GLOBAL SOLUTIONS OF THE GRAVITY-CAPILLARY WATER WAVE SYSTEM IN 3 DIMENSIONS, II: DISPERSIVE ANALYSIS

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Abstract. In this paper and its companion [32] we prove global regularity for the full water waves system in 3 dimensions for small data, under the influence of both gravity and surface tension. The main difficulties are the weak, and far from integrable, pointwise decay of solutions, together with the presence of a full codimension one set of quadratic resonances. To overcome these difficulties we use a combination of improved energy estimates and dispersive analysis.

In this paper we prove the dispersive estimates, while the energy estimates are proved in [32]. The dispersive estimates rely on analysis of the Duhamel formula in a carefully constructed weighted norm, taking into account the nonlinear contribution of special frequencies, such as the space-time resonances, and the slowly decaying frequencies.

Contents

1. Introduction

1.1. **Free boundary Euler equations and water waves.** The evolution of an inviscid perfect fluid that occupies a domain $\Omega_t \subset \mathbb{R}^n$, for $n \geq 2$, at time $t \in \mathbb{R}$, is described by the free boundary incompressible Euler equations. If v and p denote respectively the velocity and the pressure of the fluid (with constant density equal to 1) at time t and position $x \in \Omega_t$, these equations are

$$(\partial_t + v \cdot \nabla)v = -\nabla p - ge_n, \qquad \nabla \cdot v = 0, \qquad x \in \Omega_t, \tag{1.1}$$

where g is the gravitational constant. The first equation in (1.1) is the conservation of momentum equation, while the second is the incompressibility condition. The free surface $S_t := \partial\Omega_t$ moves

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with the normal component of the velocity according to the kinematic boundary condition

$$\partial_t + v \cdot \nabla \text{ is tangent to } \bigcup_t S_t \subset \mathbb{R}^{n+1}_{x,t}. \tag{1.2}$$

The pressure on the interface is given by

$$p(x,t) = \sigma\kappa(x,t), \qquad x \in S_t, \tag{1.3}$$

where κ is the mean-curvature of S_t and $\sigma \geq 0$ is the surface tension coefficient. At liquid-air interfaces, the surface tension force results from the greater attraction of water molecules to each other than to the molecules in the air.

In the case of irrotational flows, $\operatorname{curl} \mathrm{v} = 0$, one can reduce (1.1)-(1.3) to a system on the boundary. Indeed, assume also that $\Omega_t \subset \mathbb{R}^n$ is the region below the graph of a function $h : \mathbb{R}^{n-1}_x \times I_t \to \mathbb{R}$, that is

$$\Omega_t = \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \leq h(x,t)\} \quad \text{and} \quad S_t = \{(x,y) : y = h(x,t)\}.$$

Let Φ denote the velocity potential, $\nabla_{x,y}\Phi(x,y,t) = v(x,y,t)$, for $(x,y) \in \Omega_t$. If $\phi(x,t) := \Phi(x,h(x,t),t)$ is the restriction of Φ to the boundary S_t, the equations of motion reduce to the following system for the unknowns $h, \phi : \mathbb{R}^{n-1}_x \times I_t \to \mathbb{R}$:

$$\begin{cases} \partial_t h = G(h)\phi, \\ \partial_t \phi = -gh + \sigma \operatorname{div}\Big[\dfrac{\nabla h}{(1+|\nabla h|^2)^{1/2}}\Big] - \dfrac{1}{2}|\nabla\phi|^2 + \dfrac{(G(h)\phi + \nabla h \cdot \nabla\phi)^2}{2(1+|\nabla h|^2)}. \end{cases} \tag{1.4}$$

Here

$$G(h) := \sqrt{1+|\nabla h|^2}\mathcal{N}(h), \tag{1.5}$$

and $\mathcal{N}(h)$ is the Dirichlet-Neumann map associated to the domain Ω_t. We refer to [65, chap. 11] or the book of Lannes [54] for the derivation of (1.4).

One generally refers to (1.4) as the gravity water waves system when $g > 0$ and $\sigma = 0$, as the capillary water waves system when $g = 0$ and $\sigma > 0$, and as the gravity-capillary water waves system when $g > 0$ and $\sigma > 0$.

The Cauchy problem associated to water wave systems has been studied extensively. The local existence theory is well understood both in 2 and 3 dimensions, at a suitable level of generality, see for example [57, 75, 22, 71, 72, 9, 14, 56, 53, 20, 60, 61, 12, 8, 13, 1, 2, 28]. On the other hand, the long term/global existence theory is much more limited: the only results are in the case of "small" data with trivial vorticity, in dimension 3, see [36, 74, 37, 69, 70], and in dimension 2, see [73, 46, 3, 4, 40, 41, 48, 42, 68]. Moreover, large perturbations can lead to breakdown in finite time, such as the "splash" singularities in [10, 21]. We refer the reader to the introduction of the companion paper [32] for a more extensive discussion of the history and previous work on the water waves problem.

1.2. **The main theorem.** Our results in this paper and its companion [32] concern the gravity-capillary water waves system (1.4), in the case $n = 3$. In this case h and ϕ are real-valued functions defined on $\mathbb{R}^2 \times I$.

To state our main theorem we first introduce some notation. The rotation vector-field

$$\Omega := x_1\partial_{x_2} - x_2\partial_{x_1} \tag{1.6}$$

commutes with the linearized system. For $N \geq 0$ let H^N denote the standard Sobolev spaces on $\mathbb{R}^2$. More generally, for $N, N' \geq 0$ and $b \in [-1/2, 1/2]$, $b \leq N$, we define the norms

$$\|f\|_{H^{N',N}_\Omega} := \sum_{j \leq N'} \|\Omega^j f\|_{H^N}, \qquad \|f\|_{\dot{H}^{N,b}} := \big\|(|\nabla|^N + |\nabla|^b) f\big\|_{L^2}. \tag{1.7}$$

For simplicity of notation, we sometimes let $H^{N'}_\Omega := H^{N',0}_\Omega$. Our main theorem is the following:

Theorem 1.1 (Global Regularity)**.** *Assume that $g, \sigma > 0$, $\delta > 0$ is sufficiently small, and N_0, N_1, N_3, N_4 are sufficiently large*[1] *(for example $\delta = 1/2000$, $N_0 := 4170$, $N_1 := 2070$, $N_3 := 30$, $N_4 := 70$, compare with Definition 2.1). Assume that the data (h_0, ϕ_0) satisfies*

$$\begin{aligned} &\|\mathcal{U}_0\|_{H^{N_0} \cap H^{N_1,N_3}_\Omega} + \sup_{2m+|\alpha| \leq N_1+N_4} \|(1+|x|)^{1-50\delta} D^\alpha \Omega^m \mathcal{U}_0\|_{L^2} = \varepsilon_0 \leq \overline{\varepsilon_0}, \\ &\mathcal{U}_0 := (g - \sigma\Delta)^{1/2} h_0 + i|\nabla|^{1/2}\phi_0, \end{aligned} \tag{1.8}$$

where $\overline{\varepsilon_0}$ is a sufficiently small constant and $D^\alpha = \partial_1^{\alpha^1}\partial_2^{\alpha^2}$, $\alpha = (\alpha^1, \alpha^2)$. Then, there is a unique global solution $(h, \phi) \in C\big([0,\infty) : H^{N_0+1} \times \dot{H}^{N_0+1/2,1/2}\big)$ of the system (1.4), with $(h(0), \phi(0)) = (h_0, \phi_0)$. In addition

$$(1+t)^{-\delta^2}\|\mathcal{U}(t)\|_{H^{N_0} \cap H^{N_1,N_3}_\Omega} \lesssim \varepsilon_0, \qquad (1+t)^{5/6-3\delta^2}\|\mathcal{U}(t)\|_{L^\infty} \lesssim \varepsilon_0, \tag{1.9}$$

for any $t \in [0,\infty)$, where $\mathcal{U} := (g - \sigma\Delta)^{1/2} h + i|\nabla|^{1/2}\phi$.

Remark 1.2. *(i) One can derive additional information about the global solution (h, ϕ). Indeed, by rescaling we may assume that $g = 1$ and $\sigma = 1$. Let*

$$\mathcal{U}(t) := (1-\Delta)^{1/2} h + i|\nabla|^{1/2}\phi, \qquad \mathcal{V}(t) := e^{it\Lambda}\mathcal{U}(t), \qquad \Lambda(\xi) := \sqrt{|\xi| + |\xi|^3}. \tag{1.10}$$

Here Λ is the linear dispersion relation, and $\mathcal{V}$ is the profile of the solution $\mathcal{U}$. The proof of the theorem gives the strong uniform bound

$$\sup_{t \in [0,\infty)} \|\mathcal{V}(t)\|_Z \lesssim \varepsilon_0, \tag{1.11}$$

see Definition 2.1. The pointwise decay bound in (1.9) follows from this and the linear estimates in Lemma 3.6 below.

(ii) The global solution $\mathcal{U}$ scatters in the Z norm as $t \to \infty$, i.e. there is $\mathcal{V}_\infty \in Z$ such that

$$\lim_{t \to \infty} \|e^{it\Lambda}\mathcal{U}(t) - \mathcal{V}_\infty\|_Z = 0.$$

However, the asymptotic behavior is somewhat nontrivial since $|\widehat{\mathcal{U}}(\xi, t)| \gtrsim \log t \to \infty$ for frequencies ξ on a circle in $\mathbb{R}^2$ (the set of space-time resonance outputs) and for some data. This unusual behavior is due to the presence of a large set of space-time resonances.

(iii) The function $\mathcal{U} := (g - \sigma\Delta)^{1/2} h + i|\nabla|^{1/2}\phi$ is called the "Hamiltonian variable", due to its connection to the Hamiltonian of the system. This variable is important in order to keep track correctly of the relative weights of the functions h and ϕ during the proof.

The proof of Theorem 1.1 relies on two main steps:

(1) Propagate control of high order Sobolev and weighted norms;

[1]The values of N_0 and N_1, the total number of derivatives we assume under control, can certainly be decreased by reworking parts of the argument. We prefer, however, to simplify the argument wherever possible instead of aiming for such improvements. For convenience, we arrange that $N_1 - N_4 = (N_0 - N_3)/2 - N_4 = 1/\delta$.

(2) Prove dispersion/decay over time by propagating control of a suitable Z norm.

The interplay of these two aspects has been present since the seminal work of Klainerman [51, 52] on nonlinear wave equations and vector-fields, Shatah [59] on 3d Klein-Gordon and normal forms, Christodoulou-Klainerman [15] on the stability of Minkowski space, and Delort [29] on 1d Klein-Gordon. In our problem, high order energy control was proved in [32], using a suitable bootstrap argument. The main result in this paper is the following proposition, which gives the desired dispersive control, thus completing the proof of the main theorem.

Proposition 1.3. *(Improved dispersive control) Assume that $T \geq 1$ and $(h, \phi) \in C([0, T] : H^{N_0+1} \times \dot{H}^{N_0+1/2,1/2})$ is a solution of the system (1.4) with $g = 1$ and $\sigma = 1$, with initial data (h_0, ϕ_0). Assume that, with $\mathcal{U}$ and $\mathcal{V}$ defined as in (1.10),*

$$\|\mathcal{U}_0\|_{H^{N_0} \cap H_\Omega^{N_1,N_3}} + \|\mathcal{V}_0\|_Z \leq \varepsilon_0 \ll 1 \tag{1.12}$$

and, for any $t \in [0, T]$,

$$(1+t)^{-\delta^2} \|\mathcal{U}(t)\|_{H^{N_0} \cap H_\Omega^{N_1,N_3}} + \|\mathcal{V}(t)\|_Z \leq \varepsilon_1 \ll 1, \tag{1.13}$$

where the Z norm is as in Definition 2.1. Then, for any $t \in [0, T]$,

$$\|\mathcal{V}(t)\|_Z \lesssim \varepsilon_0 + \varepsilon_1^2. \tag{1.14}$$

This corresponds to Proposition 2.3 in [32]; see also Proposition 2.2 in [32] for the other part of the bootstrap argument, concerning energy norms.

The rest of the paper is concerned with the proof of Proposition 1.3.

1.3. **Main ideas.** In the last few years new methods have emerged in the study of global solutions of quasilinear evolutions, inspired by the advances in semilinear theory. The basic idea is to combine the classical energy and vector-fields methods with refined analysis of the Duhamel formula, using the Fourier transform. This is the essence of the "method of space-time resonances" of Germain-Masmoudi-Shatah [36, 37, 35], see also Gustafson-Nakanishi-Tsai [39], and of the refinements in [43, 44, 38, 45, 46, 47, 48, 31, 30], using atomic decompositions and more sophisticated norms.

We emphasize that the proof of Theorem 1.1 in this paper and its companion [32] is different and substantially more difficult than the previous work on global solutions in water wave models (or any other time-reversible quasilinear evolutions, as far as we know). As explained in the longer discussion in the subsection 1.4 in [32], the main new difficulty is the combination of slow (at best $|t|^{-5/6}$) pointwise decay of solutions, and the presence of a large, codimension 1 set of quadratic time resonances without matching null structure.

We remark that this combination was not present in any of the earlier global regularity results on water waves described above. More precisely, in all the previous global results in 3 dimensions (2D interface) in [36, 74, 37, 69, 70] it was possible to prove $1/t$ pointwise decay of the nonlinear solutions. This decay allowed for high order energy estimates with slow growth.

On the other hand, in all the previous long term/global results in 2 dimensions (1D interface) in [73, 46, 3, 4, 40, 41, 48, 42, 68] the starting point was an identity of the form

$$\partial_t \mathcal{E}(t) = \text{ quartic semilinear term,}$$

where $\mathcal{E}$ is a suitable energy functional and the quartic expression in the right-hand side does not lose derivatives. An energy inequality of this form was first proved by Wu [73] for the gravity water wave model, and led to an almost-global existence result. Such an inequality (which is related to normal form transformations) is possible only when there are no time resonances for the quadratic terms. This is essentially the situation in all the 2D results mentioned above.[2]

1.3.1. *A simplified model and dispersive analysis.* To illustrate the main ideas in the proof of Proposition 1.3, consider the initial-value problem

$$\begin{aligned} &(\partial_t + i\Lambda)U = \nabla V \cdot \nabla U + (1/2)\Delta V \cdot U, \qquad U(0) = U_0, \\ &\Lambda(\xi) := \sqrt{|\xi| + |\xi|^3}, \qquad V := P_{[-10,10]}\Re U. \end{aligned} \tag{1.15}$$

At the level of energy estimates, this simplified model was analyzed in subsection 1.5 in [32]. Compared to the full equation, this model has the same linear part. The precise nonlinearity is not so important in dispersive analysis; in particular, the L^2 conservation satisfied by the solution U does not play a role here.

The specific dispersion relation $\Lambda(\xi) = \sqrt{|\xi| + |\xi|^3}$ in (1.15) is, however, important. It is radial and has stationary points when $|\xi| = \gamma_0 := (2/\sqrt{3} - 1)^{1/2} \approx 0.393$ (see Figure 1 below). As a result, linear solutions can only have $|t|^{-5/6}$ pointwise decay, i.e.

$$\|e^{it\Lambda}\phi\|_{L^\infty} \approx |t|^{-5/6},$$

even for Schwartz functions ϕ whose Fourier transforms do not vanish on the sphere $\{|\xi| = \gamma_0\}$.

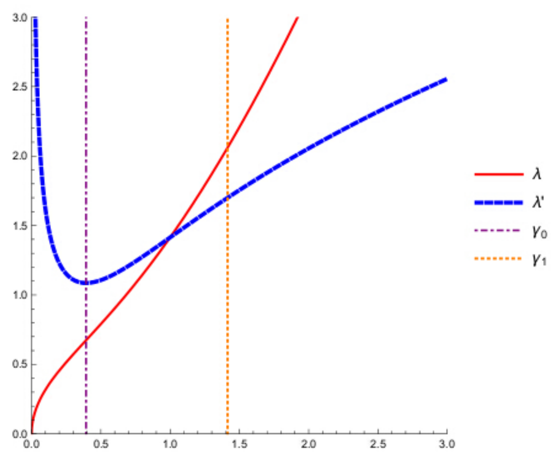

FIGURE 1. The curves represent the dispersion relation $\lambda(r) = \sqrt{r^3 + r}$ and the group velocity λ', for $g = 1 = \sigma$. The frequency γ_1 corresponds to the space-time resonant sphere. Notice that while the slower decay at γ_0 is due to some degeneracy in the linear problem, γ_1 is unremarkable from the point of view of the linear dispersion.

[2]More precisely, the only time resonances are at the 0 frequency, but they are cancelled by a suitable null structure. Some additional ideas are needed in the case of capillary waves [48] where certain singularities arise. Morevoer, new ideas, which exploit the Hamiltonian structure of the system as in [46], are needed to prove global (as opposed to almost-global) regularity.

In the case of the evolution (1.15), the analogue of Proposition 1.3 is the following partial bootstrap estimate for the Z norm:

$$\begin{aligned} &\text{if } \sup_{t\in[0,T]}\big[(1+t)^{-\delta^2}\|U(t)\|_{H^{N_0}\cap H^{N_1,N_3}_\Omega}+\|e^{it\Lambda}U(t)\|_Z\big]\leq\varepsilon_1\\ &\text{then } \sup_{t\in[0,T]}\|e^{it\Lambda}U(t)\|_Z\lesssim\varepsilon_0+\varepsilon_1^2. \end{aligned}\tag{1.16}$$

This can be complemented by a suitable energy estimate to close the full bootstrap argument.

The first main issue is to define an effective Z norm. We use the Duhamel formula, written in terms of the profile $u=u_+=e^{it\Lambda}U$, $u_-=\overline{u}$,

$$\widehat{u}(\xi,t)=\widehat{u}(\xi,0)+\sum_{+,-}\int_0^t\int_{\mathbb{R}^2}e^{is[\Lambda(\xi)\mp\Lambda(\xi-\eta)\mp\Lambda(\eta)]}m_{\pm\pm}(\xi,\eta)\widehat{u_\pm}(\xi-\eta,s)\widehat{u_\pm}(\eta,s)\,d\eta ds,\tag{1.17}$$

where the sum is taken over choices of the signs $+,-$, and $m_{\pm\pm}$ are suitable smooth multipliers.

1.3.2. *Space-time resonances and the Z-norm.* The idea is to estimate the function $\widehat{u}$ using the Duhamel formula (1.17), by integrating by parts either in s or in η. According to [36], the main contribution is expected to come from the *set of space-time resonances* (the stationary points of the integral)

$$\mathcal{SR}:=\{(\xi,\eta):\,\Phi(\xi,\eta)=0,\,(\nabla_\eta\Phi)(\xi,\eta)=0,\,m(\xi,\eta)\neq 0\},\tag{1.18}$$

where

$$\Phi(\xi,\eta)=\Lambda(\xi)\mp\Lambda(\xi-\eta)\mp\Lambda(\eta)$$

is the so-called phase or modulation, and $m=m_{\pm\pm}$. In our case, space-time resonances are present only for the phase $\Phi(\xi,\eta)=\Lambda(\xi)-\Lambda(\xi-\eta)-\Lambda(\eta)$ and the space-time resonant set is

$$\{(\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}^2:|\xi|=\gamma_1=\sqrt{2},\,\eta=\xi/2\}.\tag{1.19}$$

Moreover, the space-time resonant points are *nondegenerate* (according to the terminology introduced in [44]), in the sense that the Hessian of the matrix $\nabla^2_{\eta\eta}\Phi(\xi,\eta)$ is non-singular at these points. To gain intuition, consider the first iteration of the formula (1.17), i.e. assume that the functions $u_\pm$ in the right-hand side are Schwartz function supported at frequency ≈ 1, independent of s. Assume that $s\approx 2^m$. Integration by parts in η and s shows that the main contribution comes from a small neighborhood of the stationary points where $|\nabla_\eta\Phi(\xi,\eta)|\leq 2^{-m/2+\delta m}$ and $|\Phi(\xi,\eta)|\leq 2^{-m+\delta m}$, up to negligible errors. Thus, the main contribution comes from space-time resonant points as in (1.18). A simple calculation shows that the main contribution to the second iteration is of the type

$$\widehat{u_{(2)}}(\xi)\approx c(\xi)\varphi_{\leq -m}(|\xi|-\gamma_1),$$

up to smaller contributions, where we have also ignored factors of $2^{\delta m}$, and c is smooth.

We are now ready to describe more precisely the crucial choice of the Z space. We use the framework introduced by two of the authors in [43], which was later refined by some of the authors in [44, 38, 31]. The idea is to decompose the profile as a superposition of atoms, using localization in both space and frequency,

$$f=\sum\nolimits_{j,k}Q_{jk}f,\qquad Q_{jk}f=\varphi_j(x)\cdot P_kf(x).$$

The Z norm is then defined by measuring suitably every atom.

In our case, the Z space should include all Schwartz functions. It also has to include functions like $\widehat{u}(\xi)=\varphi_{\leq -m}(|\xi|-\gamma_1)$, due to the considerations above, for any m large. It should measure localization in both space and frequency, and be strong enough, at least, to recover the $t^{-5/6+}$ pointwise decay. We define

$$\|f\|_{Z_1} = \sup_{j,k} 2^j \cdot \| ||\xi| - \gamma_1|^{1/2} \widehat{Q_{jk} f}(\xi) \|_{L^2_\xi} \tag{1.20}$$

up to small corrections (see Definition 2.1 for the precise formula, including the small but important δ-corrections), and then we define the Z norm by applying a suitable number of vector-fields D and Ω.

We remark that the dispersive analysis in the Z norm in this paper is more subtle than in the earlier papers mentioned above. It has some similarities to the analysis in the recent paper [31] of three of the authors on the Euler–Maxwell system in 2D, but it is more involved because of the presence of the frequencies of slow decay $|\xi| = \gamma_0$.

To illustrate how this analysis works in our problem, we consider the contribution of the integral over $s \approx 2^m \gg 1$ in (1.17), and assume that the frequencies are ≈ 1.

1.3.3. *Small modulations.* Start with the contribution of small modulations,

$$\widehat{u'}(\xi) := \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2} \varphi_{\leq l}(\Phi(\xi,\eta)) e^{is\Phi(\xi,\eta)} m_{++}(\xi,\eta) \widehat{u}(\xi-\eta,s) \widehat{u}(\eta,s)\, d\eta ds, \tag{1.21}$$

where $l = -m + \delta m$ (δ is a small constant) and $q_m(s)$ restricts the time integral to $s \approx 2^m$, and, for simplicity, we consider only the phase $\Phi(\xi,\eta) = \Lambda(\xi) - \Lambda(\xi-\eta) - \Lambda(\eta)$. In this case the considerations above, leading to the definition of the Z norm, are still relevant: one can integrate by parts in η, identify the main contributions as coming from small $2^{-m/2+\delta m}$ neighborhoods of the stationary points, and estimate these contributions in the Z norm.

1.3.4. *Higher modulations and iterated resonances.* Consider now the contributions of the modulations of size 2^l, $l \geq -m + \delta m$. We start from a formula similar to (1.21) and integrate by parts in s. The main case is when d/ds hits one of the profiles u. Using again the equation (see (1.17)), we have to estimate cubic expressions of the form

$$\begin{aligned}\widehat{h_{m,l}}(\xi) := \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2\times\mathbb{R}^2} &\frac{\varphi_l(\Phi(\xi,\eta))}{\Phi(\xi,\eta)} e^{is\Phi(\xi,\eta)} m_{++}(\xi,\eta) \widehat{u}(\xi-\eta,s) \\ &\times e^{is\Phi'(\eta,\sigma)} n(\eta,\sigma) \widehat{\overline{u}}(\eta-\sigma,s) \widehat{u}(\sigma,s)\, d\eta d\sigma ds,\end{aligned} \tag{1.22}$$

where $\Phi'(\eta,\sigma) = \Lambda(\eta) + \Lambda(\eta-\sigma) - \Lambda(\sigma)$. Assume again that the three functions u are Schwartz functions supported at frequency ≈ 1. We combine Φ and Φ' into a combined phase,

$$\widetilde{\Phi}(\xi,\eta,\sigma) := \Phi(\xi,\eta) + \Phi'(\eta,\sigma) = \Lambda(\xi) - \Lambda(\xi-\eta) + \Lambda(\eta-\sigma) - \Lambda(\sigma).$$

We need to estimate $h_{m,l}$ according to the Z_1 norm. Integration by parts in ξ (approximate finite speed of propagation) shows that the main contribution in $Q_{jk} h'_{m,l}$ is when $2^j \lesssim 2^m$.

We have two main cases: if l is not too small, say $l \geq -m/14$, then we use first multilinear Hölder-type estimates, placing two of the factors $e^{is\Lambda}u$ in L^∞ and one in L^2, together with analysis of the stationary points of $\widetilde{\Phi}$ in η and σ. This suffices is most cases, except when all the variables are close to γ_0. In this case we need a key algebraic property, of the form

$$\text{if} \quad \nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma) = 0 \quad \text{and} \quad \widetilde{\Phi}(\xi,\eta,\sigma) = 0 \quad \text{then} \quad \nabla_\xi \widetilde{\Phi}(\xi,\eta,\sigma) = 0, \tag{1.23}$$

if $|\xi-\eta|, |\eta-\sigma|, |\sigma|$ are all close to γ_0.

On the other and, if l is very small, $l \leq -m/14$, then the denominator $\Phi(\xi,\eta)$ in (1.22) is dangerous. However, we can restrict to small neighborhoods of the stationary points of $\widetilde{\Phi}$ in η

and σ, thus to space-time resonances. This is the most difficult case in the dispersive analysis. We need to rely on one more algebraic property, of the form

$$\text{if} \quad \nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma)=0 \quad \text{and} \quad |\Phi(\xi,\eta)|+|\Phi'(\eta,\sigma)|\ll 1 \quad \text{then} \quad \nabla_{\xi}\widetilde{\Phi}(\xi,\eta,\sigma)=0. \tag{1.24}$$

See Lemma 7.6 for the precise quantitative claims for both (1.23) and (1.24).

The point of both (1.23) and (1.24) is that in the resonant region for the cubic integral we have that $\nabla_\xi\widetilde{\Phi}(\xi,\eta,\sigma)=0$. We call them *slow propagation of iterated resonances properties*; as a consequence the resulting function is essentially supported when $|x|\ll 2^m$, using the approximate finite speed of propagation. This gain is reflected in the factor 2^j in (1.20).

We remark that the analogous property for quadratic resonances

$$\text{if} \quad \nabla_\eta\Phi(\xi,\eta)=0 \quad \text{and} \quad \Phi(\xi,\eta)=0 \quad \text{then} \quad \nabla_\xi\Phi(\xi,\eta)=0$$

fails. In fact, in our case $|\nabla_\xi\Phi(\xi,\eta)|\approx 1$ on the space-time resonant set.

In proving (1.16), there are, of course, many cases to consider. The full proof covers sections 4 and 5. The type of arguments presented above are typical in the proof: we decompose our profiles in space and frequency, localize to small sets in the frequency space, keeping track in particular of the special frequencies of size $\gamma_0,\gamma_1,\gamma_1/2,2\gamma_0$, use integration by parts in ξ to control the location of the output, and use multilinear Hölder-type estimates to bound L^2 norms.

