_a} \lesssim \epsilon_0 + M(t)^3. \tag{3.35}$$

This also yields (since $m > 1/2$)

$$\left(\int_0^t \|\mathcal{P}_{c_0}v_2(t)\|^2_{H^2_a}\right)^{\frac{1}{2}} \lesssim \epsilon_0 + M(t)^3. \tag{3.36}$$

Next, we apply $\mathcal{Q}_{c_0}$ to (3.34):

$$\mathcal{Q}_{c_0}v_2(t) = \int_0^t S_{c_0}(t-\tau)\mathcal{Q}_{c_0}\left(\partial_y\mathcal{N}(v) + e_Q + \partial_y\mathfrak{e}_Q\right)(\tau)\, d\tau.$$

Thanks to Theorem 3.2, we obtain

$$\|\mathcal{Q}_{c_0}v_2(t)\|_{H^s_a} \lesssim \int_0^t e^{-b(t-\tau)}\max\left(1,(t-\tau)^{-1/2}\right)(\|e_Q\|_{H^2_a} + \|\mathfrak{e}_Q\|_{H^s_a} + \|\mathcal{N}(v)\|_{H^s_a})\, d\tau, \quad s = 1,\, 2. \tag{3.37}$$

To estimate the right hand side above, we recall the definition (3.12) and observe that

$$\|e_Q(t)\|_{H^2_a} \lesssim |\dot{c}| + |\dot{h}|.$$

Therefore, using Proposition 3.5, we obtain that on $[0,T]$

$$\|e_Q(t)\|_{H^2_a} \lesssim \langle t\rangle^{-2m}M(t)^2. \tag{3.38}$$

Next, we observe that

$$\|\mathfrak{e}_Q(t)\|_{H^1_a} \lesssim (|c(t)-c_0| + |\dot{h}(t)|)\|v_2(t)\|_{H^1_a} \lesssim \langle t\rangle^{-m}M(t)^3, \tag{3.39}$$

and in a similar way, we obtain

$$\int_0^t \|\mathfrak{e}_Q(t)\|^2_{H^2_a} \lesssim \tilde{\epsilon}_1^2 \int_0^t \|v_2\|^2_{H^2_a}. \tag{3.40}$$

To estimate $\mathcal{N}(v)$, we recall its definition in (3.17), and write

$$\mathcal{N}(v) = -\left(3Q^2_{c(t)}v_1 + 3Q_{c(t)}v_1^2 + 3Q_{c(t)}v_2^2 + 6Q_{c(t)}v_1v_2 + 3v_1^2v_2 + 3v_1v_2^2 + v_2^3\right). \tag{3.41}$$

Using the localization provided by Q_c we get that, on $[0,T]$,

$$\|\mathcal{N}(v)(t)\|_{H^1_a} \lesssim (1 + \|v_1\|_{H^1})\|v_1\|_{H^1_w} + (\|v_1\|_{H^1} + \|v_2\|_{H^1})(1 + \|v_1\|_{H^1} + \|v_2\|_{H^1})\|v_2\|_{H^1_a}.$$

Therefore, by using Proposition 3.6 and Proposition 3.3, we get that

$$\|\mathcal{N}(v)(t)\|_{H^1_a} \lesssim \epsilon_0\langle t\rangle^{-m} + \tilde{\epsilon}_1\langle t\rangle^{-m}M(t). \tag{3.42}$$

In a similar way, we obtain

$$\int_0^t \|\mathcal{N}(v)\|^2_{H^2_a} \lesssim (1+\tilde{\epsilon}_1)\int_0^t \|v_1\|^2_{H^2_w} + \tilde{\epsilon}_1^2\int_0^t \|v_2\|^2_{H^2_a}. \tag{3.43}$$

Putting together (3.35), (3.37), (3.38), (3.39) and (3.42), we see that for all $t \in [0,T]$

$$\langle t\rangle^m \|v_2(t)\|_{H^1_a} \lesssim \epsilon_0 + \tilde{\varepsilon}_1 M(t) + (\varepsilon_0 + \tilde{\epsilon}_1 M(t)) \int_0^t \langle t\rangle^m e^{-b(t-\tau)} \max\left(1, (t-\tau)^{-1/2}\right) \frac{1}{\langle \tau\rangle^m}\, d\tau.$$

Since

$$\sup_{t\geq 0} \int_0^t \langle t\rangle^m e^{-b(t-\tau)} \max\left(1, (t-\tau)^{-1/2}\right) \frac{1}{\langle \tau\rangle^m}\, d\tau < +\infty,$$

we obtain, by using again Proposition 3.3, that

$$\langle t\rangle^m \|v_2(t)\|_{H^1_a} \lesssim \epsilon_0 + \tilde{\epsilon}_1 \sup_{[0,t]} (\langle s\rangle^m \|v_2(s)\|_{H^1_a}).$$

Taking $\tilde{\epsilon}_1$ sufficiently small, we get

$$\langle t\rangle^m \|v_2(t)\|_{H^1_a} \lesssim \epsilon_0, \quad \forall t \in [0,T]. \tag{3.44}$$

Consequently, the first part of (3.32) is proven.

To get the second part, we use Young's inequality and (3.36), (3.37) for $s=2$, (3.38), (3.39), (3.43) to obtain

$$\left(\int_0^t \|v_2(\tau)\|^2_{H^2_a}\, d\tau\right)^{1/2} \lesssim \epsilon_0 + M(t) + \left(\int_0^t \|v_1\|^2_{H^2_w}\, d\tau\right)^{\frac{1}{2}} + \tilde{\epsilon}_1 \left(\int_0^t \|v_2(\tau)\|^2_{H^2_a}\, d\tau\right)^{1/2}.$$

By using Proposition 3.6 and (3.44), this yields

$$\left(\int_0^t \|v_2(\tau)\|^2_{H^2_a}\, d\tau\right)^{1/2} \lesssim \epsilon_0 + \tilde{\epsilon}_1 \left(\int_0^t \|v_2(\tau)\|^2_{H^2_a}\, d\tau\right)^{1/2},$$

and we conclude again the proof by choosing $\tilde{\epsilon}_1$ sufficiently small. □

Step 7: Conclusion. By combining Propositions 3.6 and 3.7, we have already proven that $M(t) \lesssim \epsilon_0$ on $[0,T]$. From (3.22), we also obtain that $|c(t)-c_0| + |h(t)-h_0| \leq \epsilon_0$ (actually we even have ϵ_0^2). Finally from Proposition 3.3, we get $\|v(t)\|_{H^1} \lesssim \epsilon_0^{\frac{1}{2}}$ on $[0,T]$. Since

$$\langle t\rangle^m \|v(t)\|_{H^1_w} + \|\langle y_+\rangle^m v(t)\|_{H^1} \lesssim \langle t\rangle^m \|v_1\|_{H^1_w} + \langle t\rangle^m \|v_2\|_{H^1_a} + \|\langle y_+\rangle^m v_1\|_{H^1} + \|v_2\|_{H^1_a},$$

we obtain, by using again Proposition 3.6, that

$$N(t) \lesssim \epsilon_0^{\frac{1}{2}}, \quad \forall\ t \in [0,T].$$

By taking $\epsilon_1 \gg \epsilon_0^{1/2}$ and sufficiently small, we see, by a standard bootstrap argument, that the estimate (3.20) holds true for all times.

Moreover, from Proposition 3.5, we have that

$$|\dot{h}| + |\dot{c}| \lesssim \tilde{\epsilon}_1^2 \langle t\rangle^{-2m},$$

and, since $m > 1/2$, we deduce that there exists c_+ and h_+ such that

$$|c(t) - c_+| + |h(t) - h_+| \lesssim \tilde{\epsilon}_1^2 \langle t\rangle^{-(2m-1)}.$$

This gives (3.4). Finally, note that the estimate (3.5) follows from Proposition 3.6 and Proposition 3.7. □

3.2. **Proof of Theorem 1.5.** We now show how to obtain Theorem 1.5 from Theorem 3.1 and the first part of the paper. We start again from the decompositions (3.3) and (3.14), so that $v_1(t,y)$, $v_2(t,y)$, c and h satisfy the estimates in the proof of Theorem 3.1. We thus write

$$u(t,x) = Q_{c(t)}(y) + \tilde{v}(t,x), \quad \tilde{v}(t,x) = \tilde{v}_1(t,x) + \tilde{v}_2(t,x), \quad \tilde{v}_i(t,x) = v_i(t,y) \tag{3.45}$$

where again $y = y(t,x) = x - \int_0^t c + h(t)$. By using the estimates in Theorem 3.1, we already have that

$$\begin{aligned} \|\langle x_+\rangle^m v(t)\|_{H^1} + \langle t\rangle^m (\|v_1(t)\|_{H^1_w} + \|v_2(t)\|_{H^1_a}) + \langle t\rangle^{2m}(|\dot{c}(t)| + |\dot{h}(t)|) \\ + \langle t\rangle^{2m-1}\left(|c(t) - c_+| + |h(t) - h_+|\right) \lesssim \tilde{\epsilon}_1, \quad \forall\, t \geq 0. \end{aligned} \tag{3.46}$$

We now use the approach of the first part of the paper to estimate $\tilde{v} = \tilde{v}_1 + \tilde{v}_2$ behind the solitary wave.

Step 1: Estimates for $\tilde{v}_1$. By definition, since $v_1(t,y)$ solves (3.15), we get that $\tilde{v}_1(t,x)$ solves the mKdV equation

$$\partial_t \tilde{v}_1 + \partial_x^3 \tilde{v}_1 + \partial_x \tilde{v}_1^3 = 0, \quad (\tilde{v}_1)_{/t=0} = v_0(x).$$

Consequently, $\tilde{v}_1$ verifies the estimates of Theorem 1.1. In particular if we denote $f_1 := e^{t\partial_x^3}\tilde{v}_1$ we have

$$\|\tilde{v}_1\|_X := \sup_t \left(\langle t\rangle^{-\delta}\|xf_1\|_{H^1} + \langle t\rangle^{\frac{\alpha}{3}-\frac{1}{6}}\||\partial_x|^\alpha x f_1\|_{L^2} + \|\widehat{f_1}\|_{L^\infty}\right) \lesssim \epsilon_0, \tag{3.47}$$

see the definition of the X-norm in (2.1). In particular, this also gives the linear estimate (2.4) and (2.6), the bilinear estimates (2.11) and the trilinear estimate (2.12) for $\tilde{v}_1$:

$$\begin{aligned} &\langle t\rangle^{1/3+\beta/3-1/(3p)}\||\partial_x|^\beta \tilde{v}_1\|_{L^p} \lesssim \varepsilon_0, \qquad \text{for} \quad 0 \leq \beta < 1/2, \quad p(1/4 - \beta/2) > 1, \\ &\langle t\rangle \|\tilde{v}_1 \partial_x \tilde{v}_1\|_{L^\infty} \lesssim \varepsilon_0^2, \\ &\langle t\rangle^{5/6+\alpha/3}\||\partial_x|^\alpha \tilde{v}_1^3\|_{L^2} \lesssim \epsilon_0^3, \qquad \text{for} \quad 0 \leq \alpha < 1/2. \end{aligned} \tag{3.48}$$

Thanks to (2.17) we also have[6]

$$\|IS\tilde{v}_1\|_{L^2} + \|S\tilde{v}_1\|_{L^2} \lesssim \epsilon_0 t^{C\varepsilon_0^2}. \tag{3.49}$$