1.3.5. *The time derivative of the profile and scattering in the Z norm.* The considerations above and (1.17) can also be used to justify the approximate formula

$$(\partial_t\widehat{u})(\xi,t)\approx(1/t)\sum\nolimits_j g_j(\xi)e^{it\Phi(\xi,\eta_j(\xi))}+\text{lower order terms}, \tag{1.25}$$

as $t\to\infty$, where $\eta_j(\xi)$ denote the stationary points where $\nabla_\eta\Phi(\xi,\eta_j(\xi))=0$. This approximate formula, which holds at least as long as the stationary points are nondegenerate, is consistent with the asymptotic behavior of the solution described in Remark 1.2 (ii). Indeed, at space-time resonances $\Phi(\xi,\eta_j(\xi))=0$, which leads to logarithmic growth for $\widehat{u}(\xi,t)$, while away from these space-time resonances the oscillation of $e^{it\Phi(\xi,\eta_j(\xi))}$ leads to convergence.

1.3.6. *Additional remarks.* We list below some other issues one needs to keep in mind in the proof of the main theorem.

(1) The very low frequencies $|\xi|\ll 1$ play an important role in all the global results for water wave systems. These frequencies are not captured in the model (1.15). In our case, there is a suitable *null structure*: the multipliers of the quadratic terms are bounded by $|\xi|\min(|\eta|,|\xi-\eta|)^{1/2}$, see (2.21), which is an important ingredient in the proof of Proposition 1.3.

(2) It is important to propagate energy control of both high Sobolev norms and weighted norms using many copies of the rotation vector-field Ω. This is done in the companion paper [32], see also [30, 31]. As a result, the values of N_0 and N_1 in (1.12) are large. Because of this control, we can assume that the profiles in the dispersive part of the argument are almost radial and located at frequencies $\lesssim 1$. The linear estimates (in Lemma 3.6) and many of the bilinear estimates are much stronger because of this almost radiality property.

(3) At many stages it is important that the four spheres, the sphere of slow decay $\{|\xi|=\gamma_0\}$, the sphere of space-time resonant outputs $\{|\xi|=\gamma_1\}$, and the sphere of space-time resonant inputs $\{|\xi|=\gamma_1/2\}$, and the sphere $\{|\xi|=2\gamma_0\}$ are all separated from each other. Such separation conditions played an important role also in other papers, such as [35, 38, 31].

1.4. **Organization.** The rest of the paper is organized as follows: in section 2 we summarize the main definitions and notation in the paper, and state the main Proposition 2.2.

In sections 3–5 we prove Proposition 2.2. The key components of the proof are Lemma 3.4 (integration by parts using Ω), Lemma 3.6 (linear estimates involving the Z-norm), the precise analysis of the time derivative of the profile in Lemmas 4.1–4.2, and the analysis of the Duhamel formula, divided in several cases, in Lemmas 5.4–5.8.

In section 6 we show that Proposition 1.3 follows from Proposition 2.2 and a suitable expansion of the Dirichlet–Neumann operator, which is proved in section 9 in [32].

In section 7 we collect estimates on the dispersion relation and the phase functions. The main results are Proposition 7.2 (structure of the resonance sets), Proposition 7.4 (bounds on sublevel sets), and Lemma 7.6 (slow propagation of iterated resonances).

2. Setup and the main proposition

2.1. **Definitions and notation.** We summarize in this subsection some of the main definitions and notation we use in the paper.

2.1.1. *Fourier multipliers and the Z norm.* We start by defining several multipliers that allow us to localize in the Fourier space. We fix $\varphi : \mathbb{R} \to [0,1]$ an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For simplicity of notation, we also let $\varphi : \mathbb{R}^2 \to [0,1]$ denote the corresponding radial function on $\mathbb{R}^2$. Let

$$\varphi_k(x) := \varphi(|x|/2^k) - \varphi(|x|/2^{k-1}) \quad \text{for any} \quad k \in \mathbb{Z}, \qquad \varphi_I := \sum_{m \in I \cap \mathbb{Z}} \varphi_m \text{ for any } I \subseteq \mathbb{R},$$

$$\varphi_{\leq B} := \varphi_{(-\infty,B]}, \quad \varphi_{\geq B} := \varphi_{[B,\infty)}, \quad \varphi_{<B} := \varphi_{(-\infty,B)}, \quad \varphi_{>B} := \varphi_{(B,\infty)}.$$

For any $a < b \in \mathbb{Z}$ and $j \in [a,b] \cap \mathbb{Z}$ let

$$\varphi_j^{[a,b]} := \begin{cases} \varphi_j & \text{if } a < j < b, \\ \varphi_{\leq a} & \text{if } j = a, \\ \varphi_{\geq b} & \text{if } j = b. \end{cases} \tag{2.1}$$

For any $x \in \mathbb{Z}$ let $x_+ = \max(x,0)$ and $x_- := \min(x,0)$. Let

$$\mathcal{J} := \{(k,j) \in \mathbb{Z} \times \mathbb{Z}_+ : \, k + j \geq 0\}.$$

For any $(k,j) \in \mathcal{J}$ let

$$\widetilde{\varphi}_j^{(k)}(x) := \begin{cases} \varphi_{\leq -k}(x) & \text{if } k+j = 0 \text{ and } k \leq 0, \\ \varphi_{\leq 0}(x) & \text{if } j = 0 \text{ and } k \geq 0, \\ \varphi_j(x) & \text{if } k+j \geq 1 \text{ and } j \geq 1, \end{cases}$$

and notice that, for any $k \in \mathbb{Z}$ fixed, $\sum_{j \geq -\min(k,0)} \widetilde{\varphi}_j^{(k)} = 1$.

Let P_k, $k \in \mathbb{Z}$, denote the Littlewood–Paley projection operators defined by the Fourier multipliers $\xi \to \varphi_k(\xi)$. Let $P_{\leq B}$ (respectively $P_{>B}$) denote the operators defined by the Fourier multipliers $\xi \to \varphi_{\leq B}(\xi)$ (respectively $\xi \to \varphi_{>B}(\xi)$). For $(k,j) \in \mathcal{J}$ let Q_{jk} denote the operator

$$(Q_{jk} f)(x) := \widetilde{\varphi}_j^{(k)}(x) \cdot P_k f(x). \tag{2.2}$$

In view of the uncertainty principle the operators Q_{jk} are relevant only when $2^j 2^k \gtrsim 1$, which explains the definitions above. For $k, k_1, k_2 \in \mathbb{Z}$ let

$$\mathcal{D}_{k,k_1,k_2} := \{(\xi,\eta) \in (\mathbb{R}^2)^2 : |\xi| \in [2^{k-4}, 2^{k+4}],\ |\eta| \in [2^{k_2-4}, 2^{k_2+4}],\ |\xi-\eta| \in [2^{k_1-4}, 2^{k_1+4}]\}. \tag{2.3}$$

Let $\lambda(r) = \sqrt{|r| + |r|^3}$, $\Lambda(\xi) = \sqrt{|\xi| + |\xi|^3} = \lambda(|\xi|)$, $\Lambda : \mathbb{R}^2 \to [0, \infty)$. Let

$$\mathcal{U}_+ := \mathcal{U}, \quad \mathcal{U}_- := \overline{\mathcal{U}}, \quad \mathcal{V}(t) = \mathcal{V}_+(t) := e^{it\Lambda}\mathcal{U}(t), \quad \mathcal{V}_-(t) := e^{-it\Lambda}\mathcal{U}_-(t). \tag{2.4}$$

Let $\Lambda_+ = \Lambda$ and $\Lambda_- := -\Lambda$. For $\sigma, \mu, \nu \in \{+,-\}$, we define the associated phase functions

$$\begin{aligned} &\Phi_{\sigma\mu\nu}(\xi,\eta) := \Lambda_\sigma(\xi) - \Lambda_\mu(\xi-\eta) - \Lambda_\nu(\eta), \\ &\widetilde{\Phi}_{\sigma\mu\nu\beta}(\xi,\eta,\sigma) := \Lambda_\sigma(\xi) - \Lambda_\mu(\xi-\eta) - \Lambda_\nu(\eta-\sigma) - \Lambda_\beta(\sigma). \end{aligned} \tag{2.5}$$

For any set S let $\mathbf{1}_S$ denote its characteristic function. We will use two sufficiently large constants $\mathcal{D} \gg \mathcal{D}_1 \gg 1$ ($\mathcal{D}_1$ is only used in section 7 to prove properties of the phase functions).

Let $\gamma_0 := \sqrt{\frac{2\sqrt{3}-3}{3}}$ denote the radius of the sphere of slow decay and $\gamma_1 := \sqrt{2}$ denote the radius of the space-time resonant sphere. For $n \in \mathbb{Z}$, $I \subseteq \mathbb{R}$, and $\gamma \in (0,\infty)$ we define

$$\begin{aligned} &\widehat{A_{n,\gamma} f}(\xi) := \varphi_{-n}(2^{100}||\xi| - \gamma|) \cdot \widehat{f}(\xi), \\ &A_{I,\gamma} := \sum_{n\in I} A_{n,\gamma}, \quad A_{\le B,\gamma} := A_{(-\infty,B],\gamma}, \quad A_{\ge B,\gamma} := A_{[B,\infty),\gamma}. \end{aligned} \tag{2.6}$$

Given an integer $j \ge 0$ we define the operators $A^{(j)}_{n,\gamma}$, $n \in \{0,\dots,j+1\}$, $\gamma \ge 2^{-50}$, by

$$A^{(j)}_{j+1,\gamma} := \sum_{n'\ge j+1} A_{n',\gamma}, \qquad A^{(j)}_{0,\gamma} := \sum_{n'\le 0} A_{n',\gamma}, \qquad A^{(j)}_{n,\gamma} := A_{n,\gamma} \ \text{ if } \ 1 \le n \le j. \tag{2.7}$$

These operators localize to thin anuli of width 2^{-n} around the circle of radius γ. Most of the times, for us $\gamma = \gamma_0$ or $\gamma = \gamma_1$. We are now ready to define the main Z norm.

Definition 2.1. *Assume that δ, N_0, N_1, N_4 are as in Theorem 1.1. We define*

$$Z_1 := \{f \in L^2(\mathbb{R}^2) : \|f\|_{Z_1} := \sup_{(k,j)\in\mathcal{J}} \|Q_{jk} f\|_{B_j} < \infty\}, \tag{2.8}$$

where

$$\|g\|_{B_j} := 2^{(1-50\delta)j} \sup_{0\le n\le j+1} 2^{-(1/2-49\delta)n} \|A^{(j)}_{n,\gamma_1} g\|_{L^2}. \tag{2.9}$$

Then we define, with $D^\alpha := \partial_1^{\alpha^1}\partial_2^{\alpha^2}$, $\alpha = (\alpha^1, \alpha^2)$,

$$Z := \big\{f \in L^2(\mathbb{R}^2) : \|f\|_Z := \sup_{2m+|\alpha|\le N_1+N_4,\, m\le N_1/2+20} \|D^\alpha \Omega^m f\|_{Z_1} < \infty\big\}. \tag{2.10}$$

We remark that the Z norm is used to estimate the *linear profile* of the solution, which is $\mathcal{V}(t) := e^{it\Lambda}\mathcal{U}(t)$, not the solution itself.

2.2. **The Duhamel formula and the main proposition.** In this subsection we start the proof of Proposition 1.3. With $\mathcal{U} = \langle\nabla\rangle h + i|\nabla|^{1/2}\phi$, assume that $\mathcal{U}$ is a solution of the equation

$$(\partial_t + i\Lambda)\mathcal{U} = \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_{\ge 4}, \tag{2.11}$$

on some time interval $[0,T]$, $T \ge 1$, where $\mathcal{N}_2$ is a quadratic nonlinearity in $\mathcal{U}, \overline{\mathcal{U}}$, $\mathcal{N}_3$ is a cubic nonlinearity, and $\mathcal{N}_{\ge 4}$ is a higher order nonlinearity. Such an equation will be verified below, see section 6, starting from the main system (1.4) and using the expansion of the Dirichlet–Neumann operator. The nonlinearity $\mathcal{N}_2$ is of the form

$$\mathcal{N}_2 = \sum_{\mu,\nu\in\{+,-\}} \mathcal{N}_{\mu\nu}(\mathcal{U}_\mu,\mathcal{U}_\nu), \qquad \big(\mathcal{F}\mathcal{N}_{\mu\nu}(f,g)\big)(\xi) = \int_{\mathbb{R}^2} \mathfrak{m}_{\mu\nu}(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta, \tag{2.12}$$

where $\mathcal{U}_+ = \mathcal{U}$ and $\mathcal{U}_- = \overline{\mathcal{U}}$. The cubic nonlinearity is of the form

$$\begin{aligned} \mathcal{N}_3 &= \sum_{\mu,\nu,\beta\in\{+,-\}} \mathcal{N}_{\mu\nu\beta}(\mathcal{U}_\mu,\mathcal{U}_\nu,\mathcal{U}_\beta),\\ \left(\mathcal{F}\mathcal{N}_{\mu\nu\beta}(f,g,h)\right)(\xi) &= \int_{\mathbb{R}^2} \mathfrak{n}_{\mu\nu\beta}(\xi,\eta,\sigma)\widehat{f}(\xi-\eta)\widehat{g}(\eta-\sigma)\widehat{h}(\sigma)\,d\eta. \end{aligned} \tag{2.13}$$

The multipliers $\mathfrak{m}_{\mu\nu}$ and $\mathfrak{n}_{\mu\nu\beta}$ satisfy suitable symbol-type estimates. We define the profiles $\mathcal{V}_\sigma(t) = e^{it\Lambda_\sigma}\mathcal{U}_\sigma(t)$, $\sigma \in \{+,-\}$, as in (1.10). The Duhamel formula is

$$(\partial_t\widehat{\mathcal{V}})(\xi,s) = e^{is\Lambda(\xi)}\widehat{\mathcal{N}_2}(\xi,s) + e^{is\Lambda(\xi)}\widehat{\mathcal{N}_3}(\xi,s) + e^{is\Lambda(\xi)}\widehat{\mathcal{N}_{\geq 4}}(\xi,s), \tag{2.14}$$

or, in integral form,

$$\widehat{\mathcal{V}}(\xi,t) = \widehat{\mathcal{V}}(\xi,0) + \widehat{W_2}(\xi,t) + \widehat{W_3}(\xi,t) + \int_0^t e^{is\Lambda(\xi)}\widehat{\mathcal{N}_{\geq 4}}(\xi,s)\,ds, \tag{2.15}$$

where, with the definitions in (2.5),

$$\widehat{W_2}(\xi,t) := \sum_{\mu,\nu\in\{+.-\}} \int_0^t \int_{\mathbb{R}^2} e^{is\Phi_{+\mu\nu}(\xi,\eta)}\mathfrak{m}_{\mu\nu}(\xi,\eta)\widehat{\mathcal{V}_\mu}(\xi-\eta,s)\widehat{\mathcal{V}_\nu}(\eta,s)\,d\eta ds, \tag{2.16}$$

$$\begin{aligned} \widehat{W_3}(\xi,t) := \sum_{\mu,\nu,\beta\in\{+.-\}} \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} & e^{is\widetilde{\Phi}_{+\mu\nu\beta}(\xi,\eta,\sigma)}\mathfrak{n}_{\mu\nu\beta}(\xi,\eta,\sigma)\\ &\times \widehat{\mathcal{V}_\mu}(\xi-\eta,s)\widehat{\mathcal{V}_\nu}(\eta-\sigma,s)\widehat{\mathcal{V}_\beta}(\sigma,s)\,d\eta d\sigma ds. \end{aligned} \tag{2.17}$$

The vector-field Ω acts on the quadratic part of the nonlinearity according to the identity

$$\Omega_\xi\widehat{\mathcal{N}_2}(\xi,s) = \sum_{\mu,\nu\in\{+,-\}} \int_{\mathbb{R}^2} (\Omega_\xi+\Omega_\eta)\big[\mathfrak{m}_{\mu\nu}(\xi,\eta)\widehat{\mathcal{U}_\mu}(\xi-\eta,s)\widehat{\mathcal{U}_\nu}(\eta,s)\big]\,d\eta.$$

A similar formula holds for $\Omega_\xi\widehat{\mathcal{N}_3}(\xi,s)$. Therefore, for $1 \leq a \leq N_1$, letting $\mathfrak{m}^b_{\mu\nu} := (\Omega_\xi+\Omega_\eta)^b\mathfrak{m}_{\mu\nu}$ and $\mathfrak{n}^b_{\mu\nu\beta} := (\Omega_\xi+\Omega_\eta+\Omega_\sigma)^b\mathfrak{n}_{\mu\nu\beta}$ we have

$$\Omega^a_\xi(\partial_t\widehat{\mathcal{V}})(\xi,s) = e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}_2}(\xi,s) + e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}_3}(\xi,s) + e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}_{\geq 4}}(\xi,s), \tag{2.18}$$

where

$$\begin{aligned} e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}_2}(\xi,s) = \sum_{\mu,\nu\in\{+,-\}}\sum_{a_1+a_2+b=a} \int_{\mathbb{R}^2} & e^{is\Phi_{+\mu\nu}(\xi,\eta)}\mathfrak{m}^b_{\mu\nu}(\xi,\eta)\\ &\times (\Omega^{a_1}\widehat{\mathcal{V}_\mu})(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}_\nu})(\eta,s)\,d\eta \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}_3}(\xi,s) = \sum_{\mu,\nu,\beta\in\{+,-\}}\sum_{a_1+a_2+a_3+b=a} \int_{\mathbb{R}^2\times\mathbb{R}^2} & e^{is\widetilde{\Phi}_{+\mu\nu\beta}(\xi,\eta,\sigma)}\mathfrak{n}^b_{\mu\nu\beta}(\xi,\eta,\sigma)\\ &\times (\Omega^{a_1}\widehat{\mathcal{V}_\mu})(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}_\nu})(\eta-\sigma,s)(\Omega^{a_3}\widehat{\mathcal{V}_\beta})(\sigma,s)\,d\eta d\sigma. \end{aligned} \tag{2.20}$$

To state our main proposition we need to make suitable assumptions on the nonlinearities $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_{\geq 4}$. Recall the class of symbols S^∞ defined in (3.5).

• Concerning the multipliers defining $\mathcal{N}_2$, we assume that $(\Omega_\xi + \Omega_\eta)m(\xi,\eta) \equiv 0$ and

$$\begin{aligned} \|m^{k,k_1,k_2}\|_{S^\infty} &\lesssim 2^k 2^{\min(k_1,k_2)/2}, \\ \|D^\alpha_\eta m^{k,k_1,k_2}\|_{L^\infty} &\lesssim_{|\alpha|} 2^{(|\alpha|+3/2)\max(|k_1|,|k_2|)}, \\ \|D^\alpha_\xi m^{k,k_1,k_2}\|_{L^\infty} &\lesssim_{|\alpha|} 2^{(|\alpha|+3/2)\max(|k|,|k_1|,|k_2|)}, \end{aligned} \tag{2.21}$$

for any $k,k_1,k_2 \in \mathbb{Z}$ and $m \in \{\mathfrak{m}_{\mu\nu} : \mu,\nu \in \{+,-\}\}$, where

$$m^{k,k_1,k_2}(\xi,\eta) := m(\xi,\eta)\cdot\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta).$$

• Concerning the multipliers defining $\mathcal{N}_3$, we assume that $(\Omega_\xi + \Omega_\eta + \Omega_\sigma)n(\xi,\eta,\sigma) \equiv 0$ and

$$\begin{aligned} \|n^{k,k_1,k_2,k_3}\|_{S^\infty} &\lesssim 2^{\min(k,k_1,k_2,k_3)/2} 2^{3\max(k,k_1,k_2,k_3,0)}, \\ \|D^\alpha_{\eta,\sigma} n^{k,k_1,k_2,k_3;l}\|_{L^\infty} &\lesssim_{|\alpha|} 2^{|\alpha|\max(|k_1|,|k_2|,|k_3|,|l|)} 2^{(7/2)\max(|k_1|,|k_2|,|k_3|)}, \\ \|D^\alpha_\xi n^{k,k_1,k_2,k_3}\|_{L^\infty} &\lesssim_{|\alpha|} 2^{(|\alpha|+7/2)\max(|k|,|k_1|,|k_2|,|k_3|)}, \end{aligned} \tag{2.22}$$

for any $k,k_1,k_2,k_3,l \in \mathbb{Z}$ and $n \in \{\mathfrak{n}_{\mu\nu\beta} : \mu,\nu \in \{+,-\}\}$, where

$$\begin{aligned} n^{k,k_1,k_2,k_3}(\xi,\eta,\sigma) &:= n(\xi,\eta,\sigma)\cdot\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta-\sigma)\varphi_{k_3}(\sigma), \\ n^{k,k_1,k_2,k_3;l}(\xi,\eta,\sigma) &:= n(\xi,\eta,\sigma)\cdot\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta-\sigma)\varphi_{k_3}(\sigma)\varphi_l(\eta). \end{aligned}$$

Our main result is the following:

Proposition 2.2. *Assume that $\mathcal{U}$ is a solution of the equation*

$$(\partial_t + i\Lambda)\mathcal{U} = \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_{\geq 4}, \tag{2.23}$$

on some time interval $[0,T]$, $T \geq 1$, with initial data $\mathcal{U}_0$. Define, as before, $\mathcal{V}(t) = e^{it\Lambda}\mathcal{U}(t)$. With δ as in Definition 2.1, assume that

$$\|\mathcal{U}_0\|_{H^{N_0}\cap H^{N_1,N_3}_\Omega} + \|\mathcal{V}_0\|_Z \leq \varepsilon_0 \ll 1 \tag{2.24}$$

and

$$\begin{aligned} &(1+t)^{-\delta^2}\|\mathcal{U}(t)\|_{H^{N_0}\cap H^{N_1,N_3}_\Omega} + \|\mathcal{V}(t)\|_Z \leq \varepsilon_1 \ll 1, \\ &(1+t)^2\|\mathcal{N}_{\geq 4}(t)\|_{H^{N_0-N_3}\cap H^{N_1,0}_\Omega} + (1+t)^{1+\delta^2}\|e^{it\Lambda}\mathcal{N}_{\geq 4}(t)\|_Z \leq \varepsilon_1^2, \end{aligned} \tag{2.25}$$

for all $t \in [0,T]$. Moreover, assume that the nonlinearities $\mathcal{N}_2$ and $\mathcal{N}_3$ satisfy (2.12)–(2.13) *and* (2.21)–(2.22)*. Then, for any $t \in [0,T]$*

$$\|\mathcal{V}(t)\|_Z \lesssim \varepsilon_0 + \varepsilon_1^2. \tag{2.26}$$

We will show in section 6 below how to use this proposition and a suitable expansion of the Dirichlet–Neumann operator to complete the proof of the main Proposition 1.3.

3. Some lemmas

In this section we collect several important lemmas which are used often in the proofs in the next two sections. Let $\Phi = \Phi_{\sigma\mu\nu}$ as in (2.5).

3.1. **Fourier multipliers and Schur's lemma.** We will work with bilinear and trilinear multipliers. Many of the simpler estimates can be proved using the following basic result (see [46, Lemma 5.2] for the proof).

Lemma 3.1. *(i) Assume $l \geq 2$, $f_1, \dots, f_l, f_{l+1} \in L^2(\mathbb{R}^2)$, and $m : (\mathbb{R}^2)^l \to \mathbb{C}$ is a continuous compactly supported function. Then*

$$\begin{aligned}\Big| \int_{(\mathbb{R}^2)^l} m(\xi_1, \dots, \xi_l)\widehat{f_1}(\xi_1) \cdot \ldots \cdot \widehat{f_l}(\xi_l) \cdot \widehat{f_{l+1}}(-\xi_1 - \ldots - \xi_l)\, d\xi_1 \dots d\xi_l \Big| \\ \lesssim \big\| \mathcal{F}^{-1}(m) \big\|_{L^1} \|f_1\|_{L^{p_1}} \cdot \ldots \cdot \|f_{l+1}\|_{L^{p_{l+1}}},\end{aligned} \tag{3.1}$$

for any exponents $p_1, \dots p_{l+1} \in [1, \infty]$ satisfying $\frac{1}{p_1} + \ldots + \frac{1}{p_{l+1}} = 1$.

(ii) Assume $l \geq 2$ and L_m is the multilinear operator defined by

$$\mathcal{F}\{L_m[f_1, \dots, f_l]\}(\xi) = \int_{(\mathbb{R}^2)^{l-1}} m(\xi, \eta_2, \dots, \eta_l)\widehat{f_1}(\xi - \eta_2) \cdot \ldots \cdot \widehat{f_{l-1}}(\eta_{l-1} - \eta_l)\widehat{f_l}(\eta_l)\, d\eta_2 \dots d\eta_l.$$

Then, for any exponents $p, q_1, \dots q_l \in [1, \infty]$ satisfying $\frac{1}{q_1} + \ldots + \frac{1}{q_l} = \frac{1}{p}$, we have

$$\big\| L_m[f_1, \dots, f_l] \big\|_{L^p} \lesssim \big\| \mathcal{F}^{-1}(m) \big\|_{S^\infty} \|f_1\|_{L^{q_1}} \cdot \ldots \cdot \|f_l\|_{L^{q_l}}. \tag{3.2}$$

Given a multiplier $m : (\mathbb{R}^2)^2 \to \mathbb{C}$, we define the bilinear operator M by the formula

$$\mathcal{F}[M[f, g])](\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} m(\xi, \eta)\widehat{f}(\xi - \eta)\widehat{g}(\eta)\, d\eta. \tag{3.3}$$

With $\Omega = x_1\partial_2 - x_2\partial_1$, we notice the formula

$$\Omega M[f, g] = M[\Omega f, g] + M[f, \Omega g] + \widetilde{M}[f, g], \tag{3.4}$$

where $\widetilde{M}$ is the bilinear operator defined by the multiplier $\widetilde{m}(\xi, \eta) = (\Omega_\xi + \Omega_\eta)m(\xi, \eta)$.

For simplicity of notation, we define the following classes of bilinear multipliers:

$$\begin{aligned}&S^\infty := \{m : (\mathbb{R}^2)^n \to \mathbb{C} : m \text{ continuous and } \|m\|_{S^\infty} := \|\mathcal{F}^{-1}m\|_{L^1} < \infty\}, \\ &S^\infty_\Omega := \{m : (\mathbb{R}^2)^2 \to \mathbb{C} : m \text{ continuous and } \|m\|_{S^\infty_\Omega} := \sup_{l \leq N_1} \|(\Omega_\xi + \Omega_\eta)^l m\|_{S^\infty} < \infty\}.\end{aligned} \tag{3.5}$$

We will often need to analyze bilinear operators more carefully, by localizing in the frequency space. We therefore define, for any symbol m,

$$m^{k,k_1,k_2}(\xi, \eta) := \varphi_k(\xi)\varphi_{k_1}(\xi - \eta)\varphi_{k_2}(\eta)m(\xi, \eta). \tag{3.6}$$

We will often use the Schur's test:

Lemma 3.2 (Schur's lemma)**.** *Consider the operator T given by*

$$Tf(\xi) = \int_{\mathbb{R}^2} K(\xi, \eta)f(\eta)d\eta.$$

Assume that

$$\sup_\xi \int_{\mathbb{R}^2} |K(\xi, \eta)|d\eta \leq K_1, \qquad \sup_\eta \int_{\mathbb{R}^2} |K(\xi, \eta)|d\xi \leq K_2.$$

Then

$$\|Tf\|_{L^2} \lesssim \sqrt{K_1K_2}\|f\|_{L^2}.$$

3.2. **Integration by parts.** In this subsection we state two lemmas that are used in the paper in integration by parts arguments. We start with an oscillatory integral estimate. See [44, Lemma 5.4] for the proof.