For later use, we improve these latter estimates in front of the solitary wave:

Lemma 3.8. *Let us set $H_1(t,y) = (IS\tilde{v}_1)(t,x)$, again with $y = x - \int_0^t c(s)\,ds + h(t)$. Then we have the estimates*

$$\|H_1(t)\|_{L^2_w} \lesssim \frac{\epsilon_0}{\langle t\rangle^{m-1-\frac{1}{2}C\varepsilon_0^2}}, \qquad \int_0^t \|H_1(\tau)\|^2_{H^1_{w'}}\,d\tau \lesssim \epsilon_0^2.$$

Proof. We observe that since v_1 solves the free equation (3.15), and S commutes with the equation as in (2.16), then

$$\partial_t H_1 - \tilde{c}\partial_y H_1 + \partial_y^3 H_1 + 3v_1^2 \partial_y H_1 = 0.$$

[6]Note that the second part of this estimate, was not explicitly written down, but it is a direct consequence of (2.16) and Gronwall's inequality.

Then we can use a virial type computation similar to (3.28), with the same ϕ_k defined in (3.27), to find

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}}\phi_k H_1^2\,dy+\frac{1}{2}(\tilde{c}-\sigma-C\delta^2-C\varepsilon_0^2)\int_{\mathbb{R}}\phi_k' H_1^2\,dy+\frac{3}{2}\int_{\mathbb{R}}\phi_k'|\partial_y H_1|^2\,dy\\ =3\int_{\mathbb{R}}|v_1\partial_y v_1|\phi_k H_1^2\,dy\lesssim\varepsilon_0^2\frac{1}{\langle t\rangle}\int_{\mathbb{R}}\phi_k H_1^2\,dy,\end{aligned}\tag{3.50}$$

where we have used (3.48) to obtain the above inequality. Note that for ϵ_0 sufficiently small, we can ensure that $\tilde{c}-\sigma-C\delta^2-C\epsilon_0^2\geq\frac{c_0}{2}$. Consequently, integrating in time we get

$$\int_{\mathbb{R}}\phi_k(t,y)|H_1(t,y)|^2\,dy\lesssim\langle t\rangle^{C\epsilon_0^2}\int_{\mathbb{R}}\phi_k(0)|H_1(0,y)|^2\,dy.$$

Moreover, by observing that $H_1(0,y)=yv_0$ and taking the parameters in the weight so that $\sigma>0$, $x_0=-\sigma\tau$, and $k\leq m-1$, we get in particular that for every $\tau\geq0$,

$$\|\langle x_+\rangle^k H_1(\tau)\|_{L^2_w}^2\lesssim\frac{\epsilon_0^2\langle\tau\rangle^{C\epsilon_0^2}}{\langle\tau\rangle^{2(m-1)}}.$$

This proves the first part of the estimates by taking $k=0$.

Next, by using (3.50) with $\sigma=0$, $x_0=0$ and $k=0$ in the weight, and integrating in time we get, using that $w'^2\lesssim\chi_{0,\delta}^2$,

$$\int_0^t\|H_1\|^2_{H^1_{w'}}\,ds\lesssim\epsilon_0^2+\epsilon_0^4\int_0^t\frac{1}{\langle s\rangle^{1+2(m-1)-C\epsilon_0^2}}\,ds\lesssim\epsilon_0^2+\epsilon_0^4,$$

for ϵ_0 sufficiently small,since we assumed $m>3/2$. □

Step 2: Weighted estimates for $IS\tilde{v}_2$. In this section we shall use in a crucial way that

$$\partial_c Q_c=\frac{1}{2c}\partial_y(yQ_c).\tag{3.51}$$

Lemma 3.9. *Set $H_2(t,y)=(IS\tilde{v}_2)(t,x)$ again with $y=x-\int_0^t c+h$. Then*

$$\|H_2\|^2_{L^2_a}+\int_0^\infty\|H_2\|^2_{H^1_a}\,d\tau\lesssim\tilde{\varepsilon}_1^2.$$

Proof. We shall first estimate $\partial_y H_2(t,y)=(S\tilde{v}_2)(t,y)$. Commuting the vector field S with the equation $\partial_t v+\partial_x^3 v=F+\partial_x G$ gives $\partial_t Sv+\partial_x^3 Sv=(S+3)F+\partial_x(S+2)G$. Applying this identity to (3.16), we get that $\partial_y H_2$ solves

$$\begin{aligned}\partial_t\partial_y H_2+\mathcal{L}_{c_+}\partial_y H_2=\partial_y\big(S(\tilde{\mathcal{N}}(v))(t,y)+\mathcal{N}(v)(t,y)+S\tilde{e}_Q(t,y)+3e_Q(t,y)\big)\\ -\partial_y\big((S\tilde{Q}_c^2)v_2\big)-\partial_y(Q_c^2v_2)+\partial_y\mathfrak{e}_Q(t,y):=F(t,y),\end{aligned}\tag{3.52}$$

where we have set

$$\begin{aligned}&\tilde{Q}_c(t,x)=Q_c(t,y),\qquad\tilde{e}_Q(t,x)=e_Q(t,y),\\&\tilde{\mathcal{N}}(v)(t,x)=\big(-(\tilde{Q}_{c(t)}+\tilde{v}_1+\tilde{v}_2)^3+\tilde{Q}_{c(t)}^3+\tilde{v}_1^3+3\tilde{Q}_{c(t)}^2\tilde{v}_2\big)(t,x),\end{aligned}\tag{3.53}$$

and

$$\mathfrak{e}_Q(t,y)=(\tilde{c}-c_+)\partial_y H_2-3((Q_c^2-Q_{c_+}^2)\partial_y H_2).$$

Let us first estimate $\mathcal{P}_{c_+}\partial_y H_2 = \mathcal{P}_{c_+}(S\tilde{v}_2)(t,y)$. By using the equation (3.16) to compute $\partial_t \tilde{v}_2$, and by putting the space derivatives on the functions $\zeta^i_{c_+}$, $\xi^i_{c_+}$, we obtain that

$$\|\mathcal{P}_{c_+}\partial_y H_2\|_{H^1_a} \lesssim \langle t\rangle \left(\|v_2\|_{L^2_a} + \|v_1\|_{L^2_w} + |\dot{c}| + |\dot{h}|\right)$$

and hence by using (3.46), we obtain

$$\|\mathcal{P}_{c_+}\partial_y H_2\|_{H^1_a} \lesssim \frac{\tilde{\epsilon}_1}{\langle t\rangle^{m-1}}, \qquad \|\mathcal{P}_{c_+}\partial_y H_2\|_{L^2_t H^1_a} \lesssim \tilde{\epsilon}_1. \tag{3.54}$$

Note that the second estimate comes from the fact that $m > 3/2$.

We can now estimate $\mathcal{Q}_{c_+}\partial_y H$. Applying $\mathcal{Q}_{c_+}$ to the integral formulation of the equation $\partial_y H_2(t) = \int_0^t S_{c_+}(t-\tau)F(\tau,y)\,d\tau$ gives

$$\mathcal{Q}_{c_+}\partial_y H_2(t) = \int_0^t S_{c_+}(t-\tau)\mathcal{Q}_{c_+}F(\tau,y)\,d\tau.$$

Using the smoothing estimates in Theorem 3.2, the estimates (3.46), and Proposition 3.5, we get

$$\begin{aligned}\|\mathcal{Q}_{c_+}\partial_y H_2(t)\|_{L^2_a} \lesssim \int_0^t \Big(&e^{-b(t-\tau)}\max\left(1,(t-\tau)^{-1/2}\right)\big(\|S\tilde{\mathcal{N}}(v)\|_{L^2_a} + \|\mathcal{N}(v)\|_{L^2_a} \\ &+\frac{\tilde{\epsilon}_1}{\langle\tau\rangle^{2m-1}}\|\partial_y H_2\|_{L^2_a} + \frac{\tilde{\epsilon}_1}{\langle\tau\rangle^m}\big) + e^{-b(t-\tau)}\langle\tau\rangle(|\ddot{c}(\tau)| + |\ddot{h}(\tau)|)\Big)\,d\tau.\end{aligned} \tag{3.55}$$

Next, we claim that from the definitions of the nonlinearities in (3.41) and (3.53), and using (3.46) and (3.49), one has

$$\|\mathcal{N}(v)\|_{L^2_a} \lesssim \tilde{\epsilon}_1\langle t\rangle^{-m}, \tag{3.56}$$

$$\|S\tilde{\mathcal{N}}(v)\|_{L^2_a} \lesssim \tilde{\epsilon}_1\langle t\rangle^{-(m-1)} + \|\partial_y H_1\|_{L^2_{w'}} + \tilde{\epsilon}_1\|\partial_y H_2\|_{L^2_a}. \tag{3.57}$$

The first estimate is a direct consequence of (3.42). Most of the estimates involved in proving (3.57) are straightforward, so we only give details about one of the most complicated terms:

$$\|v_1(S\tilde{v}_1)(t,y)v_2\|_{L^2_a} \lesssim \|v_1\|_{L^\infty}\|S\tilde{v}_1\|_{L^2}\|v_2\|_{H^1_a} \lesssim \frac{\tilde{\epsilon_1}^3}{\langle t\rangle^{m-C\epsilon_0^2}}.$$

We also have to estimate $\ddot{c}$ and $\ddot{h}$. By differentiating in time the equation (3.23), using the equations for v_1 and v_2 to express $\partial_t v_1$ and $\partial_t v_2$ and always putting the space derivatives on ζ^i_c in the scalar products using integration by parts, we obtain by using (3.46) that

$$|\ddot{h}(t)| + |\ddot{c}(t)| \lesssim \frac{\tilde{\epsilon}_1}{\langle t\rangle^m} + \frac{\tilde{\epsilon}_1}{\langle t\rangle^m}\|\partial_{yy}v\|_{L^2_{w'}}.$$

Note that the last term comes from the estimate of cubic nonlinear term that yields, after integration by parts, terms of the form $\int_{\mathbb{R}} v\partial_v v\partial_{yy}v\partial_x\zeta^i_c\,dx$, for $i = 1, 2$. Consequently, by using (3.5), we obtain that

$$\Big(\int_0^t \langle\tau\rangle^2(|\ddot{h}(\tau)|^2 + |\ddot{c}(\tau)|^2)\,d\tau\Big)^{\frac{1}{2}} \lesssim \tilde{\epsilon}_1. \tag{3.58}$$

By plugging the above estimates into (3.55), we thus get that

$$\|\mathcal{Q}_{c_+}\partial_y H_2(t)\|_{L^2_a} \lesssim \int_0^t e^{-b(t-\tau)}\langle\tau\rangle(|\ddot{c}(\tau)| + |\ddot{h}(\tau)|)\,d\tau$$
$$+ \int_0^t \Big(e^{-b(t-\tau)}\max(1,(t-\tau)^{-1/2})\Big(\frac{\tilde\epsilon_1}{\langle\tau\rangle^{m-1}} + \|\partial_y H_1\|_{L^2_{w'}} + \tilde\epsilon_1\|\partial_y H_2\|_{L^2_a}\Big)\,d\tau.$$