Lemma 3.3. *(i) Assume that* $0<\epsilon\leq 1/\epsilon\leq K$, $N\geq 1$ *is an integer, and* $f,g\in C^N(\mathbb{R}^2)$*. Then*

$$\Big|\int_{\mathbb{R}^2}e^{iKf}g\,dx\Big|\lesssim_N (K\epsilon)^{-N}\big[\sum_{|\alpha|\leq N}\epsilon^{|\alpha|}\|D^\alpha_x g\|_{L^1}\big],\tag{3.7}$$

provided that f *is real-valued,*

$$|\nabla_x f|\geq \mathbf{1}_{\mathrm{supp}\,g},\qquad\text{and}\qquad \|D^\alpha_x f\cdot\mathbf{1}_{\mathrm{supp}\,g}\|_{L^\infty}\lesssim_N \epsilon^{1-|\alpha|},\ 2\leq|\alpha|\leq N.\tag{3.8}$$

(ii) Similarly, if $0<\rho\leq 1/\rho\leq K$ *then*

$$\Big|\int_{\mathbb{R}^2}e^{iKf}g\,dx\Big|\lesssim_N (K\rho)^{-N}\big[\sum_{m\leq N}\rho^{m}\|\Omega^m g\|_{L^1}\big],\tag{3.9}$$

provided that f *is real-valued,*

$$|\Omega f|\geq \mathbf{1}_{\mathrm{supp}\,g},\qquad\text{and}\qquad \|\Omega^m f\cdot\mathbf{1}_{\mathrm{supp}\,g}\|_{L^\infty}\lesssim_N \rho^{1-m},\ 2\leq m\leq N.\tag{3.10}$$

We will need another result about integration by parts using the vector-field Ω. This lemma is more subtle. It is needed many times in the next two sections to localize and then estimate bilinear expressions. The point is to be able to take advantage of the fact that our profiles are "almost radial" (due to the bootstrap assumption involving many copies of Ω), and prove that for such functions one has better localization properties than for general functions.

Lemma 3.4. *Assume that* $N\geq 100$, $m\geq 0$, $p,k,k_1,k_2\in\mathbb{Z}$*, and*

$$2^{-k_1}\leq 2^{2m/5},\qquad 2^{\max(k,k_1,k_2)}\leq U\leq U^2\leq 2^{m/10},\qquad U^2+2^{3|k_1|/2}\leq 2^{p+m/2}.\tag{3.11}$$

For some $A\geq\max(1,2^{-k_1})$ *assume that*

$$\begin{split}\sup_{0\leq a\leq 100}\big[\|\Omega^a g\|_{L^2}+\|\Omega^a f\,\|_{L^2}\big]+\sup_{|\alpha|\leq N}A^{-|\alpha|}\|D^\alpha f\|_{L^2}\leq 1,\\ \sup_{\xi,\eta}\sup_{|\alpha|\leq N}(2^{-m/2}|\eta|)^{|\alpha|}|D^\alpha_\eta m(\xi,\eta)|\leq 1.\end{split}\tag{3.12}$$

Fix $\xi\in\mathbb{R}^2$ *and let, for* $t\in[2^m-1,2^{m+1}]$,

$$I_p(f,g):=\int_{\mathbb{R}^2}e^{it\Phi(\xi,\eta)}m(\xi,\eta)\varphi_p(\Omega_\eta\Phi(\xi,\eta))\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)f(\xi-\eta)g(\eta)d\eta.$$

If $2^p\leq U2^{|k_1|/2+100}$ *and* $A\leq 2^mU^{-2}$ *then*

$$|I_p(f,g)|\lesssim_N (2^{p+m})^{-N}U^{2N}\big[2^{m/2}+A2^p\big]^N+2^{-10m}.\tag{3.13}$$

In addition, assuming that $(1+\delta/4)\nu\geq -m$*, the same bound holds when* I_p *is replaced by*

$$\widetilde{I}_p(f,g):=\int_{\mathbb{R}^2}e^{it\Phi(\xi,\eta)}\varphi_\nu(\Phi(\xi,\eta))m(\xi,\eta)\varphi_p(\Omega_\eta\Phi(\xi,\eta))\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)f(\xi-\eta)g(\eta)d\eta.$$

A slightly simpler version of this integration by parts lemma was used recently in [31]. The main interest of this lemma is that we have essentially no assumption on g and very mild assumptions on f.

Proof of Lemma 3.4. We decompose first $f = \mathcal{R}_{\leq m/10}f + [I - \mathcal{R}_{\leq m/10}]f$, $g = \mathcal{R}_{\leq m/10}g + [I - \mathcal{R}_{\leq m/10}]g$, where the operators $\mathcal{R}_{\leq L}$ are defined in polar coordinates by

$$(\mathcal{R}_{\leq L}h)(r\cos\theta, r\sin\theta) := \sum_{n\in\mathbb{Z}} \varphi_{\leq L}(n)h_n(r)e^{in\theta} \quad \text{ if } \quad h(r\cos\theta, r\sin\theta) := \sum_{n\in\mathbb{Z}} h_n(r)e^{in\theta}. \tag{3.14}$$

Since Ω corresponds to $d/d\theta$ in polar coordinates, using (3.12) we have,

$$\big\|[I - \mathcal{R}_{\leq m/10}]f\big\|_{L^2} + \big\|[I - \mathcal{R}_{\leq m/10}]g\big\|_{L^2} \lesssim 2^{-10m}.$$

Therefore, using the Hölder inequality,

$$|I_p\big([I - \mathcal{R}_{\leq m/10}]f, g\big)| + |I_p\big(\mathcal{R}_{\leq m/10}f, [I - \mathcal{R}_{\leq m/10}]g\big)| \lesssim 2^{-10m}.$$

It remains to prove a similar inequality for $I_p := I_p\big(f_1, g_1\big)$, where $f_1 := \varphi_{[k_1-2,k_1+2]} \cdot \mathcal{R}_{\leq m/10}f$, $g_1 := \varphi_{[k_2-2,k_2+2]} \cdot \mathcal{R}_{\leq m/10}g$. It follows from (3.12) and the definitions that

$$\|\Omega^a g_1\|_{L^2} \lesssim_a 2^{am/10}, \qquad \|\Omega^a D^\alpha f_1\|_{L^2} \lesssim_a 2^{am/10}A^{|\alpha|}, \tag{3.15}$$

for any $a \geq 0$ and $|\alpha| \leq N$. Integration by parts gives

$$I_p = c\varphi_k(\xi)\int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta)}\Omega_\eta \left\{\frac{m(\xi,\eta)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)}{t\Omega_\eta\Phi(\xi,\eta)}\varphi_p(\Omega_\eta\Phi(\xi,\eta))f_1(\xi-\eta)g_1(\eta)\right\} d\eta.$$

Iterating N times, we obtain an integrand made of a linear combination of terms like

$$\begin{aligned} e^{it\Phi(\xi,\eta)}\varphi_k(\xi)\left(\frac{1}{t\Omega_\eta\Phi(\xi,\eta)}\right)^N &\times \Omega_\eta^{a_1}\left\{m(\xi,\eta)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)\right\} \\ &\times \Omega_\eta^{a_2}f_1(\xi-\eta)\cdot\Omega_\eta^{a_3}g_1(\eta)\cdot\Omega_\eta^{a_4}\varphi_p(\Omega_\eta\Phi(\xi,\eta))\cdot\frac{\Omega_\eta^{a_5+1}\Phi}{\Omega_\eta\Phi}\cdots\frac{\Omega_\eta^{a_q+1}\Phi}{\Omega_\eta\Phi}, \end{aligned}$$

where $\sum a_i = N$. The desired bound follows from the pointwise bounds

$$\begin{aligned} &\big|\Omega_\eta^a\left\{m(\xi,\eta)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)\right\}\big| \lesssim 2^{am/2}, \\ &\big|\Omega_\eta^a\varphi_p(\Omega_\eta\Phi(\xi,\eta))\big| + \left|\frac{\Omega_\eta^{a+1}\Phi}{\Omega_\eta\Phi}\right| \lesssim U^{2a}2^{am/2}, \end{aligned} \tag{3.16}$$

which hold in the support of the integral, and the L^2 bounds

$$\begin{aligned} \|\Omega_\eta^a g_1(\eta)\|_{L^2} &\lesssim 2^{am/4}, \\ \|\Omega_\eta^a f_1(\xi-\eta)\varphi_k(\xi)\varphi_{k_2}(\eta)\varphi_{\leq p+2}(\Omega_\eta\Phi(\xi,\eta))\|_{L^2_\eta} &\lesssim U^{2a}\big[2^{m/2} + A2^p\big]^a. \end{aligned} \tag{3.17}$$

The first bound in (3.16) is direct (see (3.11)). For the second bound we notice that

$$\begin{aligned} &\Omega_\eta(\xi\cdot\eta^\perp) = -\xi\cdot\eta, \qquad \Omega_\eta(\xi\cdot\eta) = \xi\cdot\eta^\perp, \qquad \Omega_\eta\Phi(\xi,\eta) = \frac{\lambda'_\mu(|\xi-\eta|)}{|\xi-\eta|}(\xi\cdot\eta^\perp), \\ &|\Omega_\eta^a\Phi(\xi,\eta)| \lesssim \lambda(|\xi-\eta|)\big[|\xi-\eta|^{-2a}|\xi\cdot\eta^\perp|^a + |\xi-\eta|^{-a}U^a\big]. \end{aligned} \tag{3.18}$$

Since $\lambda'(|\xi-\eta|) \approx 2^{|k_1|/2}$, in the support of the integral, we have $|\xi-\eta|^{-2}|\xi\cdot\eta^\perp| \approx 2^p 2^{-k_1-|k_1|/2}$. The second bound in (3.16) follows once we recall the assumptions in (3.11).

We turn now to the proof of (3.17). The first bound follows from the construction of g_1. For the second bound, if $2^p \gtrsim 2^{|k_1|/2+\min(k,k_2)}$ then we have the simple bound

$$\|\Omega_\eta^a f_1(\xi-\eta)\varphi_k(\xi)\varphi_{k_2}(\eta)\|_{L^2_\eta} \lesssim [A2^{\min(k,k_2)} + 2^{m/10}]^a,$$

which suffices. On the other hand, if $2^p \ll 2^{|k_1|/2+\min(k,k_2)}$ then we may assume that $\xi = (s,0)$, $s \approx 2^k$. The identities (3.18) show that $\varphi_{\leq p+2}(\Omega_\eta \Phi(\xi,\eta)) \neq 0$ only if $|\xi \cdot \eta^\perp| \leq 2^{p+20} 2^{k_1 - |k_1|/2}$, which gives $|\eta_2| \leq 2^{p+30} 2^{k_1-|k_1|/2} 2^{-k}$. Therefore $|\eta_2| \ll 2^{k_1}$, so we may assume that $|\eta_1 - s| \approx 2^{k_1}$.

We write now

$$-\Omega_\eta f_1(\xi-\eta) = (\eta_1\partial_2 f_1 - \eta_2 \partial_1 f_1)(\xi-\eta) = \frac{\eta_1}{s-\eta_1}(\Omega f_1)(\xi-\eta) - \frac{s\eta_2}{s-\eta_1}(\partial_1 f_1)(\xi-\eta).$$

By iterating this identity we see that $\Omega_\eta^a f_1(\xi-\eta)$ can be written as a sum of terms of the form

$$P(s,\eta)\cdot\Big(\frac{1}{s-\eta_1}\Big)^{c+d+e}\Big(\frac{s\eta_2}{s-\eta_1}\Big)^{|b|-d}(D^b\Omega^c f_1)(\xi-\eta),$$

where $b+c+d+e \leq a$, $|b|, c, d, e \in \mathbb{Z}_+$, $|b| \geq d$, and $P(s,\eta)$ is a polynomial of degree at most a in s, η_1, η_2. The second bound in (3.17) follows using the bounds on f_1 in (3.15) and the bounds proved earlier, $|s\eta_2| \lesssim 2^p 2^{k_1-|k_1|/2}$, $|\eta_1 - s| \approx 2^{k_1}$.

The last claim follows using the formula (3.20), as in Lemma 3.5 below. □

3.3. **Localization in modulation.** Our lemma in this subsection shows that localization with respect to the phase is often a bounded operation:

Lemma 3.5. *Let $s \in [2^m-1, 2^m]$, $m \geq 0$, and $-p \leq m - 2\delta^2 m$. Let $\Phi = \Phi_{\sigma\mu\nu}$ as in (2.5) and assume that $1/2 = 1/q + 1/r$ and χ is a Schwartz function. Then, if $\|m\|_{S^\infty} \leq 1$,*

$$\begin{aligned}\Big\|\varphi_{\leq 10m}(\xi)\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} m(\xi,\eta)\chi(2^{-p}\Phi(\xi,\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\Big\|_{L^2_\xi}& \\ \lesssim \sup_{|\rho|\leq 2^{-p+\delta^2 m}} \|e^{-i(s+\rho)\Lambda_\mu} f\|_{L^q}\|e^{-i(s+\rho)\Lambda_\nu} g\|_{L^r} + 2^{-10m}\|f\|_{L^2}\|g\|_{L^2},&\end{aligned}\tag{3.19}$$

where the constant in the inequality only depends on the function χ.

Proof. We may assume that $m \geq 10$ and use the Fourier transform to write

$$\chi(2^{-p}\Phi(\xi,\eta)) = c\int_{\mathbb{R}} e^{i\rho 2^{-p}\Phi(\xi,\eta)}\widehat{\chi}(\rho)d\rho.\tag{3.20}$$

The left-hand side of (3.19) is dominated by

$$C\int_{\mathbb{R}}|\widehat{\chi}(\rho)|\Big\|\varphi_{\leq 10m}(\xi)\int_{\mathbb{R}^2} e^{i(s+2^{-p}\rho)\Phi(\xi,\eta)} m(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\Big\|_{L^2_\xi}d\rho.$$

Using (3.2), the contribution of the integral over $|\rho| \leq 2^{\delta^2 m}$ is dominated by the first term in the right-hand side of (3.19). The contribution of the integral over $|\rho| \geq 2^{\delta^2 m}$ is arbitrarily small and is dominated by the second term in the right-hand side of (3.19). □

3.4. **Linear estimates.** We note first the straightforward estimates,

$$\|P_k f\|_{L^2} \lesssim \min\{2^{(1-50\delta)k}, 2^{-Nk}\}\|f\|_{Z_1\cap H^N},\tag{3.21}$$

for $N \geq 0$. We prove now several linear estimates for functions in $Z_1 \cap H^N_\Omega$. As in Lemma 3.4, it is important to take advantage of the fact that our functions are "almost radial". The bounds we prove here are much stronger than the bounds one would normally expect for general functions with the same localization properties, and this is important in the next two sections.

Lemma 3.6. *Assume that* $N \geq 10$ *and*

$$\|f\|_{Z_1} + \sup_{k\in\mathbb{Z},\, a\leq N} \|\Omega^a P_k f\|_{L^2} \leq 1. \tag{3.22}$$

Let $\delta' := 50\delta + 1/(2N)$. *For any* $(k,j) \in \mathcal{J}$ *and* $n \in \{0,\dots,j+1\}$ *let (recall the notation (2.1))*

$$f_{j,k} := P_{[k-2,k+2]}Q_{jk}f, \qquad \widehat{f_{j,k,n}}(\xi) := \varphi_{-n}^{[-j-1,0]}(2^{100}(|\xi|-\gamma_1))\widehat{f_{j,k}}(\xi). \tag{3.23}$$

For any $\xi_0 \in \mathbb{R}^2 \setminus \{0\}$ *and* $\kappa,\rho \in [0,\infty)$ *let* $\mathcal{R}(\xi_0;\kappa,\rho)$ *denote the rectangle*

$$\mathcal{R}(\xi_0;\kappa,\rho) := \{\xi \in \mathbb{R}^2 : \big|(\xi-\xi_0)\cdot\xi_0/|\xi_0|\big| \leq \rho,\ \big|(\xi-\xi_0)\cdot\xi_0^\perp/|\xi_0|\big| \leq \kappa\}. \tag{3.24}$$

(i) Then, for any $(k,j) \in \mathcal{J}$, $n \in [0,j+1]$, *and* $\kappa,\rho \in (0,\infty)$ *satisfying* $\kappa + \rho \leq 2^{k-10}$

$$\big\|\sup_{\theta\in\mathbb{S}^1} |\widehat{f_{j,k,n}}(r\theta)|\big\|_{L^2(rdr)} + \big\|\sup_{\theta\in\mathbb{S}^1} |f_{j,k,n}(r\theta)|\big\|_{L^2(rdr)} \lesssim 2^{(1/2-49\delta)n-(1-\delta')j}, \tag{3.25}$$

$$\int_{\mathbb{R}^2} |\widehat{f_{j,k,n}}(\xi)|\mathbf{1}_{\mathcal{R}(\xi_0;\kappa,\rho)}(\xi)\,d\xi \lesssim \kappa 2^{-j+\delta' j}2^{-49\delta n}\min(1,2^n\rho 2^{-k})^{1/2}, \tag{3.26}$$

$$\|\widehat{f_{j,k,n}}\|_{L^\infty} \lesssim \begin{cases} 2^{(\delta+(1/2N))n}2^{-(1/2-\delta')(j-n)} & \text{if } |k|\leq 10,\\ 2^{-\delta' k}2^{-(1/2-\delta')(j+k)} & \text{if } |k|\geq 10,\end{cases} \tag{3.27}$$

and

$$\|D^\beta\widehat{f_{j,k,n}}\|_{L^\infty} \lesssim_{|\beta|} \begin{cases} 2^{|\beta|j}2^{(\delta+1/(2N))n}2^{-(1/2-\delta')(j-n)} & \text{if } |k|\leq 10,\\ 2^{|\beta|j}2^{-\delta' k}2^{-(1/2-\delta')(j+k)} & \text{if } |k|\geq 10.\end{cases} \tag{3.28}$$

(ii) (Dispersive bounds) If $m \geq 0$ *and* $|t| \in [2^m-1, 2^{m+1}]$ *then*

$$\big\|e^{-it\Lambda}f_{j,k,n}\big\|_{L^\infty} \lesssim \big\|\widehat{f_{j,k,n}}\big\|_{L^1} \lesssim 2^k 2^{-j+50\delta j}2^{-49\delta n}, \tag{3.29}$$

$$\big\|e^{-it\Lambda}f_{j,k,0}\big\|_{L^\infty} \lesssim 2^{3k/2}2^{-m+50\delta j}, \quad \text{if } |k|\geq 10. \tag{3.30}$$

Recall the operators A_{n,γ_0} *defined in (2.6). If* $j \leq (1-\delta^2)m + |k|/2$ *and* $|k| + \mathcal{D} \leq m/2$ *then we have the more precise bounds*

$$\big\|e^{-it\Lambda}A_{\leq 0,\gamma_0}f_{j,k,n}\big\|_{L^\infty} \lesssim \begin{cases} 2^{-m+2\delta^2 m}2^{-(j-n)(1/2-\delta')}2^{n(\delta+1/(2N))} & \text{if } n\geq 1,\\ 2^{-m+2\delta^2 m}2^k 2^{-(1/2-\delta')j} & \text{if } n=0.\end{cases} \tag{3.31}$$

Moreover, for $l \geq 1$,

$$\big\|e^{-it\Lambda}A_{l,\gamma_0}f_{j,k,0}\big\|_{L^\infty} \lesssim \begin{cases} 2^{-m+2\delta^2 m}2^{\delta' j}2^{m/2-j/2-l/2-\max(j,l)/2} & \text{if } 2l+\max(j,l)\geq m,\\ 2^{-m+2\delta^2 m}2^{\delta' j}2^{(l-j)/2} & \text{if } 2l+\max(j,l)\leq m.\end{cases} \tag{3.32}$$

In particular, if $j \leq (1-\delta^2)m + |k|/2$ *and* $|k| + \mathcal{D} \leq m/2$ *then*

$$\begin{gathered}\big\|e^{-it\Lambda}A_{\leq 0,\gamma_0}f_{j,k}\big\|_{L^\infty} \lesssim 2^{-m+2\delta^2 m}2^k 2^{j(\delta+1/(2N))},\\ \sum_{l\geq 1}\big\|e^{-it\Lambda}A_{l,\gamma_0}f_{j,k}\big\|_{L^\infty} \lesssim 2^{-m+2\delta^2 m}2^{\delta' j}2^{(m-3j)/6}.\end{gathered} \tag{3.33}$$

For all $k \in \mathbb{Z}$ *we have the bound*

$$\begin{gathered}\big\|e^{-it\Lambda}A_{\leq 0,\gamma_0}P_k f\big\|_{L^\infty} \lesssim (2^{k/2}+2^{2k})2^{-m}\big[2^{51\delta m} + 2^{m(2\delta+1/(2N))}\big],\\ \big\|e^{-it\Lambda}A_{\geq 1,\gamma_0}P_k f\big\|_{L^\infty} \lesssim 2^{-5m/6+2\delta^2 m}.\end{gathered} \tag{3.34}$$

Proof. (i) The hypothesis gives

$$\|f_{j,k,n}\|_{L^2} \lesssim 2^{(1/2-49\delta)n-(1-50\delta)j}, \qquad \big\|\Omega^N f_{j,k,n}\big\|_{L^2} \lesssim \|\Omega^N P_k f\|_{L^2} \lesssim 1. \tag{3.35}$$

The first inequality in (3.25) follows using the interpolation inequality

$$\big\| \sup_{\theta\in\mathbb{S}^1} |h(r\theta)| \,\big\|_{L^2(rdr)} \lesssim L^{1/2}\|h\|_{L^2} + L^{1/2-N}\|\Omega^N h\|_{L^2}, \tag{3.36}$$

for any $h \in L^2(\mathbb{R}^2)$ and $L \geq 1$. This inequality follows easily using the operators $\mathcal{R}_{\leq L}$ defined in (3.14). The second inequality in (3.25) follows similarly.

Inequality (3.26) follows from (3.25). Indeed, the left-hand side is dominated by

$$C(\kappa 2^{-k}) \sup_{\theta\in\mathbb{S}^1} \int_{\mathbb{R}} |\widehat{f_{j,k,n}}(r\theta)|\mathbf{1}_{\mathcal{R}(\xi_0;\kappa,\rho)}(r\theta)\, rdr \lesssim \sup_{\theta\in\mathbb{S}^1} \big\|\widehat{f_{j,k,n}}(r\theta)\,\big\|_{L^2(rdr)}(\kappa 2^{-k})[2^k\min(\rho, 2^{k-n})]^{1/2},$$

which gives the desired result.

We now consider (3.27). For any $\theta \in \mathbb{S}^1$ fixed we have

$$\begin{aligned}\|\widehat{f_{j,k,n}}(r\theta)\|_{L^\infty} &\lesssim 2^{j/2}\|\widehat{f_{j,k,n}}(r\theta)\|_{L^2(dr)} + 2^{-j/2}\|(\partial_r\widehat{f_{j,k,n}})(r\theta)\|_{L^2(dr)}\\ &\lesssim 2^{j/2}2^{-k/2}\|\widehat{f_{j,k,n}}(r\theta)\|_{L^2(rdr)},\end{aligned}$$

using the support property of $Q_{jk}f$ in the physical space. The desired bound follows using (3.25) and the observation that $\widehat{f_{j,k,n}} = 0$ unless $n = 0$ or $k \in [-10, 10]$. The bound (3.28) follows also since differentiation in the Fourier space corresponds essentially to multiplication by factors of 2^j, due to space localization.

(ii) The bound (3.29) follows directly from Hausdorff-Young and (3.35). To prove (3.30), if $|k| \geq 10$ then the standard dispersion estimate

$$\Big|\int_{\mathbb{R}^2} e^{-it\lambda(|\xi|)}\varphi_k(\xi)e^{ix\cdot\xi}\,d\xi\Big| \lesssim 2^{2k}(1+|t|2^{k+|k|/2})^{-1} \tag{3.37}$$

gives

$$\|e^{-it\Lambda}f_{j,k,n}\|_{L^\infty} \lesssim \frac{2^{2k}}{1+|t|2^{k/2}}\|f_{j,k,n}\|_{L^1} \lesssim \frac{2^{2k}}{1+|t|2^{k/2}}2^{50\delta j}. \tag{3.38}$$

The bound (3.30) follows (in the case $m \leq 10$ and $k \geq 0$ one can use (3.29)).

We prove now (3.31). The operator $A_{\leq 0,\gamma_0}$ is important here, because the function λ has an inflection point at γ_0, see (7.3). Using Lemma 3.3 (i) and the observation that $|(\nabla\Lambda)(\xi)| \approx 2^{|k|/2}$ if $|\xi| \approx 2^k$, it is easy to see that

$$\big|\big(e^{-it\Lambda}A_{\leq 0,\gamma_0}f_{j,k,n}\big)(x)\big| \lesssim 2^{-10m} \qquad \text{unless } |x| \approx 2^{m+|k|/2}.$$

Also, letting $f'_{j,k,n} := \mathcal{R}_{\leq m/5}f_{j,k,n}$, see (3.14), we have $\|f_{j,k,n} - f'_{j,k,n}\|_{L^2} \lesssim 2^{-m(N/5)}$ therefore

$$\big\|e^{-it\Lambda}A_{\leq 0,\gamma_0}(f_{j,k,n} - f'_{j,k,n})\big\|_{L^\infty} \lesssim \big\|\widehat{f_{j,k,n}} - \widehat{f'_{j,k,n}}\big\|_{L^1} \lesssim 2^{-2m}2^k. \tag{3.39}$$

On the other hand, if $|x| \approx 2^{m+|k|/2}$ then, using again Lemma 3.3 and (3.28),

$$\begin{aligned}\big(e^{-it\Lambda}A_{\leq 0,\gamma_0}f'_{j,k,n}\big)(x) = C\int_{\mathbb{R}^2} &e^{i\Psi(\xi)}\varphi(\kappa_r^{-1}\nabla_\xi\Psi)\varphi(\kappa_\theta^{-1}\Omega_\xi\Psi)\\ &\times \widehat{f'_{j,k,n}}(\xi)\varphi_{\geq -100}(|\xi|-\gamma_0)d\xi + O(2^{-10m}),\end{aligned} \tag{3.40}$$

where

$$\Psi := -t\Lambda(\xi) + x\cdot\xi, \quad \kappa_r := 2^{\delta^2 m}\big(2^{(m+|k|/2-k)/2} + 2^j\big), \quad \kappa_\theta := 2^{\delta^2 m}2^{(m+k+|k|/2)/2}. \tag{3.41}$$

We notice that the support of the integral in (3.40) is contained in a $\kappa\times\rho$ rectangle in the direction of the vector x, where $\rho\lesssim\frac{\kappa_r}{2^{m+|k|/2-k}}$ and $\kappa\lesssim\frac{\kappa_\theta}{2^{m+|k|/2}}$, $\kappa\lesssim\rho$. This is because the function λ'' does not vanish in the support of the integral, so $\lambda''(|\xi|)\approx 2^{|k|/2-k}$. Therefore we can estimate the contribution of the integral in (3.40) using either (3.26) or (3.27). More precisely, if $j\leq(m+|k|/2-k)/2$ then we use (3.27) while if $j\geq(m+|k|/2-k)/2$ then we use (3.26) (and estimate $\min(1,2^n\rho 2^{-k})\leq 2^n\rho 2^{-k}$); in both cases the desired estimate follows.

We prove now (3.32). We may assume that $|k|\leq 10$ and $m\geq\mathcal{D}$. As before, we may assume that $|x|\approx 2^m$ and replace $f_{j,k,0}$ with $f'_{j,k,0}$. As in (3.40), we have

$$\begin{split}\big(e^{-it\Lambda}A_{l,\gamma_0}f_{j,k,0}\big)(x)=C\int_{\mathbb{R}^2}&e^{i\Psi(\xi)}\varphi(2^{-m/2-\delta^2m}\Omega_\xi\Psi)\\&\times\widehat{f'_{j,k,0}}(\xi)\varphi_{-l-100}(|\xi|-\gamma_0)d\xi+O(2^{-2m}),\end{split}\tag{3.42}$$

where Ψ is as in (3.41). The support of the integral above is contained in a $\kappa\times\rho$ rectangle in the direction of the vector x, where $\rho\lesssim 2^{-l}$ and $\kappa\lesssim 2^{-m/2+\delta^2m}$. Since $|\widehat{f'_{j,k,0}}(\xi)|\lesssim 2^{-j/2+\delta' j}$ in this rectangle (see (3.27)), the bound in the first line of (3.32) follows if $l\geq j$. On the other hand, if $l\leq j$ then we use (3.26) to show that the absolute value of the integral in (3.42) is dominated by $C2^{-j+\delta' j}\kappa\rho^{1/2}$, which gives again the bound in the first line of (3.32).

It remains to prove the stronger bound in the second line of (3.32) in the case $2l+\max(j,l)\leq m$. We notice that $\lambda''(|\xi|)\approx 2^{-l}$ in the support of the integral. Assume that $x=(x_1,0)$, $x_1\approx 2^m$, and notice that we can insert an additional cutoff function of the form

$$\varphi[\kappa_r^{-1}(x_1-t\lambda'(|\xi_1|)\,\mathrm{sgn}\,(\xi_1))]\qquad\text{where}\qquad\kappa_r:=2^{\delta^2m}(2^{(m-l)/2}+2^j+2^l),$$

in the integral in (3.42), at the expense of an acceptable error. This can be verified using Lemma 3.3 (i). The support of the integral is then contained in a $\kappa\times\rho$ rectangle in the direction of the vector x, where $\rho\lesssim\kappa_r 2^{-m}2^l$ and $\kappa\lesssim 2^{-m/2+\delta^2m}$. The desired estimate then follows as before, using the L^∞ bound (3.27) if $2j\leq m-l$ and the integral bound (3.26) if $2j\geq m-l$.

The bounds in (3.33) follow from (3.31) and (3.32) by summation over n and l respectively. Finally, the bounds in (3.34) follow by summation (use (3.29) if $j\geq(1-\delta^2)m$ or $m\leq 4\mathcal{D}$, use (3.30) if $j\leq(1-\delta^2)m$ and $|k|\geq 10$, and use (3.33) if $j\leq(1-\delta^2)m$ and $|k|\leq 10$). $\square$

Remark 3.7. *We notice that we also have the bound (with no loss of $2^{2\delta^2m}$, used only in* [32]*)*

$$\big\|e^{-it\Lambda}A_{\leq 2\mathcal{D},\gamma_0}A_{\leq 2\mathcal{D},\gamma_1}f_{j,k}\big\|_{L^\infty}\lesssim_{\mathcal{D}}2^{-m}2^k2^{-(1/2-\delta'-\delta)j},\tag{3.43}$$

provided that $j\leq(1-\delta^2)m+|k|/2$ and $|k|+\mathcal{D}\leq m/2$. Indeed, this follows from (3.31) *if $j\geq m/10$. On the other hand, if $j\leq m/10$ then we write $e^{-it\Lambda}A_{\leq 2\mathcal{D},\gamma_0}A_{\leq 2\mathcal{D},\gamma_1}f_{j,k}$ as in* (3.40). *The contribution of $|\nabla_\xi\Psi|\leq\kappa\approx 2^{(m+|k|/2-k)/2}$ is estimated as before, using* (3.27), *while the contribution of $|\nabla_\xi\Psi|\geq\kappa$ is estimated using integration by parts in ξ.*

4. Dispersive analysis, I: the function $\partial_t\mathcal{V}$

In this section we prove several lemmas describing the function $\partial_t\mathcal{V}$. These lemmas rely on the Duhamel formula (2.18),

$$\Omega^a_\xi(\partial_t\widehat{\mathcal{V}})(\xi,s)=e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}_2}(\xi,s)+e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}_3}(\xi,s)+e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}_{\geq 4}}(\xi,s),\tag{4.1}$$

where

$$e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}_2}(\xi,s)=\sum_{\mu,\nu\in\{+,-\}}\sum_{a_1+a_2=a}\int_{\mathbb{R}^2}e^{is\Phi_{+\mu\nu}(\xi,\eta)}\mathfrak{m}_{\mu\nu}(\xi,\eta)(\Omega^{a_1}\widehat{\mathcal{V}_\mu})(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}_\nu})(\eta,s)\,d\eta \tag{4.2}$$

and

$$\begin{split}e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}_3}(\xi,s)=\sum_{\mu,\nu,\beta\in\{+,-\}}\sum_{a_1+a_2+a_3=a}\int_{\mathbb{R}^2\times\mathbb{R}^2}e^{is\Phi_{+\mu\nu\beta}(\xi,\eta,\sigma)}\mathfrak{n}_{\mu\nu\beta}(\xi,\eta,\sigma)\\ \times(\Omega^{a_1}\widehat{\mathcal{V}_\mu})(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}_\nu})(\eta-\sigma,s)(\Omega^{a_3}\widehat{\mathcal{V}_\beta})(\sigma,s)\,d\eta d\sigma.\end{split} \tag{4.3}$$

Recall also the assumptions on the nonlinearity $\mathcal{N}_{\geq 4}$ and the profile $\mathcal{V}$ (see (2.25)),

$$\begin{split}&\|\mathcal{V}(t)\|_{H^{N_0}\cap H^{N_1,N_3}_\Omega}\leq\varepsilon_1(1+t)^{\delta^2},\qquad\|\mathcal{V}(t)\|_Z\leq\varepsilon_1,\\ &\|\mathcal{N}_{\geq 4}(t)\|_{H^{N_0-N_3}\cap H^{N_1}_\Omega}\lesssim\varepsilon_1^2(1+t)^{-2},\end{split} \tag{4.4}$$

and the symbol-type bounds (2.21) on the multipliers $\mathfrak{m}_{\mu\nu}$. Given $\Phi=\Phi_{\sigma\mu\nu}$ as in (2.5) let

$$\begin{split}&\Xi=\Xi_{\mu\nu}(\xi,\eta):=(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)=(\nabla\Lambda_\mu)(\xi-\eta)-(\nabla\Lambda_\nu)(\eta),\qquad\Xi:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2,\\ &\Theta=\Theta_\mu(\xi,\eta):=(\Omega_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)=\frac{\lambda'_\mu(|\xi-\eta|)}{|\xi-\eta|}(\xi\cdot\eta^\perp),\qquad\Theta:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}.\end{split} \tag{4.5}$$

In this section we prove three lemmas describing the function $\partial_t\mathcal{V}$.

Lemma 4.1. *(i) Assume (4.1)–(4.4), $m\geq 0$, $s\in[2^m-1,2^{m+1}]$, $k\in\mathbb{Z}$, $\sigma\in\{+,-\}$. Then*

$$\big\|(\partial_t\mathcal{V}_\sigma)(s)\big\|_{H^{N_0-N_3}\cap H^{N_1}_\Omega}\lesssim\varepsilon_1^2 2^{-5m/6+6\delta^2 m}, \tag{4.6}$$

$$\sup_{a\leq N_1/2+20,\,2a+|\alpha|\leq N_1+N_4}\|e^{-is\Lambda_\sigma}P_kD^\alpha\Omega^a(\partial_t\mathcal{V}_\sigma)(s)\|_{L^\infty}\lesssim\varepsilon_1^2 2^{-5m/3+6\delta^2 m}. \tag{4.7}$$

(ii) In addition, if $a\leq N_1/2+20$ and $2a+|\alpha|\leq N_1+N_4$, then we may decompose

$$P_kD^\alpha\Omega^a(\partial_t\mathcal{V}_\sigma)=\varepsilon_1^2\sum_{a_1+a_2=a,\,\alpha_1+\alpha_2=\alpha,\,\mu,\nu\in\{+,-\}}\sum_{[(k_1,j_1),(k_2,j_2)]\in X_{m,k}}A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}+\varepsilon_1^2P_kE^{a,\alpha}_\sigma, \tag{4.8}$$

where

$$\|P_kE^{a,\alpha}_\sigma(s)\|_{L^2}\lesssim 2^{-3m/2+5\delta m}. \tag{4.9}$$

Moreover, with $\mathfrak{m}_{+\mu\nu}(\xi,\eta):=\mathfrak{m}_{\mu\nu}(\xi,\eta)$, $\mathfrak{m}_{-\mu\nu}(\xi,\eta):=\overline{\mathfrak{m}_{(-\mu)(-\nu)}(-\xi,-\eta)}$, we have

$$\mathcal{F}\{A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\}(\xi,s):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\mathfrak{m}_{\sigma\mu\nu}(\xi,\eta)\varphi_k(\xi)\widehat{f^\mu_{j_1,k_1}}(\xi-\eta,s)\widehat{f^\nu_{j_2,k_2}}(\eta,s)d\eta, \tag{4.10}$$

where

$$f^\mu_{j_1,k_1}=\varepsilon_1^{-1}P_{[k_1-2,k_1+2]}Q_{j_1k_1}D^{\alpha_1}\Omega^{a_1}\mathcal{V}_\mu,\qquad f^\nu_{j_2,k_2}=\varepsilon_1^{-1}P_{[k_2-2,k_2+2]}Q_{j_2k_2}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\nu.$$

The sets $X_{m,k}$ and the functions $A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}$ have the following properties:

(1) $X_{m,k}=\emptyset$ unless $m\geq\mathcal{D}^2$, $k\in[-3m/4,m/N'_0]$ and

$$X_{m,k}\subseteq\big\{[(k_1,j_1),(k_2,j_2)]\in\mathcal{J}\times\mathcal{J}:k_1,k_2\in[-3m/4,m/N'_0],\,\max(j_1,j_2)\leq 2m\big\}. \tag{4.11}$$

(2) If $[(k_1,j_1),(k_2,j_2)]\in X_{m,k}$ and $\min(k_1,k_2)\leq -2m/N'_0$, then

$$\max(j_1,j_2)\leq(1-\delta^2)m-|k|,\quad\max(|k_1-k|,|k_2-k|)\leq 100,\quad\mu=\nu, \tag{4.12}$$

and

$$\big\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)\big\|_{L^2}\lesssim 2^{2k}2^{-m+6\delta^2 m}. \tag{4.13}$$

(3) If $[(k_1,j_1),(k_2,j_2)]\in X_{m,k}$*,* $\min(k_1,k_2)\geq -5m/N_0'$*, and* $k\leq\min(k_1,k_2)-200$*, then*

$$\max(j_1,j_2)\leq(1-\delta^2)m-|k|,\quad \max(|k_1|,|k_2|)\leq 10,\quad \mu=-\nu, \tag{4.14}$$

and

$$\big\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)\big\|_{L^2}\lesssim 2^{k}2^{-m+4\delta m}. \tag{4.15}$$

(4) If $[(k_1,j_1),(k_2,j_2)]\in X_{m,k}$ *and* $\min(k,k_1,k_2)\geq -6m/N_0'$ *then*

$$\textit{either}\quad j_1\leq 5m/6\quad\textit{or}\quad |k_1|\leq 10, \tag{4.16}$$

$$\textit{either}\quad j_2\leq 5m/6\quad\textit{or}\quad |k_2|\leq 10, \tag{4.17}$$

and

$$\min(j_1,j_2)\leq(1-\delta^2)m. \tag{4.18}$$

Moreover,

$$\big\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)\big\|_{L^2}\lesssim 2^{k}2^{-m+4\delta m}, \tag{4.19}$$

and

$$\textit{if}\quad \max(j_1,j_2)\geq(1-\delta^2)m-|k|\quad\textit{then}\quad \big\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)\big\|_{L^2}\lesssim 2^{-4m/3+4\delta m}. \tag{4.20}$$

(iii) As a consequence of (4.9)*,* (4.13)*,* (4.15)*,* (4.19)*, if* $a\leq N_1/2+20$*, and* $2a+|\alpha|\leq N_1+N_4$ *then we have the* L^2 *bound*

$$\big\|P_kD^\alpha\Omega^a(\partial_t\mathcal{V}_\sigma)\big\|_{L^2}\lesssim\varepsilon_1^2\big[2^k2^{-m+5\delta m}+2^{-3m/2+5\delta m}\big]. \tag{4.21}$$

Proof. (i) We consider first the quadratic part of the nonlinearity. Let $I^{\sigma\mu\nu}$ denote the bilinear operator defined by

$$\begin{split}&\mathcal{F}\left\{I^{\sigma\mu\nu}[f,g]\right\}(\xi):=\int_{\mathbb{R}^2}e^{is\Phi_{\sigma\mu\nu}(\xi,\eta)}m(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta,\\ &\|m^{k,k_1,k_2}\|_{S^\infty}\leq 2^k2^{\min(k_1,k_2)/2},\qquad \|D^\alpha_\eta m^{k,k_1,k_2}\|_{L^\infty}\lesssim_{|\alpha|}2^{(|\alpha|+3/2)\max(|k_1|,|k_2|)},\end{split} \tag{4.22}$$

where, for simplicity of notation, $m=\mathfrak{m}_{\sigma\mu\nu}$. For simplicity, we often write Φ, Ξ, and Θ instead of $\Phi_{\sigma\mu\nu}$, $\Xi_{\mu\nu}$, and Θ_μ in the rest of this proof.

We define the operators P_k^+ for $k\in\mathbb{Z}_+$ by $P_k^+:=P_k$ for $k\geq 1$ and $P_0^+:=P_{\leq 0}$. In view of Lemma 3.1 (ii), (4.4), and (3.34), for any $k\geq 0$ we have

$$\begin{split}\|P_k^+I^{\sigma\mu\nu}[\mathcal{V}_\mu,\mathcal{V}_\nu](s)\|_{H^{N_0-N_3}}&\lesssim 2^{(N_0-N_3)k}\sum_{0\leq k_1\leq k_2,\,k_2\geq k-10}2^k2^{k_1/2}\|P^+_{k_2}\mathcal{V}(s)\|_{L^2}\|e^{-is\Lambda}P^+_{k_1}\mathcal{V}(s)\|_{L^\infty}\\ &\lesssim\varepsilon_1^22^{-k}2^{-5m/6+6\delta^2m},\end{split} \tag{4.23}$$

which is consistent with (4.6). Similarly,

$$\|P_k^+I^{\sigma\mu\nu}[\Omega^{a_2}\mathcal{V}_\mu,\Omega^{a_3}\mathcal{V}_\nu](s)\|_{L^2}\lesssim 2^{-k}\varepsilon_1^22^{-5m/6+6\delta^2m},\qquad a_2+a_3\leq N_1 \tag{4.24}$$

by placing the factor with less than $N_1/2$ Ω-derivatives in L^∞, and the other factor in L^2. Finally, using L^∞ estimates on both factors,

$$\|e^{-is\Lambda_\sigma}P_k^+I^{\sigma\mu\nu}[D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\mu,D^{\alpha_3}\Omega^{a_3}\mathcal{V}_\nu](s)\|_{L^\infty}\lesssim\begin{cases}\varepsilon_1^22^{-5m/3+6\delta^2m} & \text{if } k\leq 20,\\ \varepsilon_1^22^{4k}2^{-11m/6+52\delta m} & \text{if } k\geq 20,\end{cases} \tag{4.25}$$

provided that $a_2+a_3=a$ and $\alpha_2+\alpha_3=\alpha$. The conclusions in part (i) follow for the quadratic components.

The conclusions for the cubic components follow by the same argument, using the assumption (2.22) instead of (2.21), and the formula (4.3). The contributions of the higher order nonlinearity $\mathcal{N}_{\geq}4$ are estimated using directly the bootstrap hypothesis (4.4).

(ii) We assume that s is fixed and, for simplicity, drop it from the notation. In view of (4.4) and using interpolation, the functions $f^\mu := \varepsilon_1^{-1}D^{\alpha_2}\Omega^{a_2}V_\mu$ and $f^\nu := \varepsilon_1^{-1}D^{\alpha_3}\Omega^{a_3}V_\nu$ satisfy

$$\|f^\mu\|_{H^{N_0'}\cap Z_1\cap H_\Omega^{N_1'}} + \|f^\nu\|_{H^{N_0'}\cap Z_1\cap H_\Omega^{N_1'}} \lesssim 2^{\delta^2 m}. \tag{4.26}$$

where, compare with the notation in Theorem 1.1,

$$N_1' := (N_1-N_4)/2 = 1/(2\delta), \qquad N_0' := (N_0-N_3)/2 - N_4 = 1/\delta. \tag{4.27}$$

In particular, the dispersive bounds (3.29)–(3.34) hold with $N = N_1' = 1/(2\delta)$.

The contributions of the higher order nonlinearities $\mathcal{N}_3$ and $\mathcal{N}_{\geq 4}$ can all be estimated as part of the error term $P_kE_\sigma^{a,\alpha}$, so we focus on the quadratic nonlinearity $\mathcal{N}_2$. Notice that

$$A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2} = P_kI^{\sigma\mu\nu}(f^\mu_{j_1,k_1}, f^\nu_{j_2,k_2}).$$

Proof of property (1). In view of Lemma 3.1 and (3.33), we have the general bound

$$\big\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\big\|_{L^2} \lesssim 2^{k+\min(k_1,k_2)/2}\cdot 2^{-5m/6+5\delta^2 m}\min\big[2^{-(1/2-\delta)\max(j_1,j_2)}, 2^{-N_0'\max(k_1,k_2)}\big].$$

This bound suffices to prove the claims in (1). Indeed, if $k\geq m/N_0'$ or if $k\leq -3m/4+\mathcal{D}^2$ then the sum of all the terms can be bounded as in (4.9). Similarly, if $k\in[-3m/4+\mathcal{D}^2, m/N_0']$ then the sums of the L^2 norms corresponding to $\max(k_1,k_2)\geq m/N_0'$, or $\max(j_1,j_2)\geq 2m$, or $\min(k_1,k_2)\leq -3m/4+\mathcal{D}^2$ are all bounded by $2^{-3m/2}$ as desired.

Proof of property (2). Assume now that $\min(k_1,k_2)\leq -2m/N_0'$ and $j_2=\max(j_1,j_2)\geq (1-\delta^2)m-|k|$. Then, using the $L^2\times L^\infty$ estimate as before

$$\big\|P_kI^{\sigma\mu\nu}[f^\mu_{j_1,k_1}, A_{\leq 0,\gamma_1}f^\nu_{j_2,k_2}]\big\|_{L^2} \lesssim 2^{k+\min(k_1,k_2)/2}2^{-5m/6+5\delta^2 m}2^{-j_2(1-50\delta)} \lesssim 2^{-3m/2}.$$

Moreover, we notice that if $A_{\geq 1,\gamma_1}f^\nu_{j_2,k_2}$ is nontrivial then $|k_2|\leq 10$ and $k_1\leq -2m/N_0'$, therefore

$$\big\|P_kI^{\sigma\mu\nu}[f^\mu_{j_1,k_1}, A_{\geq 1,\gamma_1}f^\nu_{j_2,k_2}]\big\|_{L^2} \lesssim 2^{k+k_1/2}2^{-m+5\delta^2 m}2^{-j_2(1/2-\delta)} \lesssim 2^{-3m/2+3\delta m},$$

if $j_1\leq(1-\delta^2)m$, using (3.31) if $k_1\geq -m/2$ and (3.30) if $k_1\leq -m/2$. On the other hand, if $j_1\geq(1-\delta^2)m$ then we use again the $L^2\times L^\infty$ estimate (placing $f^\mu_{j_1,k_1}$ in L^2) to conclude that

$$\big\|P_kI^{\sigma\mu\nu}[f^\mu_{j_1,k_1}, A_{\geq 1,\gamma_1}f^\nu_{j_2,k_2}]\big\|_{L^2} \lesssim 2^{k+k_1/2}2^{-j_1+50\delta j_1}2^{-m+52\delta m} \lesssim 2^{-3m/2}.$$

The last three bounds show that

$$\big\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\big\|_{L^2} \lesssim 2^{-3m/2+3\delta m} \quad \text{if} \quad \max(j_1,j_2)\geq (1-\delta^2)m-|k|. \tag{4.28}$$

Assume now that

$$k_1=\min(k_1,k_2)\leq -2m/N_0' \quad \text{and} \quad \max(j_1,j_2)\leq(1-\delta^2)m-|k|.$$

If $k_2\geq k_1+20$ then $|\nabla_\eta\Phi(\xi,\eta)|\gtrsim 2^{|k_1|/2}$, so $\big\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\big\|_{L^2}\lesssim 2^{-3m}$ in view of Lemma 3.3 (i). On the other hand, if $k,k_2\leq k_1+30$ then, using again the $L^2\times L^\infty$ argument as before,

$$\big\|P_kI^{\sigma\mu\nu}[f^\mu_{j_1,k_1}, f^\nu_{j_2,k_2}]\big\|_{L^2} \lesssim 2^{k+k_1}2^{-m+5\delta^2 m}. \tag{4.29}$$

The L^2 bound in (4.9) follows if $k+k_1\leq -m/2$. On the other hand, if $k+k_1\geq -m/2$ and

$$\max(|k_1-k|,|k_2-k|)\geq 100 \quad \text{or} \quad \mu=-\nu$$

then $|\nabla_\eta\Phi(\xi,\eta)|\gtrsim 2^{k-\max(k_1,k_2)}$ in the support of the integral, in view of (7.18). Therefore $\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\|_{L^2}\lesssim 2^{-3m}$ in view of Lemma 3.3 (i). The inequalities in (4.12) follow. The bound (4.13) then follows from (4.29).

Proof of property (3). Assume first that

$$\min(k_1,k_2)\geq -5m/N_0',\quad k\leq \min(k_1,k_2)-200,\quad \max(j_1,j_2)\geq (1-\delta^2)m-|k|-|k_2|. \tag{4.30}$$

We may assume that $j_2\geq j_1$. Using the $L^2\times L^\infty$ estimate and Lemma 3.6 (ii) as before

$$\big\|P_kI^{\sigma\mu\nu}[f^\mu_{j_1,k_1},A^{(j_2)}_{n_2,\gamma_1}f^\nu_{j_2,k_2}]\big\|_{L^2}\lesssim 2^{k+k_1/2}2^{-5m/6+5\delta^2m}2^{-j_2(1-50\delta)}\lesssim 2^{-3m/2}$$

if $n_2\leq\mathcal{D}$. On the other hand, if $n_2\in[\mathcal{D},j_2]$ then

$$P_kI^{\sigma\mu\nu}[f^\mu_{j_1,k_1},A^{(j_2)}_{n_2,\gamma_1}f^\nu_{j_2,k_2}]=P_kI^{\sigma\mu\nu}[A_{\geq 1,\gamma_1}f^\mu_{j_1,k_1},A^{(j_2)}_{n_2,\gamma_1}f^\nu_{j_2,k_2}].$$

If $j_1\leq (1-\delta^2)m$ then we estimate

$$\big\|P_kI^{\sigma\mu\nu}[A_{\geq 1,\gamma_1}f^\mu_{j_1,k_1},A^{(j_2)}_{n_2,\gamma_1}f^\nu_{j_2,k_2}]\big\|_{L^2}\lesssim 2^k2^{-m+5\delta^2m+2\delta m}2^{-j_2(1/2-\delta)}\lesssim 2^{-3m/2+3\delta m+8\delta^2m}.$$

Finally, if $j_2\geq j_1\geq (1-\delta^2)m$ then we use Schur's lemma in the Fourier space and estimate

$$\begin{aligned}\big\|P_kI^{\sigma\mu\nu}[A^{(j_1)}_{n_1,\gamma_1}f^\mu_{j_1,k_1},A^{(j_2)}_{n_2,\gamma_1}f^\nu_{j_2,k_2}]\big\|_{L^2}&\lesssim 2^k2^{-\max(n_1,n_2)/2}\big\|A^{(j_1)}_{n_1,\gamma_1}f^\mu_{j_1,k_1}\big\|_{L^2}\big\|A^{(j_2)}_{n_2,\gamma_1}f^\nu_{j_2,k_2}\big\|_{L^2}\\&\lesssim 2^k2^{2\delta^2m}2^{-\max(n_1,n_2)/2}2^{-j_1(1-50\delta)}2^{(1/2-49\delta)n_1}\cdot 2^{-j_2(1-50\delta)}2^{(1/2-49\delta)n_2}\\&\lesssim 2^{2\delta^2m}2^{\min(n_1,n_2)/2}2^{-j_1(1-50\delta)}2^{-49\delta(n_1+n_2)}2^{-j_2(1-50\delta)}\\&\lesssim 2^{2\delta^2m}2^{-(2-2\delta^2)(1-50\delta)m}2^{(1/2-98\delta)m}\end{aligned} \tag{4.31}$$

for any $n_1\in[1,j_1+1]$, $n_2\in[1,j_2+1]$. Therefore, if (4.30) holds then

$$\big\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\big\|_{L^2}\lesssim 2^{-3m/2+4\delta m}. \tag{4.32}$$

Assume now that

$$\min(k_1,k_2)\geq -9m/N_0,\quad k\leq \min(k_1,k_2)-200,\quad \max(j_1,j_2)\leq (1-\delta^2)m-|k|-|k_2|. \tag{4.33}$$

If, in addition, $\max(|k_1|,|k_2|)\geq 11$ or $\mu=\nu$ then $|\nabla_\eta\Phi(\xi,\eta)|\gtrsim 2^{k-k_2}$ in the support of the integral. Indeed, this is a consequence of (7.18) if $k\leq -100$ and it follows easily from the formula (7.22) if $k\geq -100$. Therefore, $\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\|_{L^2}\lesssim 2^{-3m}$, using Lemma 3.3 (i). As a consequence, the functions $A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}$ can be absorbed into the error term $P_kE^{a,\alpha}_\sigma$ unless all the inequalities in (4.14) hold.