By using Young's inequality, (3.54) and Lemma 3.8, we thus get that

$$\|\partial_y H_2\|_{L^2_t L^2_a} \lesssim \tilde\epsilon_1 + \epsilon_0 + \tilde\epsilon_1\|\partial_y H_2\|_{L^2_t L^2_a}$$

and hence for $\tilde\epsilon_1$ sufficiently small that

$$\|\partial_y H_2\|_{L^2_t L^2_a} \lesssim \tilde\epsilon_1. \tag{3.59}$$

It remains to estimate $\|H_2\|_{L^2_a}$. Since $H_2 = \int_{-\infty}^y \partial_y H_2$, integrating in y (3.52), we get

$$\begin{aligned}\partial_t H_2 - \tilde c\partial_y H_2 + Q_c^2\partial_y H_2 + \partial_y^3 H_2 &= S(\tilde{\mathcal{N}}(v))(t,y) + 2\mathcal{N}(v)(t,y)\\ &\quad - \int_{-\infty}^y [e_Q(t,y') + 3(S\tilde e_Q)(t,y')]\,dy - \big(S\tilde{Q}_c^2 v_2\big) - Q_c^2 v_2\\ &= G(t,y).\end{aligned}$$

By a weighted energy estimate, we get that for some $b>0$, we have

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} e^{ay}|H_2|^2\,dy + b\|H_2\|^2_{H^1_a} \lesssim \|G\|_{L^2_a}\|H_2\|_{L^2_a} + \|\partial_y H_2\|_{L^2_a}\|H_2\|_{L^2_a}.$$

Next, we use (3.51) to write

$$e_Q = \frac{\dot c}{2c}\partial_y(yQ_c) + \dot h\partial_y Q_c,$$

and see that

$$\int_{-\infty}^y \big(e_Q(t,y') + (S\tilde e_Q)(t,y')\big)\,dy'$$

is an exponentially decreasing function at $\pm\infty$. Therefore, thanks to (3.46) and (3.58), we obtain

$$\Big\|\int_{-\infty}^y \big(e_Q(t,y') + (S\tilde e_Q)(t,y')\big)\,dy'\Big\|_{L^2_a} \lesssim \frac{\tilde\epsilon_1}{\langle t\rangle^{m-1}}.$$

Consequently, by using (3.56), (3.57), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} e^{ay}|H_2|^2\,dy + b\|H_2\|^2_{H^1_a} \lesssim \Big(\frac{\tilde\epsilon_1}{\langle t\rangle^{m-1}} + \|\partial_y H_1\|_{L^2_{w'}} + \|\partial_y H_2\|_{L^2_a}\Big)\|H_2\|_{L^2_a}.$$

By integrating in time and using Young's inequality, we obtain

$$\|H_2(t)\|^2_{L^2_a} + b\int_0^t \|H_2\|^2_{H^1_a} \lesssim \tilde\epsilon_1^2 + \int_0^t \big(\|\partial_y H_1\|^2_{L^2_{w'}} + \|\partial_y H_2\|^2_{L^2_a}\big)\,d\tau.$$

By using (3.59) and Lemma 3.8, we finally get that

$$\int_0^t \|H_2\|^2_{H^1_a} \lesssim \tilde\epsilon_1^2.$$

This completes the proof of the Lemma. $\square$

Step 3: Estimates of $\widetilde{v}$. Let us recall that $\tilde{v}(t,x)$ is defined in (3.45). We can write the equation for $\tilde{v}$ under the form

$$\partial_t \tilde{v} + \partial_x^3 \tilde{v} + \partial_x(\tilde{v}^3) = -\partial_x\big((\tilde{Q}_c + \tilde{v})^3 - \tilde{Q}_c^3 - \tilde{v}^3\big) + \dot{h}\partial_x \tilde{Q}_c + \dot{c}\partial_c \tilde{Q}_c = \partial_x K. \tag{3.60}$$

where

$$K := -(\tilde{Q}_c + \tilde{v})^3 + \tilde{Q}_c^3 + \tilde{v}^3 + \dot{h}\tilde{Q}_c + \frac{\dot{c}}{2c}\widetilde{(yQ_c)}(t,x). \tag{3.61}$$

Above we have used the notation $\widetilde{(yQ_c)}(t,x) = (x - \int_0^t c + h)Q_c(x - \int_0^t c + h)$. Note that in order to write the right hand side of (3.60) as the derivative of a localized function, we have used (3.51).

We now study the profile f of $\tilde{v}$ defined by $f = e^{t\partial_x^3}\tilde{v}$. By taking ϵ_1 sufficiently small but such that $\epsilon_0 \ll \tilde{\epsilon}_1 \ll \epsilon_1$, we will prove the following:

Lemma 3.10. *We have the estimate*

$$\|\tilde{v}\|_X := \sup_t \left(\|\tilde{v}\|_{H^1} + \langle t\rangle^{-\frac{1}{6}}\big\|xf\big\|_{H^1} + \langle t\rangle^{\frac{\alpha}{3}-\frac{1}{6}}\big\||\partial_x|^\alpha xf\big\|_{L^2} + \big\|\widehat{f}(\xi)\big\|_{L^\infty}\right) \le \epsilon_1.$$

Proof. We follow the same steps as in the first part of the paper. Note that the norm $\|\tilde{v}\|_{H^1}$ is already estimated in view of (3.46). Moreover, by combining Lemma 3.8 and Lemma 3.9 we have that

$$\int_0^t \|(IS\tilde{v})(\tau,y)\|_{H^1_{w'}}^2\, d\tau \lesssim \tilde{\epsilon}_1^2. \tag{3.62}$$

Still following the steps of the first part of the paper, we shall first estimate $\|xf\|_{H^1}$, using again $IS\tilde{v}$. We note that $IS\tilde{v}$ solves

$$\partial_t IS\tilde{v} + \partial_x^3 IS\tilde{v} + 3\tilde{v}^2\partial_x IS\tilde{v} = (S+2)K$$