Assume now that (4.14) holds and we are looking to prove (4.15). It suffices to prove that

$$\big\|P_kI^{\sigma\mu\nu}[A_{\geq 1,\gamma_0}f^\mu_{j_1,k_1},A_{\geq 1,\gamma_0}f^\nu_{j_2,k_2}]\big\|_{L^2}\lesssim 2^k2^{-m+4\delta m}, \tag{4.34}$$

after using (3.31) and the $L^2\times L^\infty$ argument. We may assume that $\max(j_1,j_2)\leq m/3$; otherwise (4.34) follows from the $L^2\times L^\infty$ estimate. Using (3.27) and the more precise bound (3.32),

$$\|A_{p,\gamma_0}h\|_{L^2}\lesssim 2^{\delta^2m}2^{-p/2},\qquad \|e^{-it\Lambda}A_{p,\gamma_0}h\|_{L^\infty}\lesssim 2^{-m+3\delta^2m}\min\big(2^{p/2},2^{m/2-p}\big),$$

where $h\in\{f_{j_1,k_1},g_{j_2,k_2}\}$, $p\geq 1$. Therefore, using Lemma 3.1,

$$\big\|P_kI^{\sigma\mu\nu}[A_{p_1,\gamma_0}f^\mu_{j_1,k_1},A_{p_2,\gamma_0}f^\nu_{j_2,k_2}]\big\|_{L^2}\lesssim 2^k2^{-m+5\delta^2m}2^{-\max(p_1,p_2)/2}2^{\min(p_1,p_2)/2}.$$

The desired bound (5.80) follows, using also the simple estimate

$$\big\|P_kI^{\sigma\mu\nu}[A_{p_1,\gamma_0}f^{\mu}_{j_1,k_1},A_{p_2,\gamma_0}f^{\nu}_{j_2,k_2}]\big\|_{L^2}\lesssim 2^k2^{2\delta^2m}2^{-(p_1+p_2)/2}.$$

This completes the proof of (4.15).

Proof of property (4). The same argument as in the proof of (4.32), using just $L^2\times L^\infty$ estimates shows that $\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\|_{L^2}\lesssim 2^{-3m/2+4\delta m}$ if either (4.16) or (4.18) do not hold. The bounds (4.20) follow in the same way. The same argument as in the proof of (4.34), together with $L^2\times L^\infty$ estimates using (3.33) and (3.29), gives (4.19). □

In our second lemma we give a more precise description of the basic functions $A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)$ in the case $\min(k,k_1,k_2)\geq -6m/N_0'$.

Lemma 4.2. *Assume $[(k_1,j_1),(k_2,j_2)]\in X_{m,k}$ and $k,k_1,k_2\in[-6m/N_0',m/N_0']$ (as in Lemma 4.1 (ii) (4)), and recall the functions $A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)$ defined in* (4.10).

(i) We can decompose

$$A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}=\sum_{i=1}^{3}A^{a_1,\alpha_1;a_2,\alpha_2;[i]}_{k;k_1,j_1;k_2,j_2}=\sum_{i=1}^{3}G^{[i]},\tag{4.35}$$

$$\mathcal{F}A^{a_1,\alpha_1;a_2,\alpha_2;[i]}_{k;k_1,j_1;k_2,j_2}(\xi,s):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\mathfrak{m}_{\sigma\mu\nu}(\xi,\eta)\varphi_k(\xi)\chi^{[i]}(\xi,\eta)\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s)\widehat{f^{\nu}_{j_2,k_2}}(\eta,s)d\eta,\tag{4.36}$$

where $\chi^{[i]}$ are defined as

$$\begin{aligned}\chi^{[1]}(\xi,\eta)&=\varphi(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{30\delta m}\nabla_\eta\Phi(\xi,\eta))\mathbf{1}_{[0,5m/6]}(\max(j_1,j_2)),\\ \chi^{[2]}(\xi,\eta)&=\varphi_{\geq1}(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{20\delta m}\Omega_\eta\Phi(\xi,\eta)),\\ \chi^{[3]}&=1-\chi^{[1]}-\chi^{[2]}.\end{aligned}$$

The functions $A^{a_1,\alpha_1;a_2,\alpha_2;[1]}_{k;k_1,j_1;k_2,j_2}(s)$ are nontrivial only when $\max(|k|,|k_1|,|k_2|)\leq 10$. Moreover

$$\big\|G^{[1]}(s)\big\|_{L^2}\lesssim 2^{-m+4\delta m}2^{-(1-50\delta)\max(j_1,j_2)},\tag{4.37}$$

$$\big\|G^{[2]}(s)\big\|_{L^2}\lesssim 2^k2^{-m+4\delta m},\qquad \big\|G^{[3]}(s)\big\|_{L^2}\lesssim 2^{-3m/2+4\delta m}.\tag{4.38}$$

(ii) We have

$$\big\|\mathcal{F}\{A_{\leq\mathcal{D},2\gamma_0}A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\}(s)\big\|_{L^\infty}\lesssim(2^{-k}+2^{3k})2^{-m+14\delta m}.\tag{4.39}$$

As a consequence, if $k\geq -6m/N_0'+\mathcal{D}$ then we can decompose

$$\begin{aligned}&A_{\leq\mathcal{D}-10,2\gamma_0}\partial_tf^{\sigma}_{j,k}=h_2+h_\infty,\\ &\|h_2(s)\|_{L^2}\lesssim 2^{-3m/2+5\delta m},\qquad \|\widehat{h_\infty}(s)\|_{L^\infty}\lesssim(2^{-k}+2^{3k})2^{-m+15\delta m}.\end{aligned}\tag{4.40}$$

(iii) If $j_1,j_2\leq m/2+\delta m$ then we can write

$$\begin{aligned}&\widehat{G^{[1]}}(\xi,s)=e^{is[\Lambda_\sigma(\xi)-2\Lambda_\sigma(\xi/2)]}g^{[1]}(\xi,s)\varphi(2^{3\delta m}(|\xi|-\gamma_1))+h^{[1]}(\xi,s),\\ &\|D^{\alpha}_{\xi}g^{[1]}(s)\|_{L^\infty}\lesssim_\alpha 2^{-m+4\delta m}2^{|\alpha|(m/2+4\delta m)},\quad \|\partial_sg^{[1]}(s)\|_{L^\infty}\lesssim 2^{-2m+18\delta m},\\ &\|h^{[1]}(s)\|_{L^\infty}\lesssim 2^{-4m}.\end{aligned}\tag{4.41}$$

Proof. (i) To prove the bounds (4.37)–(4.38) we decompose

$$A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2} = \sum_{i=1}^{5} A_i, \qquad A_i := P_k I_i[f^{\mu}_{j_1,k_1}, f^{\nu}_{j_2,k_2}], \tag{4.42}$$

$$\mathcal{F}\{I_i[f,g]\}(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} m(\xi,\eta)\chi_i(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta, \tag{4.43}$$

where $m = \mathfrak{m}^{a_1}_{\sigma\mu\nu}$ and χ_i are defined as

$$\begin{aligned}
&\chi_1(\xi,\eta) := \varphi_{\geq 1}(2^{20\delta m}\Theta(\xi,\eta)),\\
&\chi_2(\xi,\eta) := \varphi_{\geq 1}(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{20\delta m}\Theta(\xi,\eta)),\\
&\chi_3(\xi,\eta) := \varphi(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{20\delta m}\Theta(\xi,\eta))\mathbf{1}_{(5m/6,\infty)}(\max(j_1,j_2)),\\
&\chi_4(\xi,\eta) := \varphi(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{20\delta m}\Theta(\xi,\eta))\varphi_{\geq 1}(2^{30\delta m}\Xi(\xi,\eta))\mathbf{1}_{[0,5m/6]}(\max(j_1,j_2)),\\
&\chi_5(\xi,\eta) := \varphi(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{20\delta m}\Theta(\xi,\eta))\varphi(2^{30\delta m}\Xi(\xi,\eta))\mathbf{1}_{[0,5m/6]}(\max(j_1,j_2)).
\end{aligned}\tag{4.44}$$

Notice that $A_2 = G^{[2]}$, $A_5 = G^{[1]}$, and $A_1 + A_3 + A_4 = G^{[3]}$. We will show first that

$$\|A_1\|_{L^2} + \|A_3\|_{L^2} + \|A_4\|_{L^2} \lesssim 2^{-3m/2+4\delta m}. \tag{4.45}$$

It follows from Lemma 3.4 and (4.16)–(4.18) that $\|A_1\|_{L^2} \lesssim 2^{-2m}$, as desired. Also, $\|A_4\|_{L^2} \lesssim 2^{-4m}$, as a consequence of Lemma 3.3 (i). It remains to prove that

$$\|A_3\|_{L^2} \lesssim 2^{-3m/2+4\delta m}. \tag{4.46}$$

Assume that $j_2 > 5m/6$ (the proof of (4.46) when $j_1 > 5m/6$ is similar). We may assume that $|k_2| \leq 10$ (see (4.17)), and then $|k|, |k_1| \in [0,100]$ (due to the restrictions $|\Phi(\xi,\eta)| \lesssim 2^{-10\delta m}$ and $|\Theta(\xi,\eta)| \lesssim 2^{-20\delta}$, see also (7.6)). We show first that

$$\big\|P_k I_3[f^{\mu}_{j_1,k_1}, A_{\leq 0,\gamma_1} f^{\nu}_{j_2,k_2}]\big\|_{L^2} \lesssim 2^{-3m/2+4\delta m}. \tag{4.47}$$

Indeed, we notice that, as a consequence of the $L^2 \times L^\infty$ argument,

$$\big\|P_k I^{\sigma\mu\nu}[f^{\mu}_{j_1,k_1}, A_{\leq 0,\gamma_1} f^{\nu}_{j_2,k_2}]\big\|_{L^2} \lesssim 2^{-3m/2},$$

where $I^{\sigma\mu\nu}$ is defined as in (4.22). Let $I^{||}$ be defined by

$$\mathcal{F}\{I^{||}[f,g]\}(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} m(\xi,\eta)\varphi(2^{20\delta m}\Theta(\xi,\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta. \tag{4.48}$$

Using Lemma 3.4 and (4.18), it follows that

$$\big\|P_k I^{||}[f^{\mu}_{j_1,k_1}, A_{\leq 0,\gamma_1} f^{\nu}_{j_2,k_2}]\big\|_{L^2} \lesssim 2^{-3m/2}.$$

The same averaging argument as in the proof of Lemma 3.5 gives (4.47).

We show now that

$$\big\|P_k I_3[f^{\mu}_{j_1,k_1}, A_{\geq 1,\gamma_1} f^{\nu}_{j_2,k_2}]\big\|_{L^2} \lesssim 2^{-3m/2+4\delta m}. \tag{4.49}$$

Recall that $|k_2| \leq 10$ and $k, |k_1| \in [0,100]$. It follows that $|\nabla_\eta \Phi(\xi,\eta)| \geq 2^{-\mathcal{D}}$ in the support of the integral (otherwise $|\eta|$ would be close to $\gamma_1/2$, as a consequence of Proposition 7.2 (iii), which is not the case). The bound (4.49) (in fact rapid decay) follows using Lemma 3.3 (i) unless

$$j_2 \geq (1-\delta^2)m. \tag{4.50}$$

Finally, assume that (4.50) holds. Notice that $P_kI_3[A_{\geq 1,\gamma_0}f^{\mu}_{j_1,k_1}, A_{\geq 1,\gamma_1}f^{\nu}_{j_2,k_2}]\equiv 0$. This is due to the fact that $|\lambda(\gamma_1)\pm\lambda(\gamma_0)\pm\lambda(\gamma_1\pm\gamma_0)|\gtrsim 1$, see Lemma 7.1 (iv). Moreover,

$$\big\|P_kI^{\sigma\mu\nu}[A_{\leq 0,\gamma_0}f^{\mu}_{j_1,k_1}, A_{\geq 1,\gamma_1}f^{\nu}_{j_2,k_2}]\big\|_{L^2}\lesssim 2^{-3m/2+3\delta m+6\delta^2 m}$$

as a consequence of the $L^2\times L^\infty$ argument and the bound (3.33). Therefore, using Lemma 3.4,

$$\big\|P_kI^{||}[A_{\leq 0,\gamma_0}f^{\mu}_{j_1,k_1}, A_{\geq 1,\gamma_1}f^{\nu}_{j_2,k_2}]\big\|_{L^2}\lesssim 2^{-3m/2+3\delta m+6\delta^2 m}.$$

The same averaging argument as in the proof of Lemma 3.5 shows that

$$\big\|P_kI_3[A_{\leq 0,\gamma_0}f^{\mu}_{j_1,k_1}, A_{\geq 1,\gamma_1}f^{\nu}_{j_2,k_2}]\big\|_{L^2}\lesssim 2^{-3m/2+3\delta m+6\delta^2 m},$$

and the desired bound (4.49) follows in this case as well. This completes the proof of (4.46).

We prove now the bounds (4.37). We notice that $|\eta|$ and $|\xi-\eta|$ are close to $\gamma_1/2$ in the support of the integral, due to Proposition 7.2 (iii), so

$$\widehat{G^{[1]}}(\xi)=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}m(\xi,\eta)\varphi_k(\xi)\chi^{[1]}(\xi,\eta)\widehat{A_{\geq 1,\gamma_1/2}f^{\mu}_{j_1,k_1}}(\xi-\eta)\widehat{A_{\geq 1,\gamma_1/2}f^{\nu}_{j_2,k_2}}(\eta)d\eta.$$

Then we notice that the factor $\varphi(2^{30\delta m}\nabla_\eta\Phi(\xi,\eta))$ can be removed at the expense of negligible errors (due to Lemma 3.3 (i)). The bound follows using the $L^2\times L^\infty$ argument and Lemma 3.5.

The bound on $G^{[2]}(s)$ in (4.38) follows using (4.19), (4.37), and (4.45).

(ii) The plan is to localize suitably, in the Fourier space both in the radial and the angular directions, and use (3.26) or (3.27). More precisely, let

$$B_{\kappa_\theta,\kappa_r}(\xi):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}m(\xi,\eta)\varphi_k(\xi)\varphi(\kappa_r^{-1}\Xi(\xi,\eta))\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta)\widehat{f^{\nu}_{j_2,k_2}}(\eta)d\eta,\quad (4.51)$$

where κ_θ and κ_r are to be fixed.

Let $\overline{j}:=\max(j_1,j_2)$. If

$$\min(k_1,k_2)\geq -2m/N_0',\qquad \overline{j}\leq m/2$$

then we set $\kappa_r=2^{2\delta m-m/2}$ (we do not localize in the angular variable in this case). Notice that $|\mathcal{F}\{A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\}(\xi)-B_{\kappa_\theta,\kappa_r}(\xi)|\lesssim 2^{-4m}$ in view of Lemma 3.3 (i). If $||\xi|-2\gamma_0|\geq 2^{-2\mathcal{D}}$ then we use Proposition 7.2 (ii) and conclude that the integration in η is over a ball of radius $\lesssim 2^{|k|}\kappa_r$. Therefore

$$|B_{\kappa_\theta,\kappa_r}(\xi)|\lesssim 2^{k+\min(k_1,k_2)/2}(2^{|k|}\kappa_r)^2\|\widehat{f^{\mu}_{j_1,k_1}}\|_{L^\infty}\|\widehat{f^{\nu}_{j_2,k_2}}\|_{L^\infty}\lesssim (2^{-k}+2^{3k})2^{-m+10\delta m}.\qquad (4.52)$$

If

$$\min(k_1,k_2)\geq -2m/N_0',\qquad \overline{j}\in[m/2,m-10\delta m]$$

then we set $\kappa_r=2^{2\delta m+\overline{j}-m}$, $\kappa_\theta=2^{3\delta m-m/2}$. Notice that $|\mathcal{F}\{A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\}(\xi)-B_{\kappa_\theta,\kappa_r}(\xi)|\lesssim 2^{-2m}$ in view of Lemma 3.3 (i) and Lemma 3.4. If $||\xi|-2\gamma_0|\geq 2^{-2\mathcal{D}}$ then we use Proposition 7.2 (ii) (notice that the hypothesis (7.16) holds in our case) to conclude that the integration in η in the integral defining $B_{\kappa_\theta,\kappa_r}(\xi)$ is over a $O(\kappa\times\rho)$ rectangle in the direction of the vector ξ, where $\kappa:=2^{|k|}2^{\delta m}\kappa_\theta$, $\rho:=2^{|k|}\kappa_r$. Then we use (3.26) for the function corresponding to the larger j and (3.27) to the other function to estimate

$$|B_{\kappa_\theta,\kappa_r}(\xi)|\lesssim 2^k\kappa 2^{-\overline{j}+51\delta\overline{j}}\rho^{49\delta}2^{2\delta\overline{j}}2^{8\delta^2 m}\lesssim (2^{-k}+2^{3k})2^{-m+10\delta m}.\qquad (4.53)$$

If

$$\min(k_1,k_2)\geq -2m/N_0',\qquad \overline{j}\geq m-10\delta m$$

then we have two subcases: if $\min(j_1, j_2) \le m - 10\delta m$ then we still localize in the angular direction (with $\kappa_\theta = 2^{3\delta m - m/2}$ as before) and do not localize in the radial direction. The same argument as above, with $\rho \lesssim 2^{2\delta m}$, gives the same pointwise bound (4.53). On the other hand, if $\min(j_1, j_2) \ge m - 10\delta m$ then the desired conclusion follows by Hölder's inequality. The bound (4.39) follows if $\min(k_1, k_2) \ge -2m/N_0'$.

On the other hand, if $\min(k_1, k_2) \le -2m/N_0'$ then $2^k \approx 2^{k_1} \approx 2^{k_2}$ (due to (4.12)) and the bound (4.39) can be proved in a similar way. The decomposition (4.40) is a consequence of (4.39) and the L^2 bounds (4.9).

(iii) We prove now the decomposition (4.41). With $\kappa := 2^{-m/2+\delta m+\delta^2 m}$ we define

$$\begin{aligned}
g^{[1]}(\xi, s) &:= \int_{\mathbb{R}^2} e^{is\Phi'(\xi,\eta)} m(\xi, \eta)\varphi_k(\xi)\chi^{[1]}(\xi, \eta)\widehat{f^\mu_{j_1,k_1}}(\xi - \eta, s)\widehat{f^\nu_{j_2,k_2}}(\eta, s)\varphi(\kappa^{-1}\Xi(\xi, \eta))d\eta,\\
h^{[1]}(\xi, s) &:= \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} m(\xi, \eta)\varphi_k(\xi)\chi^{[1]}(\xi, \eta)\widehat{f^\mu_{j_1,k_1}}(\xi - \eta, s)\widehat{f^\nu_{j_2,k_2}}(\eta, s)\varphi_{\ge 1}(\kappa^{-1}\Xi(\xi, \eta))d\eta,
\end{aligned} \tag{4.54}$$

where $\Phi'(\xi, \eta) = \Phi_{\sigma\mu\nu}(\xi, \eta) - \Lambda_\sigma(\xi) + 2\Lambda_\sigma(\xi/2)$. In view of Proposition 7.2 (iii) and the definition of $\chi^{[1]}$, the function $G^{[1]}$ is nontrivial only when $\mu = \nu = \sigma$, and it is supported in the set $\{||\xi| - \gamma_1| \lesssim 2^{-10\delta m}\}$. The conclusion $\|h^{[1]}(s)\|_{L^\infty} \lesssim 2^{-4m}$ in (4.41) follows from Lemma 3.3 (i) and the assumption $j_1, j_2 \le m/2 + \delta m$.

To prove the bounds on $g^{[1]}$ we notice that $\Phi'(\xi, \eta) = 2\Lambda_\sigma(\xi/2) - \Lambda_\sigma(\xi - \eta) - \Lambda_\sigma(\eta)$ and $|\eta - \xi/2| \lesssim \kappa$ (due to (7.21)). Therefore $|\Phi'(\xi, \eta)| \lesssim \kappa^2$, $|(\nabla_\xi\Phi')(\xi, \eta)| \lesssim \kappa$, and $|(D^\alpha_\xi\Phi')(\xi, \eta)| \lesssim_{|\alpha|} 1$ in the support of the integral. The bounds on $\|\mathcal{D}^\alpha_\xi g^{[1]}(s)\|_{L^\infty}$ in (4.41) follow using L^∞ bounds on $\widehat{f^\mu_{j_1,k_1}}(s)$ and $\widehat{f^\nu_{j_2,k_2}}(s)$. The bounds on $\|\partial_s g^{[1]}(s)\|_{L^\infty}$ follow in the same way, using also the decomposition (4.40) when the s-derivative hits either $\widehat{f^\mu_{j_1,k_1}}(s)$ or $\widehat{f^\nu_{j_2,k_2}}(s)$ (the contribution of the L^2 component is estimated using Hölder's inequality). This completes the proof. □

Our last lemma concerning $\partial_t\mathcal{V}$ is a refinement of Lemma 4.2 (ii). It is only used in the proof of Lemma 5.4 in [32].

Lemma 4.3. *For $s \in [2^m - 1, 2^{m+1}]$ and $k \in [-10, 10]$ we can decompose*

$$\mathcal{F}\{P_k A_{\le\mathcal{D},2\gamma_0}(D^\alpha\Omega^a\partial_t\mathcal{V}_\sigma)(s)\}(\xi) = g_d(\xi) + g_\infty(\xi) + g_2(\xi) \tag{4.55}$$

provided that $a \le N_1/2 + 20$ and $2a + |\alpha| \le N_1 + N_4$, where

$$\begin{aligned}
&\|g_2\|_{L^2} \lesssim \varepsilon_1^2 2^{-3m/2+20\delta m}, \qquad \|g_\infty\|_{L^\infty} \lesssim \varepsilon_1^2 2^{-m-4\delta m},\\
&\sup_{|\rho|\le 2^{7m/9+4\delta m}} \|\mathcal{F}^{-1}\{e^{-i(s+\rho)\Lambda_\sigma} g_d\}\|_{L^\infty} \lesssim \varepsilon_1^2 2^{-16m/9-4\delta m}.
\end{aligned} \tag{4.56}$$

Proof. Starting from Lemma 4.1 (ii), we notice that the error term $E^{a,\alpha}_\sigma$ can be placed in the L^2 component g_2 (due to (4.9)). It remains to decompose the functions $A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}$. We may assume that we are in case (4), $k_1, k_2 \in [-2m/N_0', m/N_0']$. We define the functions $B_{\kappa_\theta,\kappa_r}$ as in (4.51). We notice that the argument in Lemma 4.2 (ii) already gives the desired conclusion if $\overline{j} = \max(j_1, j_2) \ge m/2 + 20\delta m$ (without having to use the function g_d).

It remains to decompose the functions $A_{\le\mathcal{D},2\gamma_0} A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)$ when

$$\overline{j} = \max(j_1, j_2) \le m/2 + 20\delta m. \tag{4.57}$$

As in (4.51) let

$$B_{\kappa_r}(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} m(\xi,\eta)\varphi_k(\xi)\varphi(\kappa_r^{-1}\Xi(\xi,\eta))\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta)\widehat{f^{\nu}_{j_2,k_2}}(\eta)d\eta, \tag{4.58}$$

where $\kappa_r := 2^{30\delta m - m/2}$ (we do not need angular localization here). In view of Lemma 3.3 (i), $|\mathcal{F}A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(\xi) - B_{\kappa_r}(\xi)| \lesssim 2^{-4m}$. It remains to prove that

$$\big\|\mathcal{F}^{-1}\big\{e^{-i(s+\rho)\Lambda_\sigma(\xi)}\varphi_{\geq -\mathcal{D}}(2^{100}||\xi|-2\gamma_0|)B_{\kappa_r}(\xi)\big\}\big\|_{L^\infty} \lesssim 2^{-16m/9-5\delta m} \tag{4.59}$$

for any $k, j_1, k_1, j_2, k_2, \rho$ fixed, $|\rho| \leq 2^{7m/9+4\delta m}$.

In proving (4.59), we may assume that $m \geq \mathcal{D}^2$. The condition $|\Xi(\xi,\eta)| \leq 2\kappa_r$ shows that the variable η is localized to a small ball. More precisely, using Lemma 7.2, we have

$$|\eta - p(\xi)| \lesssim \kappa_r, \qquad \text{for some} \qquad p(\xi) \in P_{\mu\nu}(\xi), \tag{4.60}$$

provided that $||\xi|-2\gamma_0| \gtrsim 1$. The sets $P_{\mu\nu}(\xi)$ are defined in (7.15) and contain two or three points. We parametrize these points by $p_\ell(\xi) = q_\ell(|\xi|)\xi/|\xi|$, where $q_1(r) = r/2, q_2(r) = p_{++2}(r), q_3(r) = r - p_{++2}(r)$ if $\mu = \nu$, or $q_1(r) = p_{+-1}(r), q_2(r) = r - p_{+-1}(r)$ if $\mu = -\nu$. Then we rewrite

$$B_{\kappa_r}(\xi) = \sum_\ell e^{is\Lambda_\sigma(\xi)} e^{-is[\Lambda_\mu(\xi - p_\ell(\xi)) + \Lambda_\nu(p_\ell(\xi))]} H_\ell(\xi) \tag{4.61}$$

where

$$\begin{split} H_\ell(\xi) := \int_{\mathbb{R}^2} & e^{is[\Phi(\xi,\eta)-\Phi(\xi,p_\ell(\xi))]} m(\xi,\eta)\varphi_k(\xi)\varphi(\kappa_r^{-1}\Xi(\xi,\eta)) \\ & \widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta)\widehat{f^{\nu}_{j_2,k_2}}(\eta)\varphi(2^{m/2-31\delta m}(\eta - p_\ell(\xi))d\eta. \end{split} \tag{4.62}$$

Clearly, $|\Phi(\xi,\eta) - \Phi(\xi,p_\ell(\xi)| \lesssim |\eta - p_\ell(\xi)|^2$, $|\nabla_\xi[\Phi(\xi,\eta) - \Phi(\xi,p_\ell(\xi)]| \lesssim |\eta - p_\ell(\xi)|$. Therefore

$$|D^\beta H_\ell(\xi)| \lesssim_\beta 2^{-m+70\delta m} 2^{|\beta|(m/2+35\delta m)} \qquad \text{if} \quad ||\xi| - 2\gamma_0| \gtrsim 1. \tag{4.63}$$

We can now prove (4.59). Notice that the factor $e^{is\Lambda_\sigma(\xi)}$ simplifies and that the remaining phase $\xi \to \Lambda_\mu(\xi - p_\ell(\xi)) + \Lambda_\nu(p_\ell(\xi))$ is radial. Let $\Gamma_l = \Gamma_{l;\mu\nu}$ be defined such that $\Gamma_l(|\xi|) = \Lambda_\mu(\xi - p_\ell(\xi)) + \Lambda_\nu(p_\ell(\xi))$. Standard stationary phase estimates, using also (4.63), show that (4.59) holds provided that

$$|\Gamma'_\ell(r)| \approx 1 \ \text{ and } \ |\Gamma''_\ell(r)| \approx 1 \qquad \text{if} \qquad r \in [2^{-20}, 2^{20}],\, |r - 2\gamma_0| \geq 2^{-3\mathcal{D}/2}. \tag{4.64}$$

To prove (4.64), assume first that $\mu = \nu$. If $\ell = 1$ then $p_\ell(\xi) = \xi/2$ and the desired conclusion is clear. If $\ell \in \{2,3\}$ then $\pm\Gamma_\ell(r) = \lambda(r - p_{++2}(r)) + \lambda(p_{++2}(r))$. In view of Proposition 7.2 (i), $r - 2\gamma_0 \geq 2^{-2\mathcal{D}}$, $p_{++2}(r) \in (0, \gamma_0 - 2^{-2\mathcal{D}}]$, and $\lambda'(r - p_{++2}(r)) = \lambda'(p_{++2}(r))$. Therefore

$$|\Gamma'_\ell(r)| = \lambda'(r - p_{++2}(r)), \qquad |\Gamma''_\ell(r)| = |\lambda''(r - p_{++2}(r))(1 - p'_{++2}(r))|.$$

The desired conclusions in (4.64) follow since $|1 - p'_{++2}(r)| \approx 1$ in the domain of r (due to the identity $\lambda''(r - p_{++2}(r))(1 - p'_{++2}(r)) = \lambda''(p_{++2}(r))p'_{++2}(r)$).

The proof of (4.64) in the case $\mu = -\nu$ is similar. This completes the proof of the lemma. $\square$

5. Dispersive analysis, II: proof of Proposition 2.2

5.1. **Quadratic interactions.** In this section we prove Proposition 2.2. We start with the quadratic component in the Duhamel formula (2.15) and show how to control its Z norm.

Proposition 5.1. *With the hypothesis in Proposition 2.2, for any $t \in [0,T]$ we have*

$$\sup_{0\leq a\leq N_1/2+20,\, 2a+|\alpha|\leq N_1+N_4} \|D^\alpha\Omega^a W_2(t)\|_{Z_1} \lesssim \varepsilon_1^2. \tag{5.1}$$

The rest of this section is concerned with the proof of this proposition. Notice first that

$$\Omega_\xi^a\widehat{W_2}(\xi,t) = \sum_{\mu,\nu\in\{+,-\}}\sum_{a_1+a_2=a}\int_0^t\int_{\mathbb{R}^2} e^{is\Phi_{+\mu\nu}(\xi,\eta)}\mathfrak{m}_{\mu\nu}(\xi,\eta)(\Omega^{a_1}\widehat{\mathcal{V}_\mu})(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}_\nu})(\eta,s)\,d\eta ds. \tag{5.2}$$

Given $t \in [0,T]$, we fix a suitable decomposition of the function $\mathbf{1}_{[0,t]}$, i.e. we fix functions $q_0,\dots,q_{L+1}:\mathbb{R}\to[0,1]$, $|L-\log_2(2+t)|\leq 2$, with the properties

$$\begin{aligned}
&\operatorname{supp} q_0 \subseteq [0,2], \quad \operatorname{supp} q_{L+1}\subseteq [t-2,t], \quad \operatorname{supp} q_m \subseteq [2^{m-1},2^{m+1}] \text{ for } m\in\{1,\dots,L\},\\
&\sum_{m=0}^{L+1} q_m(s) = \mathbf{1}_{[0,t]}(s), \qquad q_m\in C^1(\mathbb{R}) \text{ and } \int_0^t |q_m'(s)|\,ds \lesssim 1 \text{ for } m\in\{1,\dots,L\}.
\end{aligned} \tag{5.3}$$

For $\mu,\nu\in\{+,-\}$, and $m\in[0,L+1]$ we define the operator $T_{m,b}^{\mu\nu}$ by

$$\mathcal{F}\{T_{m,b}^{\mu\nu}[f,g]\}(\xi) := \int_{\mathbb{R}} q_m(s)\int_{\mathbb{R}^2} e^{is\Phi_{+\mu\nu}(\xi,\eta)}\mathfrak{m}_{\mu\nu}(\xi,\eta)\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)d\eta ds. \tag{5.4}$$

In view of Definition 2.1, Proposition 5.1 follows from Proposition 5.2 below:

Proposition 5.2. *Assume that $t\in[0,T]$ is fixed and define the operators $T_{m,b}^{\mu\nu}$ as above. If $a_1+a_2=a$, $\alpha_1+\alpha_2=\alpha$, $\mu,\nu\in\{+,-\}$, $m\in[0,L+1]$, and $(k,j)\in\mathcal{J}$, then*

$$\sum_{k_1,k_2\in\mathbb{Z}} \big\|Q_{jk}T_{m,b}^{\mu\nu}[P_{k_1}D^{\alpha_1}\Omega^{a_1}\mathcal{V}_\mu, P_{k_2}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\nu]\big\|_{B_j} \lesssim 2^{-\delta^2 m}\varepsilon_1^2. \tag{5.5}$$

Assume that a_1, a_2, b, α_1, α_2, μ, ν are fixed and let, for simplicity of notation,

$$f^\mu := \varepsilon_1^{-1}D^{\alpha_1}\Omega^{a_1}\mathcal{V}_\mu, \quad f^\nu := \varepsilon_1^{-1}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\nu, \quad \Phi := \Phi_{+\mu\nu}, \quad m_0 := \mathfrak{m}_{\mu\nu}, \quad T_m := T_{m,b}^{\mu\nu}. \tag{5.6}$$

The bootstrap assumption (2.25) gives, for any $s\in[0,t]$,

$$\|f^\mu(s)\|_{H^{N_0'}\cap Z_1\cap H_\Omega^{N_1'}} + \|f^\nu(s)\|_{H^{N_0'}\cap Z_1\cap H_\Omega^{N_1'}} \lesssim (1+s)^{\delta^2}. \tag{5.7}$$

We recall also the symbol-type bounds, which hold for any $k,k_1,k_2\in\mathbb{Z}$, $|\alpha|\geq 0$,

$$\begin{aligned}
\|m_0^{k,k_1,k_2}\|_{S^\infty} &\lesssim 2^k 2^{\min(k_1,k_2)/2},\\
\|D_\eta^\alpha m_0^{k,k_1,k_2}\|_{L^\infty} &\lesssim_{|\alpha|} 2^{(|\alpha|+3/2)\max(|k_1|,|k_2|)},\\
\|D_\xi^\alpha m_0^{k,k_1,k_2}\|_{L^\infty} &\lesssim_{|\alpha|} 2^{(|\alpha|+3/2)\max(|k_1|,|k_2|,|k|)},
\end{aligned} \tag{5.8}$$

where $m_0^{k,k_1,k_2}(\xi,\eta) = m_0(\xi,\eta)\cdot\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)$.

We consider first a few simple cases before moving to the main analysis in the next subsections. Recall (see (3.34)) that, for any $k\in\mathbb{Z}$, $m\in\{0,\dots,L+1\}$, and $s\in I_m:=\operatorname{supp}q_m$,

$$\begin{split}\|P_kf^{\mu}(s)\|_{L^2}+\|P_kf^{\nu}(s)\|_{L^2}\lesssim 2^{\delta^2 m}\min\{2^{(1-50\delta)k},2^{-N_0'k}\},\\ \|P_ke^{-is\Lambda_\mu}f^{\mu}(s)\|_{L^\infty}+\|P_ke^{-is\Lambda_\nu}f^{\nu}(s)\|_{L^\infty}\lesssim 2^{3\delta^2 m}\min\{2^{(2-50\delta)k},2^{-5m/6}\}.\end{split}\tag{5.9}$$

Lemma 5.3. *Assume that f^μ, f^ν are as in* (5.6) *and let $(k,j)\in\mathcal{J}$. Then*

$$\sum_{\max\{k_1,k_2\}\geq 1.01(j+m)/N_0'-\mathcal{D}^2}\|Q_{jk}T_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-\delta^2 m},\tag{5.10}$$

$$\sum_{\min\{k_1,k_2\}\leq -(j+m)/2+\mathcal{D}^2}\|Q_{jk}T_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-\delta^2 m},\tag{5.11}$$

$$\textit{if } 2k\leq -j-m+49\delta j-\delta m \textit{ then } \sum_{k_1,k_2\in\mathbb{Z}}\|Q_{jk}T_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-\delta^2 m},\tag{5.12}$$

$$\textit{if } j\geq 2.1m \textit{ then } \sum_{-j\leq k_1,k_2\leq 2j/N_0'}\|Q_{jk}T_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-\delta^2 m}.\tag{5.13}$$

Proof. Using (5.9), the left-hand side of (5.10) is dominated by

$$C\sum_{\max\{k_1,k_2\}\geq 1.01(m+j)/N_0'-\mathcal{D}^2}2^{j+m}2^{2k_+}2^{\min(k_1,k_2)/2}\sup_{s\in I_m}\|P_{k_1}f^\mu(s)\|_{L^2}\|P_{k_2}f^\nu(s)\|_{L^2}\lesssim 2^{-\delta m},$$

which is acceptable. Similarly, if $k_1\leq k_2$ and $k_1\leq\mathcal{D}^2$ then

$$\begin{split}2^j\|P_kT_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{L^2}&\lesssim 2^{j+m}2^{k+k_1/2}\sup_{s\in I_m}\|\widehat{P_{k_1}f^\mu}(s)\|_{L^1}\|P_{k_2}f^\nu(s)\|_{L^2}\\&\lesssim 2^{j+m}2^{(5/2-50\delta)k_1}2^{-(N_0'-1)\max(k_2,0)},\end{split}$$

and the bound (5.11) follows by summation over $\min\{k_1,k_2\}\leq -(j+m)/2+2\mathcal{D}^2$.

To prove (5.12) we may assume that

$$2k\leq -j-m+49\delta j-\delta m,\qquad -(j+m)/2\leq k_1,k_2\leq 1.01(j+m)/N_0'.\tag{5.14}$$

Then

$$\begin{split}\|Q_{jk}T_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}&\lesssim 2^{j(1-50\delta)}\|P_kT_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{L^2}\\&\lesssim 2^{j(1-50\delta)}2^m2^{k+\min(k_1,k_2)/2}2^k\sup_{s\in I_m}\|P_{k_1}f^\mu(s)\|_{L^2}\|P_{k_2}f^\nu(s)\|_{L^2}\\&\lesssim 2^{-\delta(j+m)/2}.\end{split}$$

Summing in k_1,k_2 as in (5.14), we obtain an acceptable contribution.

Finally, to prove (5.13) we may assume that

$$j\geq 2.1m,\qquad j+k\geq j/10+\mathcal{D},\qquad -j\leq k_1,k_2\leq 2j/N_0',$$

and define

$$f^\mu_{j_1,k_1}:=P_{[k_1-2,k_1+2]}Q_{j_1k_1}f^\mu,\qquad f^\nu_{j_2,k_2}:=P_{[k_2-2,k_2+2]}Q_{j_2k_2}f^\nu.\tag{5.15}$$

If $\min\{j_1,j_2\}\geq 99j/100-\mathcal{D}$ then, using also (3.26),

$$\begin{split}\|P_kT_m[f^\mu_{j_1,k_1},f^\nu_{j_2,k_2}]\|_{L^2}&\lesssim 2^m2^{k+\min(k_1,k_2)/2}\sup_{s\in I_m}\|\widehat{f^\mu_{j_1,k_1}}(s)\|_{L^1}\|f^\nu_{j_2,k_2}(s)\|_{L^2}\\&\lesssim 2^m2^{k+3k_1/2}2^{-(1-\delta')j_1-(1/2-\delta)j_2}2^{4\delta^2 m},\end{split}$$

and therefore

$$\sum_{-j\le k_1,k_2\le 2j/N_0'}\ \sum_{\min\{j_1,j_2\}\ge 99j/100-\mathcal{D}} \|Q_{jk}T_m[f^{\mu}_{j_1,k_1},f^{\nu}_{j_2,k_2}]\|_{B_j}\lesssim 2^{-\delta m}.$$

On the other hand, if $j_1\le 99j/100-\mathcal{D}$ then we rewrite

$$\begin{aligned}&Q_{jk}T_m[f^{\mu}_{j_1,k_1},f^{\nu}_{j_2,k_2}](x)=C\widetilde{\varphi}^{(k)}_j(x)\\ &\times\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2}\Big[\int_{\mathbb{R}^2}e^{i[s\Phi(\xi,\eta)+x\cdot\xi]}\varphi_k(\xi)m_0(\xi,\eta)\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s)d\xi\Big]\widehat{f^{\nu}_{j_2,k_2}}(\eta,s)d\eta ds.\end{aligned}\tag{5.16}$$

In the support of integration, we have the lower bound $|\nabla_\xi\,[s\Phi(\xi,\eta)+x\cdot\xi]|\approx|x|\approx 2^j$. Integration by parts in ξ using Lemma 3.3 gives

$$\Big|Q_{jk}T_m[f^{\mu}_{j_1,k_1},f^{\nu}_{j_2,k_2}](x)\Big|\lesssim 2^{-10j}\tag{5.17}$$

which gives an acceptable contribution. This finishes the proof. □

5.2. **The main decomposition.** We may assume that

$$k_1,k_2\in\Big[-\frac{j+m}{2},\frac{1.01(j+m)}{N_0'}\Big],\quad k\ge\frac{-j-m+49\delta j-\delta m}{2},\quad j\le 2.1m,\quad m\ge\mathcal{D}^2/8.\tag{5.18}$$

Recall the definition (2.1). We fix $l_-:=\lfloor-(1-\delta/2)m\rfloor$, and decompose

$$\begin{aligned}&T_m[f,g]=\sum_{l_-\le l}T_{m,l}[f,g],\\ &\widehat{T_{m,l}[f,g]}(\xi):=\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\varphi^{[l_-,m]}_l(\Phi(\xi,\eta))m_0(\xi,\eta)\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)d\eta ds.\end{aligned}\tag{5.19}$$

Assuming (5.18), we notice that $T_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]\equiv 0$ if $l\ge 10m/N_0'$. When $l>l_-$, we may integrate by parts in time to rewrite $T_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]$,

$$\begin{aligned}&T_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]=i\mathcal{A}_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]+i\mathcal{B}_{m,l}[P_{k_1}\partial_sf^\mu,P_{k_2}f^\nu]+i\mathcal{B}_{m,l}[P_{k_1}f^\mu,P_{k_2}\partial_sf^\nu],\\ &\widehat{\mathcal{A}_{m,l}[f,g]}(\xi):=\int_{\mathbb{R}}q_m'(s)\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta))m_0(\xi,\eta)\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,d\eta ds,\\ &\widehat{\mathcal{B}_{m,l}[f,g]}(\xi):=\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta))m_0(\xi,\eta)\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,d\eta ds,\end{aligned}\tag{5.20}$$

where $\widetilde{\varphi}_l(x):=2^lx^{-1}\varphi_l(x)$. For s fixed let $\mathcal{I}_l$ denote the bilinear operator defined by

$$\widehat{\mathcal{I}_l[f,g]}(\xi):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta))m_0(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta.\tag{5.21}$$

It is easy to see that Proposition 5.2 follows from Lemma 5.3 and Lemmas 5.4–5.8 below.

Lemma 5.4. *Assume that* (5.18) *holds and, in addition,*

$$j\ge m+2\mathcal{D}+\max(|k|,|k_1|,|k_2|)/2.\tag{5.22}$$

Then, for $l_-\le l\le 10m/N_0'$,

$$2^{(1-50\delta)j}\|Q_{jk}T_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{L^2}\lesssim 2^{-2\delta^2m}.$$

Notice that the assumptions (5.18) and $j \le m + 2\mathcal{D} + \max(|k|, |k_1|, |k_2|)/2$ show that

$$k, k_1, k_2 \in [-4m/3 - 2\mathcal{D}, 3.2m/N_0'], \qquad m \ge \mathcal{D}^2/8. \tag{5.23}$$

Lemma 5.5. *Assume that* (5.23) *holds and, in addition,*

$$j \le m + 2\mathcal{D} + \max(|k|, |k_1|, |k_2|)/2, \qquad \min(k, k_1, k_2) \le -3.5m/N_0'. \tag{5.24}$$

Then, for $l_- \le l \le 10m/N_0'$,

$$2^{(1-50\delta)j} \|Q_{jk} T_{m,l}[P_{k_1} f^\mu, P_{k_2} f^\nu]\|_{L^2} \lesssim 2^{-2\delta^2 m}.$$

Lemma 5.6. *Assume that* (5.23) *holds and, in addition,*

$$j \le m + 2\mathcal{D} + \max(|k|, |k_1|, |k_2|)/2, \qquad \min(k, k_1, k_2) \ge -3.5m/N_0'. \tag{5.25}$$

Then, for $l_- < l \le 10m/N_0'$

$$\|Q_{jk} T_{m,l_-}[P_{k_1} f^\mu, P_{k_2} f^\nu]\|_{B_j} + \|Q_{jk} \mathcal{A}_{m,l}[P_{k_1} f^\mu, P_{k_2} f^\nu]\|_{B_j} \lesssim 2^{-2\delta^2 m}.$$

Lemma 5.7. *Assume that* (5.23) *holds and, in addition,*

$$j \le m + 2\mathcal{D} + \max(|k|, |k_1|, |k_2|)/2, \qquad \min(k, k_1, k_2) \ge -3.5m/N_0', \qquad l \ge -m/14. \tag{5.26}$$

Then

$$2^{(1-50\delta)j} \|Q_{jk} \mathcal{B}_{m,l}[P_{k_1} f^\mu, P_{k_2} \partial_s f^\nu]\|_{L^2} \lesssim 2^{-2\delta^2 m}.$$

Lemma 5.8. *Assume that* (5.23) *holds and, in addition,*

$$j \le m + 2\mathcal{D} + \max(|k|, |k_1|, |k_2|)/2, \quad \min(k, k_1, k_2) \ge -3.5m/N_0', \quad l_- < l \le -m/14. \tag{5.27}$$

Then

$$\|Q_{jk} T_{m,l}[P_{k_1} f^\mu, P_{k_2} f^\nu]\|_{B_j} \lesssim 2^{-2\delta^2 m}.$$

We prove these lemmas in the following five subsections. Lemma 5.4 takes advantage of the approximate finite of propagation. Lemma 5.5 uses the null structure at low frequencies. Lemma 5.6 controls interactions that lead to the creation of a space-time resonance. Lemmas 5.7 and 5.8 correspond to interactions that are particularly difficult to control in dimension 2 and contain the main novelty of our analysis (see also [31]). They rely on all the estimates in Lemmas 4.1 and 4.2, and on the "slow propagation of iterated resonances" properties in Lemma 7.6.

We will use repeatedly the symbol bounds (5.8) and the main assumption (5.7).

5.3. **Approximate finite speed of propagation.** In this subsection we prove Lemma 5.4. We define the functions $f^\mu_{j_1,k_1}$ and $f^\nu_{j_2,k_2}$ as before, see (5.15), and further decompose

$$f^\mu_{j_1,k_1} = \sum_{n_1=0}^{j_1+1} f^\mu_{j_1,k_1,n_1}, \qquad f^\nu_{j_2,k_2} = \sum_{n_2=0}^{j_2+1} f^\nu_{j_2,k_2,n_2} \tag{5.28}$$

as in (3.23). If $\min\{j_1, j_2\} \le j - \delta m$ then the same argument as in the proof of (5.13) leads to rapid decay, as in (5.17). To bound the sum over $\min\{j_1, j_2\} \ge j - \delta m$ we consider several cases.

Case 1. Assume first that

$$\min(k, k_1, k_2) \le -m/2. \tag{5.29}$$

Then we notice that

$$\begin{aligned}\big\|\mathcal{F}\{P_k T_{m,l}[f^\mu_{j_1,k_1}, f^\nu_{j_2,k_2}]\}\big\|_{L^\infty} &\lesssim 2^m 2^{k+\min(k_1,k_2)/2} \sup_{s\in I_m} \big[\big\|\widehat{f^\mu_{j_1,k_1}}(s)\big\|_{L^2}\big\|\widehat{f^\nu_{j_2,k_2}}(s)\big\|_{L^2}\big] \\ &\lesssim 2^m 2^{2\delta^2 m} 2^k 2^{-(1/2-\delta)(j_1+j_2)}.\end{aligned}$$

Therefore, the sum over j_1, j_2 with $\min(j_1, j_2) \geq j - \delta m$ is controlled as claimed provided that $k \leq -m/2$. On the other hand, if $k_1 = \min(k_1, k_2) \leq -m/2$ then we estimate

$$\begin{aligned} \|P_k T_{m,l}[f^{\mu}_{j_1,k_1}, f^{\nu}_{j_2,k_2}]\|_{L^2} &\lesssim 2^m 2^{k+k_1/2} \sup_{s\in I_m} \big[\big\|\widehat{f^{\mu}_{j_1,k_1}}(s)\big\|_{L^1}\big\|\widehat{f^{\nu}_{j_2,k_2}}(s)\big\|_{L^2}\big] \\ &\lesssim 2^m 2^{2\delta^2 m} 2^{k+k_1/2} 2^{k_1} 2^{-(1-50\delta)j_1} 2^{-(1/2-\delta)j_2} 2^{-4\max(k_2,0)}. \end{aligned} \tag{5.30}$$

The sum over j_1, j_2 with $\min(j_1, j_2) \geq j - \delta m$ is controlled as claimed in this case as well.

Case 2. Assume now that

$$\min(k, k_1, k_2) \geq -m/2, \qquad l \leq \min(k, k_1, k_2, 0)/2 - m/5. \tag{5.31}$$

We use Lemma 7.5: we may assume that $\min(k, k_1, k_2) + \max(k, k_1, k_2) \geq -100$ and estimate

$$\begin{aligned} \big\|P_k T_{m,l}[f^{\mu}_{j_1,k_1,n_1}, f^{\nu}_{j_2,k_2,n_2}]\big\|_{L^2} &\lesssim 2^m 2^{k+\min(k_1,k_2)/2} 2^{5\max(k_1,k_2,0)} 2^{l/2-n_1/2-n_2/2} \\ &\sup_{s\in I_m} \big[\big\|\sup_{\theta} |\widehat{f^{\mu}_{j_1,k_1,n_1}}(r\theta, s)|\big\|_{L^2(rdr)} \big\|\sup_{\theta} |\widehat{f^{\nu}_{j_2,k_2,n_2}}(r\theta, s)|\big\|_{L^2(rdr)}\big]. \end{aligned}$$

Using (3.25), (5.7), and summing over n_1, n_2, we have

$$2^{(1-50\delta)j}\big\|P_k T_{m,l}[f^{\mu}_{j_1,k_1}, f^{\nu}_{j_2,k_2}]\big\|_{L^2} \lesssim 2^{7\max(k_1,k_2,0)} 2^m 2^{2\delta^2 m} 2^{(1-50\delta)j} 2^{l/2} 2^{-(1-\delta')(j_1+j_2)}.$$

The sum over j_1, j_2 with $\min(j_1, j_2) \geq j - \delta m$ is controlled as claimed.

Case 3. Finally, assume that

$$\min(k, k_1, k_2) \geq -m/2, \qquad l \geq \min(k, k_1, k_2, 0)/2 - m/5. \tag{5.32}$$

We use the formula (5.20). The contribution of $\mathcal{A}_{m,l}$ can be estimated as in (5.30), with 2^m replaced by 2^{-l}, and we focus on the contribution of $\mathcal{B}_{m,l}[P_{k_1}f^{\mu}, P_{k_2}\partial_s f^{\nu}]$. We decompose $\partial_s f^{\nu}(s)$, according to (4.8). The contribution of $P_{k_2}E^{a_2,\alpha_2}_{\nu}$ can be estimated easily,

$$\begin{aligned} \|P_k \mathcal{B}_{m,l}[f^{\mu}_{j_1,k_1}, P_{k_2}E^{a_2,\alpha_2}_{\nu}]\|_{L^2} &\lesssim 2^m 2^{-l} 2^{k+\min(k_1,k_2)/2} \sup_{s\in I_m} \big[\big\|\widehat{f^{\mu}_{j_1,k_1}}(s)\big\|_{L^1}\big\|P_{k_2}E^{a_2,\alpha_2}_{\nu}(s)\big\|_{L^2}\big] \\ &\lesssim 2^m 2^{2\delta^2 m} 2^{m/5-\min(k,k_1,k_2,0)/2} 2^{k+k_2/2} 2^{k_1} 2^{-(1-51\delta)j_1} 2^{-3m/2+5\delta m} \\ &\lesssim 2^{-(1-51\delta)j_1} 2^{-m/4}, \end{aligned} \tag{5.33}$$

and the sum over $j_1 \geq j - \delta m$ of $2^{(1-50\delta)j}\|P_k \mathcal{B}_{m,l}[f^{\mu}_{j_1,k_1}, P_{k_2}E^{a_2,\alpha_2}_{\nu}]\|_{L^2}$ is suitably bounded.

We consider now the terms $A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)$ in (4.8), $[(k_3, j_3), (k_4, j_4)] \in X_{m,k_2}$, $\alpha_3 + \alpha_4 = \alpha_2$, $a_3 + a_4 \leq a_2$. In view of (4.12), (4.14), and (4.20), $\|A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)\|_{L^2} \lesssim 2^{-4m/3+4\delta m}$

$$\text{if} \quad \max(j_3, j_4) \geq (1-\delta^2)m - |k_2| \quad \text{or if} \quad |k_2| + \mathcal{D}/2 \leq \min(|k_3|, |k_4|).$$

The contributions of these terms can be estimated as in (5.33). On the other hand, to control the contribution of $Q_{jk}\mathcal{B}_{m,l}[f^{\mu}_{j_1,k_1}, A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}]$ when $\max(j_3, j_4) \leq (1-\delta^2)m - |k_2|$ and $|k_2| + \mathcal{D}/2 \geq |k_3|$, we simply rewrite this in the form

$$\begin{aligned} &c\widetilde{\varphi}^{(k)}_j(x) \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2} \widehat{f^{\mu}_{j_1,k_1}}(\eta, s) \Big[\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i[x\cdot\xi + s\widetilde{\Phi}'(\xi,\eta,\sigma)]} 2^{-l} \widetilde{\varphi}_l(\Phi_{\sigma\mu\nu}(\xi, \xi-\eta)) \\ &\times \varphi_k(\xi)\varphi_{k_2}(\xi-\eta)\mathfrak{m}_{\mu\nu}(\xi, \xi-\eta)\mathfrak{m}_{\nu\beta\gamma}(\xi-\eta, \sigma)\widehat{f^{\beta}_{j_3,k_3}}(\xi-\eta-\sigma, s)\widehat{f^{\gamma}_{j_4,k_4}}(\sigma, s)\, d\xi d\sigma\Big] d\eta ds, \end{aligned} \tag{5.34}$$

where $\widetilde{\Phi}'(\xi,\eta,\sigma) := \Lambda(\xi) - \Lambda_\mu(\eta) - \Lambda_\beta(\xi-\eta-\sigma) - \Lambda_\gamma(\sigma)$. Notice that

$$\big|\nabla_\xi[x\cdot\xi + s\Lambda(\xi) - s\Lambda_\mu(\eta) - s\Lambda_\beta(\xi-\eta-\sigma) - s\Lambda_\gamma(\sigma)]\big| \approx |x| \approx 2^j. \tag{5.35}$$

We can integrate by parts in ξ using Lemma 3.3 (i) to conclude that these are negligible contributions, pointwise bounded by $C2^{-5m}$. This completes the proof of the lemma.

5.4. **The case of small frequencies.** In this subsection we prove Lemma 5.5. The main point is that if $\underline{k} := \min(k,k_1,k_2) \le -3.5m/N_0'$ then $|\Phi(\xi,\eta)| \gtrsim 2^{\underline{k}/2}$ for any $(\xi,\eta) \in \mathcal{D}_{k,k_1,k_2}$, as a consequence of (7.6) and (5.23). Therefore the operators $T_{m,l}$ are nontrivial only if

$$l \ge \underline{k}/2 - \mathcal{D}. \tag{5.36}$$

Step 1. We consider first the operators $\mathcal{A}_{m,l}$. Since $l \ge -2m/3 - 2\mathcal{D}$, it suffices to prove that

$$2^{(1-50\delta)(m-\underline{k}/2)}\big\|P_k\mathcal{I}_l[f^\mu_{j_1,k_1}(s), f^\nu_{j_2,k_2}(s)]\big\|_{L^2} \lesssim 2^{-3\delta^2 m}, \tag{5.37}$$

for any $s \in I_m$ and j_1, j_2, where $\mathcal{I}_l$ are the operators defined in (5.21), and $f^\mu_{j_1,k_1}$ and $f^\nu_{j_2,k_2}$ are as in (5.15). We may assume $k_1 \le k_2$ and consider two cases.