An energy estimate yields

$$\frac{d}{dt}\frac{1}{2}\|IS\tilde{v}\|_{L^2}^2 \lesssim \frac{\epsilon_1^2}{t}\|IS\tilde{v}\|_{L^2}^2 + \int_{\mathbb{R}} (|SK| + |K|)\,|IS\tilde{v}|\, dx. \tag{3.63}$$

By using (3.46), we obtain

$$\int_{\mathbb{R}} (|SK| + |K|)\,|IS\tilde{v}| \lesssim \|IS\tilde{v}(t,y)\|_{L^2_{w'}}\left(\frac{\tilde{\epsilon}_1}{\langle t\rangle^{m-1}} + \langle t\rangle(|\ddot{c}(t)| + |\ddot{h}(t)|) + \|S\tilde{v}(t,y)\|_{L^2_{w'}}\right)$$

and thus by (3.62) and (3.58), we obtain

$$\int_0^t \int_{\mathbb{R}} (|SK| + |K|)\,|IS\tilde{v}(t,y)|\, dy\, dt \lesssim \tilde{\epsilon}_1^2.$$

Consequently, by integrating (3.63) in time, we obtain

$$\|IS\tilde{v}(t)\|_{L^2}^2 \lesssim \langle t\rangle^{C\epsilon_1^2}\left(\epsilon_0^2 + \tilde{\epsilon}_1^2\right). \tag{3.64}$$

Next, we observe that

$$xf = e^{t\partial_x^3} IS\tilde{v} - 3tI\partial_t f = e^{t\partial_x^3} IS\tilde{v} - 3te^{t\partial_x^3}\big(-\tilde{v}^3 + K\big).$$

Using (2.12) we get

$$t\|\tilde{v}^3\|_{L^2} \lesssim \epsilon_1^3\langle t\rangle^{\frac{1}{6}},$$

and thanks to (3.46) we obtain

$$t\|K\|_{L^2} \lesssim \frac{\tilde{\epsilon}_1}{\langle t\rangle^{m-1}}.$$

Combining these estimates gives

$$\|xf\|_{L^2} \lesssim \tilde{\epsilon}_1 + \epsilon_1^3 t^{\frac{1}{6}}.$$

Finally, we can estimate $S\tilde{v}$ in L^2: first, we note that $S\tilde{v}$ solves

$$\partial_t S\tilde{v} + \partial_x^3 S\tilde{v} + 3\partial_x(\tilde{v}^2 S\tilde{v}) = \partial_x(S+2)K. \tag{3.65}$$

¿From an energy estimate, we find after integrating by parts

$$\frac{d}{dt}\frac{1}{2}\|S\tilde{v}\|_{L^2}^2 \lesssim \frac{\epsilon_1^2}{t}\|S\tilde{v}\|_{L^2}^2 + \|S\tilde{v}\|_{L^2_{w'}}\left(\frac{\tilde{\epsilon}_1}{\langle t\rangle^{m-1}} + \langle t\rangle(|\ddot{c}(t)| + |\ddot{h}(t)|) + \|S\tilde{v}(t,y)\|_{L^2_{w'}}\right),$$

and hence (3.62) and (3.58) yield

$$\|S\tilde{v}(t)\|_{L^2}^2 \lesssim \langle t\rangle^{C\epsilon_1^2}\left(\epsilon_0^2 + \tilde{\epsilon}_1^2\right). \tag{3.66}$$

By using

$$|\partial_x|^\alpha(xf) = e^{t\partial_x^3}|\partial_x|^\alpha(IS\tilde{v}) - 3te^{t\partial_x^3}\Big(-|\partial_x^\alpha|\tilde{v}^3 + |\partial_x^\alpha|K\Big),$$

we get that

$$\||\partial_x|^\alpha(xf)\|_{L^2} \lesssim \|IS\tilde{v}\|_{H^1} + t\||\partial_x^\alpha|\tilde{v}^3\|_{L^2} + t\|K\|_{H^1}.$$

Then, by using (3.66), (3.64), (3.48) and

$$\|K\|_{H^1} \lesssim \frac{\tilde{\epsilon}_1}{\langle t\rangle^m} + (1+\tilde{\epsilon}_1)\|v\|_{H^1_w} \lesssim \frac{\tilde{\epsilon}_1}{\langle t\rangle^m},$$

we finally get

$$\||\partial_x|^\alpha(xf)\|_{L^2} \lesssim \tilde{\epsilon}_1 + \epsilon_1^3 t^{\frac{1}{6}-\frac{\alpha}{3}}.$$

Note that another way to get this estimate (that avoids using the H^1 regularity of the initial data in this step) would be to start from (3.65) and to follow the same steps as in the proof of (2.15) in Lemma 2.4.

It remains to estimate $\|\hat{f}\|_{L^\infty}$. We can follow the proof of Proposition 2.5. The equation for $\hat{f}$ is now

$$\partial_t\widehat{f}(t,\xi) = \frac{i}{2\pi}\iint e^{-it\phi(\xi,\eta,\sigma)}\xi\widehat{f}(\xi-\eta-\sigma)\widehat{f}(\eta)\widehat{f}(\sigma)\,d\eta\,d\sigma + G(t,\xi),$$

where

$$G(t,\xi) = \mathcal{F}\Big(e^{t\partial_x^3}\partial_x K\Big)(\xi).$$

We can estimate

$$\|G(t)\|_{L^\infty} \leq \|\mathcal{F}(\partial_x K)\|_{L^\infty} \lesssim \|\partial_x K\|_{L^1} \lesssim |\dot{c}| + |\dot{h}| + (1+\tilde{\epsilon}_1)\|v(t,y)\|_{H^1_w}$$

by using the exponential decay provided by Q_c, and hence deduce

$$\|G(t)\|_{L^\infty} \lesssim \frac{\tilde{\epsilon}_1}{\langle t\rangle^m}.$$

This term is thus integrable in time for $m > 1$ and does not affect the arguments in the proof of Proposition 2.5. This completes the proof of the statement, and gives Theorem 1.5. $\square$

Appendix A. Auxiliary Lemmas

In this appendix we gather several lemmas that are used throughout the paper. First, a lemma about stationary phase.