Case 1. If $\underline{k} = k_1$ then we estimate first the left-hand side of (5.37) by

$$\begin{aligned}&C2^{(1-50\delta)(m-\underline{k}/2)}\cdot 2^{k+\underline{k}/2}2^{-l}\big[\sup_{s\approx 2^m}\big\|e^{-it\Lambda_\mu}f^\mu_{j_1,k_1}(s)\big\|_{L^\infty}\big\|f^\nu_{j_2,k_2}(s)\big\|_{L^2} + 2^{-8m}\big]\\ &\qquad\lesssim 2^{(1-50\delta)(m-\underline{k}/2)}\cdot 2^k 2^{6\delta^2 m}\big[2^{\underline{k}}2^{-m+50\delta j_1}2^{-4k_+} + 2^{-8m}\big],\end{aligned}$$

using Lemma 3.5 and (3.30). This suffices to prove (5.37) if $j_1 \le 9m/10$. On the other hand, if $j_1 \ge 9m/10$ then we estimate the left-hand side of (5.37) by

$$\begin{aligned}&C2^{(1-50\delta)(m-\underline{k}/2)}\cdot 2^{k+\underline{k}/2}2^{-l}\big[\sup_{s\approx 2^m}\big\|f^\mu_{j_1,k_1}(s)\big\|_{L^2}\big\|e^{-it\Lambda_\nu}f^\nu_{j_2,k_2}(s)\big\|_{L^\infty} + 2^{-8m}\big]\\ &\qquad\lesssim 2^{(1-50\delta)(m-\underline{k}/2)}\cdot 2^k 2^{6\delta^2 m}\big[2^{-(1-50\delta)j_1}2^{-5m/6}2^{-2k_+} + 2^{-8m}\big],\end{aligned}$$

using Lemma 3.5 and (3.34). This suffices to prove the desired bound (5.37).

Case 2. If $\underline{k} = k$ then (5.37) follows using the $L^2 \times L^\infty$ estimate, as in Case 1, unless

$$\max(|k_1|,|k_2|) \le 20, \qquad \max(j_1,j_2) \le m/3. \tag{5.38}$$

On the other hand, if (5.38) holds then it suffices to prove that, for $|\rho| \le 2^{m-\mathcal{D}}$,

$$\begin{aligned}&2^{(1-50\delta)(m-k/2)}2^{-k/2}\big\|P_k I_0[f^\mu_{j_1,k_1}(s), f^\nu_{j_2,k_2}(s)]\big\|_{L^2} \lesssim 2^{-3\delta^2 m},\\ &\widehat{I_0[f,g]}(\xi) := \int_{\mathbb{R}^2} e^{i(s+\rho)\Phi(\xi,\eta)}m_0(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta.\end{aligned} \tag{5.39}$$

Indeed, (5.37) would follow from (5.39) and the inequality $l \ge \underline{k}/2 - \mathcal{D} \ge 2m/3 - 2\mathcal{D}$ (see (5.23)–(5.36)), using the superposition argument in Lemma 3.5. On the other hand, the proof of (5.39) is similar to the proof of (4.15) in Lemma 4.1.

Step 2. We consider now the operators $\mathcal{B}_{m,l}$. In some cases we prove the stronger bound

$$2^{(1-50\delta)(m-\underline{k}/2)}2^m\big\|P_k\mathcal{I}_l[f^\mu_{j_1,k_1}(s), P_{k_2}\partial_s f^\nu(s)]\big\|_{L^2} \lesssim 2^{-3\delta^2 m}, \tag{5.40}$$

for any $s \in I_m$ and j_1. We consider three cases.

Case 1. If $\underline{k} = k_1$ then we use the bounds

$$\begin{aligned}&\|P_{k_2}\partial_s f^\nu(s)\|_{L^2} \lesssim 2^{-m+5\delta m}(2^{k_2} + 2^{-m/2}),\\ &\|e^{-is\Lambda_\nu}P_{k_2}\partial_s f^\nu(s)\|_{L^\infty} \lesssim 2^{-5m/3+6\delta^2 m},\end{aligned} \tag{5.41}$$

see (4.21) and (4.7). We also record the bound, which can be verified easily using integration by parts and Plancherel for any $\rho \in \mathbb{R}$ and $k' \in \mathbb{Z}$,

$$\big\|e^{-i\rho\Lambda}P_{k'}\big\|_{L^\infty\to L^\infty} \lesssim \big\|\mathcal{F}^{-1}\{e^{-i\rho\Lambda(\xi)}\varphi_{k'}(\xi)\}\big\|_{L^1} \lesssim 1 + 2^{k'/2}2^{k'_+}|\rho|. \tag{5.42}$$

If

$$k_1 \geq -m/4, \qquad j_1 \leq (1-\delta^2)m \tag{5.43}$$

then we use (3.33), (5.41), and Lemma 3.5 to estimate the left-hand side of (5.40) by

$$\begin{aligned}&C2^{k+k_1/2}2^{(1-50\delta)(m-\underline{k}/2)}2^m\big[2^{-l}\sup_{|\rho|\leq 2^{m/2}}\|e^{-i(s+\rho)\Lambda_\mu}f^\mu_{j_1,k_1}(s)\|_{L^\infty}\|P_{k_2}\partial_s f^\nu(s)\|_{L^2} + 2^{-8m}\big]\\&\lesssim 2^{6k_+}2^{k_1/2}2^{-40\delta m}.\end{aligned}$$

This suffices to prove (5.40) when (5.43) holds (recall the choice of δ, N_0, N_1 in Definition 2.1). On the other hand, if

$$k_1 \geq -m/4, \qquad j_1 \geq (1-\delta^2)m \tag{5.44}$$

then we use (5.42), (3.29), (5.41), and Lemma 3.5 to estimate the left-hand side of (5.40) by

$$\begin{aligned}&C2^{k+k_1/2}2^{(1-50\delta)(m-\underline{k}/2)}2^m\big[2^{-l}\|f^\mu_{j_1,k_1}(s)\|_{L^2}\sup_{|\rho|\leq 2^{-l+4\delta^2 m}}\|e^{-i(s+\rho)\Lambda_\nu}P_{k_2}\partial_s f^\nu(s)\|_{L^\infty} + 2^{-8m}\big]\\&\lesssim 2^{10k_+}2^{-2m/3+10\delta m}2^{-2l}.\end{aligned}$$

This suffices to prove (5.40), provided that (5.44) holds.

Finally, if $k_1 \leq -m/4$ then we use the bound

$$\sup_{|\rho|\leq 2^{m-\mathcal{D}}}\|e^{-i(s+\rho)\Lambda_\mu}f^\mu_{j_1,k_1}(s)\|_{L^\infty} \lesssim 2^{(3/2-25\delta)k_1}2^{-m+50\delta m}2^{\delta^2 m},$$

which follows from (3.29)–(3.30). Then we estimate the left-hand side of (5.40) by

$$C2^{2k_++k_1/2}2^{(1-50\delta)(m-\underline{k}/2)}2^m\cdot 2^{-l}2^{(3/2-25\delta)k_1}2^{-m+51\delta m}2^{-m+5\delta m} \lesssim 2^{6k_+}2^{10\delta m}2^{k_1}.$$

The desired bound (5.40) follows, provided that $k_1 \leq -m/4$.

Case 2. If $\underline{k} = k$ then (5.40) follows using $L^2\times L^\infty$ estimates, as in Case 1, unless

$$\max(|k_1|,|k_2|) \leq 20. \tag{5.45}$$

Assuming (5.45), we notice that

$$\begin{aligned}&\sup_{|\rho|\leq 2^{m-\mathcal{D}}}\|e^{-i(s+\rho)\Lambda_\mu}A_{\leq 0,\gamma_0}f^\mu_{j_1,k_1}(s)\|_{L^\infty} \lesssim 2^{-m+3\delta m} \qquad \text{if} \quad j_1 \leq (1-\delta^2)m,\\&\sup_{|\rho|\leq 2^{m-\mathcal{D}}}\|e^{-i(s+\rho)\Lambda_\mu}A_{\geq 1,\gamma_0}f^\mu_{j_1,k_1}(s)\|_{L^\infty} \lesssim 2^{-m} \qquad \text{if} \quad m/2 \leq j_1 \leq (1-\delta^2)m,\end{aligned} \tag{5.46}$$

as a consequence of (3.33). Therefore, using the $L^2\times L^\infty$ estimate and (5.41), as before,

$$2^{(1-50\delta)(m-k/2)}2^m\big\|P_k\mathcal{I}_l[A_{\leq 0,\gamma_0}f^\mu_{j_1,k_1}(s), P_{k_2}\partial_s f^\nu(s)]\big\|_{L^2} \lesssim 2^{-3\delta^2 m}, \tag{5.47}$$

if $j_1 \leq (1-\delta^2)m$, and

$$2^{(1-50\delta)(m-k/2)}2^m\big\|P_k\mathcal{I}_l[A_{\geq 1,\gamma_0}f^\mu_{j_1,k_1}(s), P_{k_2}\partial_s f^\nu(s)]\big\|_{L^2} \lesssim 2^{-3\delta^2 m}, \tag{5.48}$$

if $m/2 \leq j_1 \leq (1-\delta^2)m$.

On the other hand, if $j_1 \geq (1-\delta^2)m$ then we can use the L^∞ bound $\|e^{-is\Lambda_\nu}P_{k_2}\partial_s f^\nu(s)\|_{L^\infty} \lesssim 2^{-5m/3+6\delta^2 m}$ in (5.41), together with the general bound (5.42). As in (5.28) we decompose

$f^{\mu}_{j_1,k_1} = \sum_{n_1=0}^{j_1} f^{\mu}_{j_1,k_1,n_1}$, and record the bound $\|f^{\mu}_{j_1,k_1,n_1}(s)\|_{L^2} \lesssim 2^{-j_1+50\delta j_1} 2^{n_1/2-49\delta n_1} 2^{\delta^2 m}$. Let $X := 2^{(1-50\delta)(m-\underline{k}/2)} 2^m \big\| P_k \mathcal{I}_l [f^{\mu}_{j_1,k_1,n_1}(s), P_{k_2}\partial_s f^{\nu}(s)]\big\|_{L^2}$. Using Lemma 3.5 it follows that

$$X \lesssim 2^{(1-50\delta)(m-k/2)} 2^m \big[2^k 2^{-l} \|f^{\mu}_{j_1,k_1,n_1}(s)\|_{L^2} \sup_{|\rho|\leq 2^{-l+2\delta^2 m}} \|e^{-i(s+\rho)\Lambda_\nu} P_{k_2}\partial_s f^{\nu}(s)\|_{L^\infty} + 2^{-8m}\big]$$
$$\lesssim 2^{-k/2} 2^{-2m/3} 2^{n_1/2-49\delta n_1} 2^{4\delta m}.$$

Using only L^2 bounds, see (5.41), and Cauchy–Schwarz we also have

$$X \lesssim 2^{(1-50\delta)(m-k/2)} 2^m \cdot 2^{2k} 2^{-l} \|f^{\mu}_{j_1,k_1,n_1}(s)\|_{L^2} \|P_{k_2}\partial_s f^{\nu}(s)\|_{L^2} \lesssim 2^k 2^{n_1/2-49\delta n_1} 2^{6\delta m}.$$

Finally, using (3.26), we have

$$X \lesssim 2^{(1-50\delta)(m-k/2)} 2^m \cdot 2^k 2^{-l} \|\widehat{f^{\mu}_{j_1,k_1,n_1}}(s)\|_{L^1} \|P_{k_2}\partial_s f^{\nu}(s)\|_{L^2} \lesssim 2^{-49\delta n_1} 2^{7\delta m}.$$

We can combine the last three estimates (using the last one for $n_1 \geq m/4$ and the first two for $n_1 \leq m/4$) to conclude that if $j_1 \geq (1-\delta^2)m$ then

$$2^{(1-50\delta)(m-k/2)} 2^m \big\| P_k \mathcal{I}_l [f^{\mu}_{j_1,k_1}(s), P_{k_2}\partial_s f^{\nu}(s)]\big\|_{L^2} \lesssim 2^{-3\delta^2 m}. \tag{5.49}$$

In view of (5.47)–(5.49), it remains to prove that, for $j_1 \leq m/2$,

$$2^{(1-50\delta)(m-k/2)} 2^m \big\| P_k \mathcal{I}_l [A_{\geq 1,\gamma_0} f^{\mu}_{j_1,k_1}(s), P_{k_2}\partial_s f^{\nu}(s)]\big\|_{L^2} \lesssim 2^{-3\delta^2 m}. \tag{5.50}$$

To prove (5.50) we decompose $P_{k_2}\partial_s f^{\nu}(s)$ as in (4.8). The terms that are bounded in L^2 by $2^{-4m/3+4\delta m}$ lead to acceptable contributions, using the $L^2 \times L^\infty$ argument with Lemma 3.5 and (3.34). It remains to consider the terms $A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)$ when $\max(j_3,j_4) \leq (1-\delta^2)m$ and $k_3, k_4 \in [-2m/N_0', 300]$. For these terms, it suffices to prove that

$$\big\| P_k \mathcal{I}_l [A_{\geq 1,\gamma_0} f^{\mu}_{j_1,k_1}(s), A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)]\big\|_{L^2} \lesssim 2^{-4m}. \tag{5.51}$$

Notice that $A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)$ is given by an expression similar to (4.10). Therefore

$$\begin{aligned}\mathcal{F}\{P_k \mathcal{I}_l [A_{\geq 1,\gamma_0} f^{\mu}_{j_1,k_1}(s), A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)]\}(\xi) &= \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\widetilde{\Phi}(\xi,\eta,\sigma)} \widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s) \\ &\times \varphi_{\leq -101}(|\xi-\eta|-\gamma_0) 2^{-l} \widetilde{\varphi}_l(\Phi_{+\mu\nu}(\xi,\eta)) \varphi_k(\xi)\varphi_{k_2}(\eta) \\ &\times \mathfrak{m}_{\mu\nu}(\xi,\eta)\mathfrak{m}_{\nu\beta\gamma}(\eta,\sigma) \widehat{f^{\beta}_{j_3,k_3}}(\eta-\sigma,s) \widehat{f^{\gamma}_{j_4,k_4}}(\sigma,s)\, d\sigma d\eta,\end{aligned} \tag{5.52}$$

where $\widetilde{\Phi}(\xi,\eta,\sigma) = \Lambda(\xi) - \Lambda_\mu(\xi-\eta) - \Lambda_\beta(\eta-\sigma) - \Lambda_\gamma(\sigma)$. The main observation is that either

$$\big|\nabla_\eta \widetilde{\Phi}(\xi,\eta,\sigma)\big| = \big|\nabla\Lambda_\mu(\xi-\eta) - \nabla\Lambda_\beta(\eta-\sigma)\big| \gtrsim 1, \tag{5.53}$$

or

$$\big|\nabla_\sigma \widetilde{\Phi}(\xi,\eta,\sigma)\big| = \big|\nabla\Lambda_\beta(\eta-\sigma) - \nabla\Lambda_\gamma(\sigma)\big| \gtrsim 1, \tag{5.54}$$

in the support of the integral. Indeed, $||\eta|-\gamma_0| \leq 2^{-95}$ in view of the cutoffs on the variables ξ and $\xi-\eta$. If $\big|\nabla_\sigma \widetilde{\Phi}(\xi,\eta,\sigma)\big| \leq 2^{-\mathcal{D}}$ then $\max(|k_3|,|k_4|) \leq 300$ and, using Proposition 7.2 (ii) (in particular (7.17)), it follows that $|\eta-\sigma|$ is close to either $\gamma_0/2$, or $p_{+-1}(\gamma_0) \geq 1.1\gamma_0$, or $p_{+-1}(\gamma_0) - \gamma_0 \leq 0.9\gamma_0$. In these cases the lower bound (5.53) follows. The desired bound (5.51) then follows using Lemma 3.3 (i).

Case 3. If $\underline{k} = k_2$ then we do not prove the stronger estimate (5.40). In this case the desired bound follows from Lemma 5.9 below.

Lemma 5.9. *Assume that* (5.23) *holds and, in addition,*

$$j \le m + 2\mathcal{D} + \max(|k|, |k_1|, |k_2|)/2, \qquad k_2 \le -2\mathcal{D}, \qquad 2^{-l} \le 2^{10\delta m} + 2^{-k_2/2+\mathcal{D}}. \tag{5.55}$$

Then, for any j_1,

$$2^{(1-50\delta)j}\|Q_{jk}\mathcal{B}_{m,l}[f^{\mu}_{j_1,k_1}, P_{k_2}\partial_s f^{\nu}]\|_{L^2} \lesssim 2^{-3\delta^2 m}. \tag{5.56}$$

Proof. We record the bounds

$$\begin{aligned} &\|P_{k_2}\partial_s f^{\nu}(s)\|_{L^2} \lesssim 2^{-m+5\delta m}(2^{k_2} + 2^{-m/2}),\\ &\sup_{|\rho|\le 2^{-l+2\delta^2 m}} \|e^{-i(s+\rho)\Lambda_\nu}P_{k_2}\partial_s f^{\nu}(s)\|_{L^\infty} \lesssim 2^{-5m/3+10\delta^2 m}(2^{k_2/2+10\delta m} + 1), \end{aligned} \tag{5.57}$$

see (4.7), (4.21), and (5.42). We will prove that for any $s \in \mathcal{I}_m$

$$2^{(1-50\delta)j}2^m\|Q_{jk}\mathcal{I}_l[f^{\mu}_{j_1,k_1}(s), P_{k_2}\partial_s f^{\nu}(s)]\|_{L^2} \lesssim 2^{-3\delta^2 m}. \tag{5.58}$$

Step 1. We notice the identity

$$\begin{aligned} Q_{jk}\mathcal{I}_l[f^{\mu}_{j_1,k_1}(s),P_{k_2}\partial_s f^{\nu}(s)](x) = C\widetilde{\varphi}^{(k)}_j(x)\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i[s\Phi(\xi,\eta)+x\cdot\xi]}2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta))\\ \times \varphi_k(\xi)m_0(\xi,\eta)\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s)\widehat{P_{k_2}\partial_s f}^{\nu}(\eta,s)\,d\xi d\eta. \end{aligned}$$

Therefore $\big\|Q_{jk}\mathcal{I}_l[f^{\mu}_{j_1,k_1}(s), P_{k_2}\partial_s f^{\nu}(s)]\big\|_{L^2} \lesssim 2^{-4m}$, using integration by parts in ξ and Lemma 3.3 (i), unless

$$2^j \le \max\big\{2^{j_1+\delta m}, 2^{m+\max(|k|,|k_1|)/2+\mathcal{D}}\big\}. \tag{5.59}$$

On the other hand, assuming (5.59), $L^2\times L^\infty$ bounds using Lemma 3.5, the bounds (5.57), and Lemma 3.6 show that (5.58) holds in the following cases:

$$\begin{aligned} &\text{either } k_1 \le -10 \quad\text{and}\quad j_1 \le m-\delta m,\\ &\text{or } k_1 \le -10 \quad\text{and}\quad j_1 \ge m-\delta m,\\ &\text{or } k_1 \ge 10 \quad\text{and}\quad j_1 \le 2m/3,\\ &\text{or } k_1 \ge 10 \quad\text{and}\quad j_1 \ge 2m/3. \end{aligned} \tag{5.60}$$

See the similar estimates in the proof of Lemma 5.5 above, in particular those in (Step 2, Case 1) and (Step 2, Case 2). In each case we estimate $e^{-i(s+\rho)\Lambda_\mu}f^{\mu}_{j_1,k_1}(s)$ in L^∞ and $e^{-i(s+\rho)\Lambda_\nu}P_{k_2}\partial_s f^{\nu}(s)$ in L^2 when j_1 is small, and we estimate $e^{-i(s+\rho)\Lambda_\mu}f^{\mu}_{j_1,k_1}(s)$ in L^2 and $e^{-i(s+\rho)\Lambda_\nu}P_{k_2}\partial_s f^{\nu}(s)$ in L^∞ when j_1 is large. We estimate the contribution of the symbol m_0 by $2^{(k+k_1+k_2)/2}$ in all cases.

It remains to prove the desired bound (5.58) when $k, k_1 \in [-20, 20]$. We can still prove this when $f^{\mu}_{j_1,k_1}(s)$ is replaced by $A_{\le 0,\gamma_0}f^{\mu}_{j_1,k_1}(s)$, or when $j_1 \ge m/3-\delta m$, or when $k_2 \le -m/3+\delta m$, using $L^2\times L^\infty$ estimates as before.

Step 2. To deal with the remaining cases we use the decomposition (4.8). The contribution of the error component $P_{k_2}E^{a_2,\alpha_2}_{\nu}$ can also be estimated in the same way when $j_1 \le m/3-\delta m$. After these reductions, we may assume that

$$\begin{aligned} &k, k_1 \in [-20, 20], \quad j_1 \le m/3-\delta m, \quad j \le m+2\mathcal{D}, \quad k_2 \in [-m/3+\delta m, -2\mathcal{D}],\\ &2^{-l} \lesssim 2^{10\delta m} + 2^{-k_2/2}. \end{aligned} \tag{5.61}$$

It remains to prove that for any $[(k_3, j_3), (k_4, j_4)] \in X_{m,k_2}$

$$2^{(1-50\delta)j}2^m\big\|Q_{jk}\mathcal{I}_l[A_{\ge 1,\gamma_0}f^{\mu}_{j_1,k_1}, A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}]\big\|_{L^2} \lesssim 2^{-4\delta^2 m}. \tag{5.62}$$

The $L^2\times L^\infty$ argument still works to prove (5.62) if

$$\big\|A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}(s)\big\|_{L^2}\lesssim 2^{-7m/6+10\delta m}. \tag{5.63}$$

We notice that this bound holds if $\max(j_3,j_4)\geq m/3-\delta m$. Indeed, since $k_2\leq -2\mathcal{D}$, we have

$$P_{k_2}I^{\nu\beta\gamma}[A_{\geq 1,\gamma_1}f^{\beta}_{j_3,k_3}(s),A_{\geq 1,\gamma_0}f^{\gamma}_{j_4,k_4}(s)]\equiv 0,$$

and the bound (5.63) follows by $L^2\times L^\infty$ arguments as in the proof of Lemma 4.1.

Therefore we may assume that $j_3,j_4\leq m/3-\delta m$. We examine the explicit formula (5.52). We claim that $|\mathcal{F}\{P_k\mathcal{I}_l[A_{\geq 1,\gamma_0}f^{\mu}_{j_1,k_1}(s),A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}(s)]\}(\xi)|\lesssim 2^{-10m}$ if $|k_3|\geq \mathcal{D}/10$. Indeed, in this case the η derivative of the phase $\widetilde{\Phi}$ is $\gtrsim 2^{|k_3|/2}$ in the support of the integral (recall that $|k_1|\leq 20$). Integration by parts in η, using Lemma 3.3 (i), shows that the resulting integral is negligible, as desired.

In view of Lemma 4.1 (ii) (3), it remains to prove (5.62) when, in addition to (5.61),

$$k_3,k_4\in[-10,10],\qquad j_3,j_4\leq m/3-\delta m,\qquad \beta=-\gamma. \tag{5.64}$$

We examine again the formula (5.52) and notice that the (η,σ) derivative of the phase $\widetilde{\Phi}$ is $\gtrsim 1$ unless $||\eta-\sigma|-\gamma_0|\leq 2^{-98}$ and $||\sigma|-\gamma_0|\leq 2^{-98}$. Therefore we may replace $f^{\beta}_{j_3,k_3}$ with $A_{\geq -5,\gamma_0}f^{\beta}_{j_3,k_3}$ and $f^{\gamma}_{j_4,k_4}$ with $A_{\geq -5,\gamma_0}f^{\gamma}_{j_4,k_4}$, at the expense of negligible errors. Finally, we may assume that $l\geq -\mathcal{D}$ if $\mu=-$, and we may assume that $j\leq m+k_2+\mathcal{D}$ if $\mu=+$ (otherwise the approximate finite speed of propagation argument used in the proof of (5.13) and Lemma 5.4, which relies on integration by parts in ξ, gives rapid decay). Therefore, in proving (5.62) we may assume that

$$2^{-l}2^{(1-50\delta)j}\lesssim 2^{(1-50\delta m)}(1+2^{k_2/2+10\delta m}). \tag{5.65}$$

Let $\kappa_r:=2^{\delta^2 m}2^{k_2/2-m/2}$. We observe now that if $||\eta-\sigma|-\gamma_0|+||\sigma|-\gamma_0|\leq 2^{-90}$ and $|\Xi_{\beta\gamma}(\eta,\sigma)|=|(\nabla_\theta\Phi_{\nu\beta\gamma})(\eta,\sigma)|\leq 2\kappa_r$ then

$$||\sigma|-\gamma_0|\geq 2^{k_2-10},\qquad ||\eta-\sigma|-\gamma_0|\geq 2^{k_2-10}. \tag{5.66}$$

Indeed, we may assume that $\sigma=(\sigma_1,0)$, $\eta=(\eta_1,\eta_2)$, $|\sigma_1-\gamma_0|\leq 2^{-90}$, $|\eta|\in[2^{k_2-2},2^{k_2+2}]$. Recalling that $\beta=-\gamma$ and using the formula (7.22), the condition $|\Xi_{\beta\gamma}(\eta,\sigma)|\leq 2\kappa_r$ gives

$$\Big|\lambda'(\sigma_1)-\frac{\sigma_1-\eta_1}{|\sigma-\eta|}\lambda'(|\sigma-\eta|)\Big|\leq 2\kappa_r,\qquad \frac{|\eta_2|}{|\sigma-\eta|}\lambda'(|\sigma-\eta|)\leq 2\kappa_r.$$

Since $k_2\in[-m/3+\delta m,-2\mathcal{D}]$ and $\kappa_r=2^{\delta^2 m+k_2/2-m/2}$ it follows that $|\eta_2|\leq \kappa_r 2^{\mathcal{D}}\leq 2^{k_2-\mathcal{D}}$, $|\eta_1|\in[2^{k_2-3},2^{k_2+3}]$, and $|\lambda'(\sigma_1)-\lambda'(\sigma_1-\eta_1)|\leq 4\kappa_r$. On the other hand, if $|\sigma_1-\gamma_0|\leq 2^{k_2-10}$ and $|\eta_1|\in[2^{k_2-3},2^{k_2+3}]$ then $|\lambda'(\sigma_1)-\lambda'(\sigma_1-\eta_1)|\gtrsim 2^{2k_2}$ (since $\lambda''(\gamma_0)=0$ and $\lambda'''(\gamma_0)\approx 1$), which gives a contradiction. The claims in (5.66) follow.