Lemma A.1 (Stationary phase in dimension 2). *Consider $\chi \in \mathcal{C}_0^\infty$ such that $\chi = 0$ in $B(0,2)^c$, and $|\nabla\chi| + |\nabla^2\chi| \lesssim 1$; and $\psi \in \mathcal{C}^\infty$ such that $|\det \operatorname{Hess}\psi| \geq 1$, and $|\nabla\psi| + |\nabla^2\psi| + |\nabla^3\psi| \lesssim 1$. Let*

$$I = \iint e^{i\lambda\psi(\eta,\sigma)} F(\eta,\sigma)\chi(\eta,\sigma)\, d\eta\, d\sigma.$$

Then, for any $\alpha \in [0,1]$,

(i) *If $\nabla\psi$ only vanishes at (η_0,σ_0),*

$$I = \frac{2\pi e^{i\frac{\pi}{4}s}}{\sqrt{\Delta}} \frac{e^{i\lambda\psi(\eta_0,\sigma_0)}}{\lambda} F(\eta_0,\sigma_0)\chi(\eta_0,\sigma_0) + O\left(\frac{\left\|\langle (x,y)\rangle^{2\alpha}\widehat{F}\right\|_{L^1}}{\lambda^{1+\alpha}}\right),$$

where $s = \operatorname{sign}\operatorname{Hess}\psi$ and $\Delta = |\det\operatorname{Hess}\psi|$.

(ii) *If $|\nabla\psi| \geq 1$,*

$$I = O\left(\frac{\left\|\langle (x,y)\rangle^{\alpha}\widehat{F}\right\|_{L^1}}{\lambda^{1+\alpha}}\right).$$

Proof. (i) We assume for simplicity that $\eta_0 = \sigma_0 = \psi(\eta_0,\sigma_0) = 0$. If necessary, it is possible to restrict the support of χ to an arbitrarily small neighborhood of 0, since the remainder can be treated by (ii). By Plancherel's theorem,

$$I = \frac{1}{2\pi}\iint \underbrace{\left[\iint e^{i[\lambda\psi(\eta,\sigma)-x\eta-y\sigma]}\chi(\eta,\sigma)\, d\eta\, d\sigma\right]}_{K(x,y)} \widehat{F}(x,y)\, dx\, dy. \tag{A.1}$$

The function K can be written

$$K(x,y) = \iint e^{i\lambda\Phi_{X,Y}(\eta,\sigma)}\chi(\eta,\sigma)\, d\eta\, d\sigma$$

with $\Phi_{X,Y}(\eta,\sigma) = \psi(\eta,\sigma) - X\eta - Y\sigma$ and $X = x/\lambda$, $Y = y/\lambda$. If X or Y is larger than $2\max_{\operatorname{Supp}\chi}|\nabla\psi|$, it is easy to bound K by integrating by parts in the above integral. Thus we can assume that X, Y are less than $2\max_{\operatorname{Supp}\chi}|\nabla\psi|$. Since $\operatorname{Supp}\chi$ can be chosen to be a small neighborhood of 0, we can assume that X and Y are small.

By assumption, $\operatorname{Hess}\Phi$ is non-degenerate, thus by the implicit function theorem there exists $(\bar\eta(X,Y),\bar\sigma(X,Y))$ such that

$$\nabla_{\eta,\sigma}\Phi_{X,Y}(\bar\eta(X,Y),\bar\sigma(X,Y)) = 0$$

and furthermore

$$|\bar\eta(X,Y)| + |\bar\sigma(X,Y)| \lesssim |X| + |Y| \quad \text{and} \quad |\Phi_{X,Y}(\bar\eta(X,Y),\bar\eta(X,Y))| \lesssim |X|^2 + |Y|^2. \tag{A.2}$$

By the stationary phase lemma,

$$K(x,y) = \frac{2\pi e^{i\frac{\pi}{4}s}}{\sqrt{\bar\Delta}} \frac{e^{i\lambda\Phi(\bar\eta,\bar\sigma)}}{\lambda}\chi(\bar\eta,\bar\sigma) + O\left(\frac{1}{\lambda^2}\right),$$

where $\bar{\Delta} = |\det \mathrm{Hess}\,\psi|(\bar{\eta},\bar{\sigma})$ Coming back to (A.1),

$$
\begin{aligned}
I &= 2\pi e^{i\frac{\pi}{4}s}\frac{1}{\lambda}\iint \widehat{F}(x,y)e^{i\lambda\Phi(\bar{\eta},\bar{\sigma})}\frac{1}{\bar{\Delta}}\chi(\bar{\eta},\bar{\sigma})\,dx\,dy + O\left(\frac{\|\widehat{F}\|_{L^1}}{\lambda^2}\right)\\
&= \frac{2\pi e^{i\frac{\pi}{4}s}}{\sqrt{\Delta}}\frac{1}{\lambda}F(0,0)\chi(0,0)\\
&+ \frac{e^{i\frac{\pi}{4}s}}{\sqrt{\Delta}}\frac{1}{\lambda}\iint \widehat{F}(x,y)\left[e^{i\lambda\Phi(\bar{\eta},\bar{\sigma})}\frac{\chi(\bar{\eta},\bar{\sigma})}{\bar{\Delta}} - \frac{\chi(0,0)}{\Delta}\right]\,dx\,dy + O\left(\frac{\|\widehat{F}\|_{L^1}}{\lambda^2}\right)
\end{aligned}
$$

Therefore, using (A.2), we find for any $\alpha \in [0,1]$

$$
\begin{aligned}
&\left| I - \frac{2\pi e^{i\frac{\pi}{4}s}}{\sqrt{\Delta}}\frac{1}{\lambda}F(0,0) + O\left(\frac{\|\widehat{F}\|_{L^1}}{\lambda^2}\right)\right|\\
&\lesssim \frac{1}{\lambda}\iint |\widehat{F}(x,y)|\left[|e^{i\lambda\Phi(\bar{\eta},\bar{\sigma})} - 1| + |\chi(\bar{\eta},\bar{\sigma}) - \chi(0,0)|\right]\,dxdy\\
&\lesssim \frac{1}{\lambda}\iint |\widehat{F}(x,y)|\left[\frac{|(x,y)|^{2\alpha}}{\lambda^\alpha} + \frac{|(x,y)|^{\alpha}}{\lambda^\alpha}\right]\,dxdy \lesssim \frac{1}{\lambda^{1+\alpha}}\|\langle(x,y)\rangle^{2\alpha}\widehat{F}\|_{L^1},
\end{aligned}
$$

which is the desired result.

(ii) A direct stationary phase estimate as above, using only the non-degeneracy of $\mathrm{Hess}\,\psi$, gives the bound

$$|I| \lesssim \frac{1}{\lambda}\|\widehat{F}\|_{L^1}.$$

Furthermore, since we are now assuming that ψ does not have stationary points, it is possible to integrate by parts in I before applying this stationary phase estimate. This gives

$$|I| \lesssim \frac{1}{\lambda^2}\|\langle(x,y)\rangle\widehat{F}\|_{L^1}.$$

Interpolating between these two inequalities gives the desired estimate. □

The following lemma gives some bounds on pseudo-product operators satisfying certain strong integrability conditions:

Lemma A.2 (Bounds on pseudo-product operators). *Assume that $m \in L^1(\mathbb{R}\times\mathbb{R})$ satisfies*

$$\Big\| \int_{\mathbb{R}\times\mathbb{R}} m(\eta,\sigma)e^{ix\eta}e^{iy\sigma}\,d\eta d\sigma\Big\|_{L^1_{x,y}} \le A\,, \tag{A.3}$$

for some $A > 0$. Then, for all $p,q,r \in [1,\infty]$ such that $1/p + 1/q = 1/r$ one has

$$\|T_m(f,g)\|_{L^r} \lesssim A\|f\|_{L^p}\|g\|_{L^q}. \tag{A.4}$$

Moreover, if $1/p + 1/q + 1/r = 1$

$$\Big|\int_{\mathbb{R}\times\mathbb{R}} \widehat{f}(\eta)\widehat{g}(\sigma)\widehat{h}(-\eta-\sigma)m(\eta,\sigma)\,d\eta d\sigma\Big| \lesssim A\|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}. \tag{A.5}$$

Proof. We rewrite

$$\Big| \int_{\mathbb{R}\times\mathbb{R}} \widehat{f}(\eta)\widehat{g}(\sigma)\widehat{h}(-\eta-\sigma)m(\eta,\sigma)\,d\eta d\sigma\Big| = C\Big|\int_{\mathbb{R}^3} f(x)g(y)h(z)K(z-x,z-y)\,dxdydz\Big|,$$
$$\lesssim \int_{\mathbb{R}^3} |f(z-x)g(z-y)h(z)|\,|K(x,y)|\,dxdydz,$$

where

$$K(x,y) := \int_{\mathbb{R}\times\mathbb{R}} m(\eta,\sigma)e^{ix\eta}e^{iy\sigma}\,d\eta d\sigma.$$

The desired bound (A.5) follows easily from (A.3). The bilinear estimate (A.4) can be proven similarly using a duality argument. □

Finally, for the energy estimates, we need the following lemma.

Lemma A.3. *The following commutator estimate holds:*

$$\| \left[|\partial_x|^{\alpha-1}\partial_x, P_{\ll j}w\right] P_j f\|_{L^2} \lesssim 2^{(\alpha-1)j}\|\partial_x w\|_{L^\infty}\|P_j f\|_{L^2}.$$

Proof. Denoting $\widetilde{P}_j$ for the Fourier multiplier with symbol

$$\widetilde{\chi}\left(\frac{\xi}{2^j}\right) = \frac{\xi|\xi|^{\alpha-1}}{2^{\alpha j}}\left[\chi\left(\frac{\xi}{2^{j+10}}\right) - \chi\left(\frac{\xi}{2^{j-10}}\right)\right],$$

observe that

$$\left[|\partial_x|^{\alpha-1}\partial_x, P_{\ll j}w\right] P_j f(x) = 2^{\alpha j}\left[\widetilde{P}_j, P_{\ll j}w\right] P_j f = 2^{\alpha j}\left[2^j\widehat{\widetilde{\chi}}(2^j\cdot)*, P_{\ll j}w\right] P_j f$$
$$= 2^{\alpha j}\int 2^j\widehat{\widetilde{\chi}}(2^j(x-y))\left[P_{\ll j}w(x) - P_{\ll j}w(y)\right] P_j f(y)\,dy.$$

Thus

$$\left|\left[|\partial_x|^{\alpha-1}\partial_x, P_{\ll j}w\right] P_j f(y)\right| \lesssim 2^{\alpha j}\|\partial_x w\|_\infty \int 2^j|\widehat{\widetilde{\chi}}(2^j(x-y))||x-y||P_j f(y)|\,dy,$$

and the desired result follows by Young's inequality. □

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