We examine now the formula (5.52) and recall (5.64) and (5.66). Using Lemma 3.3 (i) and integration by parts in σ, we notice that we may insert the factor $\varphi(\kappa_r^{-1}\Xi_{\beta\gamma}(\eta,\sigma))$, at the expense of a negligible error. It remains to prove that

$$2^{(1-50\delta)j}2^m\|H\|_{L^2}\lesssim 2^{-4\delta^2 m}, \tag{5.67}$$

where, with $g_1 := A_{\geq 1,\gamma_0} f^{\mu}_{j_1,k_1}(s)$, $g_3 := A_{[-20,20-k_2],\gamma_0} f^{\beta}_{j_3,k_3}(s)$, $g_4 := A_{[-20,20-k_2],\gamma_0} f^{\gamma}_{j_4,k_4}(s)$,

$$\widehat{H}(\xi) := \varphi_k(\xi) \int_{\mathbb{R}^2} e^{is[\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta)]} \widehat{g_1}(\xi-\eta) 2^{-l} \widetilde{\varphi}_l(\Phi_{+\mu\nu}(\xi,\eta)) \mathfrak{m}_{\mu\nu}(\xi,\eta) \widehat{G_2}(\eta)\, d\eta,$$

$$\widehat{G_2}(\eta) := \varphi_{k_2}(\eta) \int_{\mathbb{R}^2} e^{is[\Lambda_\nu(\eta)-\Lambda_\beta(\eta-\sigma)-\Lambda_\gamma(\sigma)]} \mathfrak{m}_{\nu\beta\gamma}(\eta,\sigma) \varphi(\kappa_r^{-1}\Xi_{\beta\gamma}(\eta,\sigma)) \widehat{g_3}(\eta-\sigma) \widehat{g_4}(\sigma)\, d\sigma.$$

We use now the more precise bound (3.32) to see that

$$\big\|e^{-is\Lambda_\beta} g_3\big\|_{L^\infty} + \big\|e^{-is\Lambda_\gamma} g_4\big\|_{L^\infty} \lesssim 2^{-m+4\delta^2 m} 2^{-k_2/2}.$$

This bound is the main reason for proving (5.66). After removing the factor $\varphi(\kappa_r^{-1}\Xi_{\beta\gamma}(\eta,\sigma))$ at the expense of a small error, and using also (3.2) and (5.42), it follows that

$$\big\|e^{-i(s+\rho)\Lambda_\nu} G_2\big\|_{L^\infty} \lesssim (1+|\rho| 2^{k_2/2}) 2^{k_2} \cdot 2^{-2m+8\delta^2 m} 2^{-k_2} \lesssim (1+|\rho| 2^{k_2/2}) 2^{-2m+8\delta^2 m},$$

for any $\rho \in \mathbb{R}$. We use now the $L^2 \times L^\infty$ argument, together with Lemma 3.5, to estimate

$$\|H\|_{L^2} \lesssim 2^{k_2/2} 2^{-l} \cdot (1+2^{-l} 2^{k_2/2}) 2^{-2m+12\delta^2 m} \lesssim 2^{-2m+12\delta^2 m} 2^{k_2/2} 2^{-l} (1+2^{10\delta m+k_2/2}).$$

The desired bound (5.67) follows using also (5.65). □

5.5. **The case of strongly resonant interactions, I.** In this subsection we prove Lemma 5.6. This is where we need the localization operators $A^{(j)}_{n,\gamma_1}$ to control the output. It is an instantaneous estimate, in the sense that the time evolution will play no role. Hence, it suffices to show the following: let $\chi \in C^\infty_c(\mathbb{R}^2)$ be supported in $[-1,1]$ and assume that j,l,s,m satisfy

$$-m + \delta m/2 \leq l \leq 10m/N_0', \qquad 2^{m-4} \leq s \leq 2^{m+4}. \tag{5.68}$$

Assume that

$$\|f\|_{H^{N_0'} \cap H^{N_1'}_\Omega \cap Z_1} + \|g\|_{H^{N_0'} \cap H^{N_1'}_\Omega \cap Z_1} \leq 1, \tag{5.69}$$

and define, with $\chi_l(x) = \chi(2^{-l}x)$,

$$\widehat{I[f,g]}(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \chi_l(\Phi(\xi,\eta)) m_0(\xi,\eta) \widehat{f}(\xi-\eta) \widehat{g}(\eta) d\eta.$$

Assume also that k,k_1,k_2,j,m satisfy (5.23) and (5.25). Then

$$2^{\delta m/2} 2^{-l} \|Q_{jk} I[P_{k_1} f, P_{k_2} g]\|_{B_j} \lesssim 2^{-5\delta^2 m}. \tag{5.70}$$

To prove (5.70) we define $f_{j_1,k_1}, g_{j_2,k_2}, f_{j_1,k_1,n_1}, g_{j_2,k_2,n_2}$ as in (3.23), $(k_1,j_1),(k_2,j_2) \in \mathcal{J}$, $n_1 \in [0,j_1+1]$, $n_2 \in [0,j_2+1]$. We will analyze several cases depending on the relative sizes of the main parameters m,l,k,j,k_1,j_1,k_2,j_2. In many cases we will prove the stronger bound

$$2^{\delta m/2} 2^{-l} 2^{(1-50\delta)j} \|Q_{jk} I[f_{j_1,k_1}, g_{j_2,k_2}]\|_{L^2} \lesssim 2^{-6\delta^2 m}. \tag{5.71}$$

However, in the main case (5.73), we can only prove the weaker bound

$$2^{\delta m/2} 2^{-l} \|Q_{jk} I[f_{j_1,k_1}, g_{j_2,k_2}]\|_{B_j} \lesssim 2^{-6\delta^2 m}. \tag{5.72}$$

These bounds clearly suffice to prove (5.70).

Case 1: We prove first the bound (5.72) under the assumption

$$\max(j_1,j_2) \leq 9m/10, \qquad 2l \leq \min(k,k_1,k_2,0) - \mathcal{D}. \tag{5.73}$$

We may assume $j_1 \leq j_2$. With

$$\kappa_\theta := 2^{-m/2+\delta^2 m}, \qquad \kappa_r := 2^{\delta^2 m}\big(2^{-m/2+3\max(|k|,|k_1|,|k_2|)/4} + 2^{j_2-m}\big)$$

we decompose

$$
\begin{aligned}
&\mathcal{F}I[f_{j_1,k_1},g_{j_2,k_2}]=\mathcal{R}_1+\mathcal{R}_2+\mathcal{NR},\\
&\mathcal{R}_1(\xi):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\varphi(\kappa_r^{-1}\Xi(\xi,\eta))\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))\widehat{f_{j_1,k_1}}(\xi-\eta)\widehat{g_{j_2,k_2}}(\eta)d\eta,\\
&\mathcal{R}_2(\xi):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\varphi(\kappa_r^{-1}\Xi(\xi,\eta))\varphi_{\geq 1}(\kappa_\theta^{-1}\Theta(\xi,\eta))\widehat{f_{j_1,k_1}}(\xi-\eta)\widehat{g_{j_2,k_2}}(\eta)d\eta,\\
&\mathcal{NR}(\xi):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\varphi_{\geq 1}(\kappa_r^{-1}\Xi(\xi,\eta))\widehat{f_{j_1,k_1}}(\xi-\eta)\widehat{g_{j_2,k_2}}(\eta)d\eta.
\end{aligned}
$$

With $\psi_1:=\varphi_{\leq(1-\delta/4)m}$ and $\psi_2:=\varphi_{>(1-\delta/4)m}$, we rewrite

$$
\begin{aligned}
&\mathcal{NR}(\xi)=C2^l[\mathcal{NR}_1(\xi)+\mathcal{NR}_2(\xi)],\\
&\mathcal{NR}_i(\xi):=\int_{\mathbb{R}}\int_{\mathbb{R}^2}e^{i(s+\lambda)\Phi(\xi,\eta)}\widehat{\chi}(2^l\lambda)\psi_i(\lambda)m_0(\xi,\eta)\varphi_{\geq 1}(\kappa_r^{-1}\Xi(\xi,\eta))\widehat{f_{j_1,k_1}}(\xi-\eta)\widehat{g_{j_2,k_2}}(\eta)\,d\eta d\lambda.
\end{aligned}
$$

Since $\widehat{\chi}$ is rapidly decreasing we have $\|\varphi_k\cdot\mathcal{NR}_2\|_{L^\infty}\lesssim 2^{-4m}$, which gives an acceptable contribution. On the other hand, in the support of the integral defining $\mathcal{NR}_1$, we have that $|s+\lambda|\approx 2^m$ and integration by parts in η (using Lemma 3.3 (i)) gives $\|\varphi_k\cdot\mathcal{NR}_1\|_{L^\infty}\lesssim 2^{-4m}$.

The contribution of $\mathcal{R}=\mathcal{R}_1+\mathcal{R}_2$ is only present if we have a space-time resonance. In particular, in view of Proposition 7.2 (iii) (notice that the assumption (7.20) is satisfied due to (5.73)) we may assume that

$$
-10\leq k,k_1,k_2\leq 10,\quad \pm(\sigma,\mu,\nu)=(+,+,+),\quad \big||\xi|-\gamma_1\big|+|\eta-\xi/2|\leq 2^{-\mathcal{D}}. \tag{5.74}
$$

Notice that, if $\mathcal{R}(\xi)\neq 0$ then

$$
\big||\xi|-\gamma_1\big|\lesssim|\Phi(\xi,\xi/2)|\lesssim|\Phi(\xi,\eta)|+|\Phi(\xi,\eta)-\Phi(\xi,\xi/2)|\lesssim 2^l+\kappa_r^2. \tag{5.75}
$$

Integration by parts using Lemma 3.4 shows that $\|\varphi_k\cdot\mathcal{R}_2\|_{L^\infty}\lesssim 2^{-5m/2}$, which gives an acceptable contribution. To bound the contribution of $\mathcal{R}_1$ we will show that

$$
2^{\delta m/2}2^{-l}\sup_{|\xi|\approx 1}\big|\big(1+2^m\big||\xi|-\gamma_1\big|\big)\mathcal{R}_1(\xi)\big|\lesssim 2^{9\delta m/10}, \tag{5.76}
$$

which is stronger than the bound we need in (5.72). Indeed for j fixed we estimate

$$
\begin{aligned}
&\sup_{0\leq n\leq j}2^{(1-50\delta)j}2^{-n/2+49\delta n}\big\|A^{(j)}_{n,\gamma_1}Q_{jk}\mathcal{F}^{-1}\mathcal{R}_1\big\|_{L^2}\\
&\quad\lesssim\sup_{0\leq n\leq j}2^{(1-50\delta)j}2^{-n/2+49\delta n}\big\|\varphi^{[-j,0]}_{-n}(2^{100}||\xi|-\gamma_1|)\mathcal{R}_1(\xi)\big\|_{L^2_\xi}\\
&\quad\lesssim\sum_{n\geq 0}2^{(1-50\delta)j}2^{-n/2-(1/2-49\delta)\min(n,j)}\big\|\varphi^{(-\infty,0]}_{-n}(2^{100}||\xi|-\gamma_1|)\mathcal{R}_1(\xi)\big\|_{L^\infty_\xi},
\end{aligned} \tag{5.77}
$$

and notice that (5.72) would follow from (5.76) and the assumption $j\leq m+3\mathcal{D}$.

Recall from Lemma 3.6 and (5.74) (we may assume $f_{j_1,k_1}=f_{j_1,k_1,0}$, $g_{j_2,k_2}=g_{j_2,k_2,0}$) that

$$
\begin{aligned}
&2^{(1/2-\delta')j_1}\|\widehat{f_{j_1,k_1}}\|_{L^\infty}+2^{(1-\delta')j_1}\sup_{\theta\in\mathbb{S}^1}\|\widehat{f_{j_1,k_1}}(r\theta)\|_{L^2(rdr)}\lesssim 1,\\
&2^{(1/2-\delta')j_2}\|\widehat{g_{j_2,k_2}}\|_{L^\infty}+2^{(1-\delta')j_2}\sup_{\theta\in\mathbb{S}^1}\|\widehat{g_{j_2,k_2}}(r\theta)\|_{L^2(rdr)}\lesssim 1.
\end{aligned} \tag{5.78}
$$

We ignore first the factor $\chi_l(\Phi(\xi,\eta))$. In view of Proposition 7.2 (ii) the η integration in the definition of $\mathcal{R}_1(\xi)$ takes place essentially over a $\kappa_\theta\times\kappa_r$ box in the neighborhood of $\xi/2$. Using (5.75) and (5.78), and estimating $\|\widehat{f_{j_1,k_1}}\|_{L^\infty}\lesssim 1$, we have, if $j_2\geq m/2$,

$$|(1+2^m||\xi|-\gamma_1|)\mathcal{R}_1(\xi)|\lesssim 2^m(2^l+\kappa_r^2)2^{-j_2+\delta' j_2}\kappa_\theta\kappa_r^{1/2}\lesssim (2^l+\kappa_r^2)2^{-j_2(1/2-\delta')}2^{2\delta^2 m}.$$

On the other hand, if $j_2\leq m/2$ we estimate $\|\widehat{f_{j_1,k_1}}\|_{L^\infty}+\|\widehat{f_{j_2,k_2}}\|_{L^\infty}\lesssim 1$ and conclude that

$$|(1+2^m||\xi|-\gamma_1|\mathcal{R}_1(\xi)|\lesssim 2^{m+l}\kappa_\theta\kappa_r\lesssim 2^l 2^{2\delta^2 m}.$$

The desired bound (5.76) follows if $\kappa_r^2 2^{-l}\leq 2^{j_2/4}$.

Assume now that $\kappa_r^2\geq 2^l 2^{j_2/4}$ (in particular $j_2\geq 11m/20$). In this case the restriction $|\Phi(\xi,\eta)|\leq 2^l$ is stronger and we have to use it. We decompose, with $p_-:=\lfloor\log_2(2^{l/2}\kappa_r^{-1})+\mathcal{D}\rfloor$,

$$\mathcal{R}_1(\xi)=\sum_{p\in[p_-,0]}\mathcal{R}_1^p(\xi),$$

$$\mathcal{R}_1^p(\xi):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\varphi_p^{[p_-,1]}(\kappa_r^{-1}\Xi(\xi,\eta))\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))\widehat{f_{j_1,k_1}}(\xi-\eta)\widehat{g_{j_2,k_2}}(\eta)d\eta.$$

As in (5.75), notice that if $\mathcal{R}_1^p(\xi)\neq 0$ then $||\xi|-\gamma_1|\lesssim 2^{2p}\kappa_r^2$. The term $\mathcal{R}_1^{p_-}(\xi)$ can be bounded as before. Moreover, using the formula (7.46), it is easy to see that if $\xi=(s,0)$ is fixed then the set of points η that satisfy the three restrictions $|\Phi(\xi,\eta)|\lesssim 2^l$, $|\nabla_\eta\Phi(\xi,\eta)|\approx 2^p\kappa_r$, $|\xi\cdot\eta^\perp|\lesssim\kappa_\theta$ is essentially contained in a union of two $\kappa_\theta\times 2^l 2^{-p}\kappa_r^{-1}$ boxes. Using (5.78), and estimating $\|\widehat{f_{j_1,k_1}}\|_{L^\infty}\lesssim 1$, we have

$$|(1+2^m||\xi|-\gamma_1|)\mathcal{R}_1^p(\xi)|\lesssim 2^{m+2p}\kappa_r^2 2^{-j_2+\delta' j_2}\kappa_\theta(2^l 2^{-p}\kappa_r^{-1})^{1/2}\lesssim 2^{3p/2}2^{-m+4\delta^2 m}2^{l/2}2^{j_2/2+\delta' j_2}.$$

This suffices to prove (5.76) since $2^p\leq 1$, $2^{-l/2}\leq 2^{m/2}$, and $2^{j_2}\leq 2^{9m/10}$, see (5.73).

Case 2. We assume now that

$$2l\geq\min(k,k_1,k_2,0)-\mathcal{D}. \tag{5.79}$$

In this case we prove the stronger bound (5.71). We can still use the standard $L^2\times L^\infty$ argument, with Lemma 3.5 and Lemma 3.6, to bound the contributions away from γ_0. For (5.71) it remains to prove that

$$2^{-l}2^{(1-50\delta)(m+|k|/2)}\|P_kI[A_{\geq 1,\gamma_0}f_{j_1,k_1},A_{\geq 1,\gamma_0}g_{j_2,k_2}]\|_{L^2}\lesssim 2^{-\delta m}. \tag{5.80}$$

The bound (5.80) follows if $\max(j_1,j_2)\geq m/3$, using the same $L^2\times L^\infty$ argument. On the other hand, if $j_1,j_2\leq m/3$ then we use (3.27) and the more precise bound (3.32) to see that

$$\|A_{p,\gamma_0}h\|_{L^2}\lesssim 2^{-p/2},\qquad \|e^{-it\Lambda}A_{p,\gamma_0}h\|_{L^\infty}\lesssim 2^{-m+2\delta^2 m}\min\big(2^{p/2},2^{m/2-p}\big),$$

where $h\in\{f_{j_1,k_1},g_{j_2,k_2}\}$, $p\geq 1$, and $t\approx 2^m$. Therefore, using Lemma 3.5,

$$\|P_kI[A_{p_1,\gamma_0}f_{j_1,k_1},A_{p_2,\gamma_0}g_{j_2,k_2}]\|_{L^2}\lesssim 2^k 2^{-\max(p_1,p_2)/2}\cdot 2^{-m+2\delta^2 m}2^{\min(p_1,p_2)/2}.$$

The desired bound (5.80) follows, using also the simple estimate

$$\|P_kI[A_{p_1,\gamma_0}f_{j_1,k_1},A_{p_2,\gamma_0}g_{j_2,k_2}]\|_{L^2}\lesssim 2^k 2^{-(p_1+p_2)/2}.$$

Case 3. Assume now that

$$\max(j_1,j_2)\geq 9m/10,\qquad j\leq\min(j_1,j_2)+m/4,\qquad 2l\leq\min(k,k_1,k_2,0)-\mathcal{D}.$$

Using Lemma 7.5 and (3.25) we estimate

$$\begin{aligned}&\|P_kI[f_{j_1,k_1,n_1},g_{j_2,k_2,n_2}]\|_{L^2}\\&\lesssim 2^{k/2}2^{30\delta m}2^{l/2-n_1/2-n_2/2}\big\|\sup_{\theta\in\mathbb{S}^1}|\widehat{f_{j_1,k_1,n_1}}(r\theta)|\big\|_{L^2(rdr)}\big\|\sup_{\theta\in\mathbb{S}^1}|\widehat{g_{j_2,k_2,n_2}}(r\theta)|\big\|_{L^2(rdr)}\\&\lesssim 2^{k/2}2^{l/2}2^{-j_1+\delta' j_1}2^{-j_2+\delta' j_2}2^{30\delta m},\end{aligned}\tag{5.81}$$

and the desired bound (5.71) follows.

Case 4. Finally, assume that

$$j_2\geq 9m/10,\qquad j\geq j_1+m/4,\qquad 2l\leq\min(k,k_1,k_2,0)-\mathcal{D}.\tag{5.82}$$

In particular, $j_1\leq 7m/8$. We decompose, with $\kappa_\theta=2^{-2m/5}$,

$$\begin{aligned}&I[f_{j_1,k_1},g_{j_2,k_2}]=I_{||}[f_{j_1,k_1},g_{j_2,k_2}]+I_\perp[f_{j_1,k_1},g_{j_2,k_2}],\\&\widehat{I_{||}[f,g]}(\xi)=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))\varphi(\kappa_\theta^{-1}\Omega_\eta\Phi(\xi,\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta,\\&\widehat{I_\perp[f,g]}(\xi)=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))(1-\varphi(\kappa_\theta^{-1}\Omega_\eta\Phi(\xi,\eta)))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta.\end{aligned}\tag{5.83}$$

Integration by parts using Lemma 3.4 shows that $\big\|\mathcal{F}P_kI_\perp[f_{j_1,k_1},g_{j_2,k_2}]\big\|_{L^\infty}\lesssim 2^{-5m/2}$. In addition, using Schur's test and Proposition 7.4 (i), (iii),

$$\|P_kI_{||}[f_{j_1,k_1},g_{j_2,k_2,n_2}]\|_{L^2}\lesssim 2^{90\delta m}2^l\kappa_\theta^{1/2}\|\widehat{f_{j_1,k_1}}\|_{L^\infty}\|\widehat{g_{j_2,k_2,n_2}}\|_{L^2}\lesssim 2^{95\delta m}2^{l-m/5}2^{-(1-50\delta)j_2}2^{n_2/2},$$

which gives an acceptable contribution if $n_2\leq\mathcal{D}$.

It remains to estimate the contribution of $I_{||}[f_{j_1,k_1},g_{j_2,k_2,n_2}]$ for $n_2\geq\mathcal{D}$. Since $|\eta|$ is close to γ_1 and $|\Phi(\xi,\eta)|$ is sufficiently small (see (5.82)), it follows from (7.6) that $\min(k,k_1,k_2)\geq-40$; moreover, the vectors ξ and η are almost aligned and $|\Phi(\xi,\eta)|$ is small, so we may also assume that $\max(k,k_1,k_2)\leq 100$. Moreover, $|\nabla_\eta\Phi(\xi,\eta)|\gtrsim 1$ in the support of integration of $I_{||}[f_{j_1,k_1},g_{j_2,k_2,n_2}]$, in view of Proposition 7.2 (iii). Integration by parts in η using Lemma 3.3 (i) then gives an acceptable contribution unless $j_2\geq(1-\delta^2)m$. We may also reset $\kappa_\theta=2^{\delta^2m-m/2}$, up to small errors, using Lemma 3.4.

To summarize, we may assume that

$$j_2\geq(1-\delta^2)m,\quad j\geq j_1+m/4,\quad k,k_1,k_2\in[-100,100],\quad n_2\geq\mathcal{D},\quad \kappa_\theta=2^{\delta^2m-m/2}.\tag{5.84}$$

We decompose, with $p_-:=\lfloor l/2\rfloor$,

$$\begin{aligned}&I_{||}[f_{j_1,k_1},g_{j_2,k_2,n_2}]=\sum_{p_-\leq p\leq\mathcal{D}}I^p_{||}[f_{j_1,k_1},g_{j_2,k_2,n_2}],\\&\widehat{I^p_{||}[f,g]}(\xi):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))\varphi_p^{[p_-,\mathcal{D}]}(\nabla_\xi\Phi(\xi,\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta.\end{aligned}$$

It suffices to prove that, for any p,

$$2^{-l}2^{(1-50\delta)j}\big\|Q_{jk}I^p_{||}[f_{j_1,k_1},g_{j_2,k_2,n_2}]\big\|_{L^2}\lesssim 2^{-\delta m}.\tag{5.85}$$

As a consequence of Proposition 7.4 (iii), under our assumptions in (5.84) and recalling that $|\nabla_\eta\Phi(\xi,\eta)|\gtrsim 1$ in the support of the integral,

$$\sup_\xi\int_{\mathbb{R}^2}|\chi_l(\Phi(\xi,\eta))|\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))\varphi_{\leq-\mathcal{D}/2}(|\eta|-\gamma_1)\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)d\eta\lesssim 2^{\delta^2m}2^l\kappa_\theta,$$

and, for any $p \geq p_-$,

$$\sup_\eta \int_{\mathbb{R}^2} |\chi_l(\Phi(\xi,\eta))| \varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))\varphi_p(\nabla_\xi\Phi(\xi,\eta))\varphi_{\leq -\mathcal{D}/2}(|\eta|-\gamma_1)\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)d\xi \lesssim 2^{\delta^2 m}2^{l-p}\kappa_\theta.$$

Using Schur's test we can then estimate, for $p \geq p_-$

$$\|P_k I^p_{||}[f_{j_1,k_1}, g_{j_2,k_2,n_2}]\|_{L^2} \lesssim 2^{-p/2}2^l 2^{-m/2+4\delta^2 m}\|\widehat{f_{j_1,k_1}}\|_{L^\infty}\|g_{j_2,k_2,n_2}\|_{L^2} \lesssim 2^{-p/2}2^l 2^{-m+5\delta m}.$$

The desired bound (5.84) follows if $j \leq m+p+4\delta m$. On the other hand, if $j \geq m+p+4\delta m$ then we use the approximate finite speed of propagation argument to show that

$$\|Q_{jk} I^p_{||}[f_{j_1,k_1}, g_{j_2,k_2,n_2}]\|_{L^2} \lesssim 2^{-3m}. \tag{5.86}$$

Indeed, we write, as in Lemma 3.5, $\chi_l(\Phi(\xi,\eta)) = c2^l\int_{\mathbb{R}}\widehat{\chi}(2^l\rho)e^{i\rho\Phi(\xi,\eta)}\,d\rho$ and notice that $\big|\nabla_\xi[x\cdot\xi+(s+\rho)\Phi(\xi,\eta)]\big| \approx 2^j$ in the support of the integral, provided that $|x|\approx 2^j$ and $|\rho|\leq 2^m$. Then we recall that $j \geq j_1+m/4$, see (5.84), and use Lemma 3.3 (i) to prove (5.86). This completes the proof of Lemma 5.6.

5.6. **The case of weakly resonant interactions.** In this subsection we prove Lemma 5.7. We decompose $P_{k_2}\partial_s f^\nu$ as in (4.8) and notice that the contribution of the error term can be estimated using the $L^2\times L^\infty$ argument as before.

To estimate the contributions of the terms $A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}$ we need more careful analysis of trilinear operators. With $\widetilde{\Phi}(\xi,\eta,\sigma) = \Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\beta(\eta-\sigma)-\Lambda_\gamma(\sigma)$ and $p\in\mathbb{Z}$ we define the trilinear operators $\mathcal{J}_{l,p}$ by

$$\begin{aligned}\widehat{\mathcal{J}_{l,p}[f,g,h]}(\xi,s) := \int_{\mathbb{R}^2\times\mathbb{R}^2} & e^{is\widetilde{\Phi}(\xi,\eta,\sigma)}\widehat{f}(\xi-\eta)2^{-l}\widetilde{\varphi}_l(\Phi_{+\mu\nu}(\xi,\eta))\varphi_p(\widetilde{\Phi}(\xi,\eta,\sigma))\\ &\times\varphi_{k_2}(\eta)\mathfrak{m}_{\mu\nu}(\xi,\eta)\mathfrak{m}_{\nu\beta\gamma}(\eta,\sigma)\widehat{g}(\eta-\sigma)\widehat{h}(\sigma)\,d\sigma d\eta.\end{aligned} \tag{5.87}$$

Let $\mathcal{J}_{l,\leq p} = \sum_{q\leq p}\mathcal{J}_{l,q}$ and $\mathcal{J}_l = \sum_{q\in\mathbb{Z}}\mathcal{J}_{l,q}$. Let

$$\mathcal{C}_{l,p}[f,g,h] := \int_{\mathbb{R}} q_m(s)\mathcal{J}_{l,p}[f,g,h](s)\,ds, \qquad \mathcal{C}_{l,\leq p} := \sum_{q\leq p}\mathcal{C}_{l,q}, \qquad \mathcal{C}_l = \sum_{q\in\mathbb{Z}}\mathcal{C}_{l,q}. \tag{5.88}$$

Notice that

$$\mathcal{B}_{m,l}[f^\mu_{j_1,k_1}, A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}] = \mathcal{C}_l[f^\mu_{j_1,k_1}, f^\beta_{j_3,k_3}, f^\gamma_{j_4,k_4}]. \tag{5.89}$$

To prove the lemma it suffices to show that

$$2^{(1-50\delta)j}\big\|Q_{jk}\mathcal{C}_l[f^\mu_{j_1,k_1}, f^\beta_{j_3,k_3}, f^\gamma_{j_4,k_4}]\big\|_{L^2} \lesssim 2^{-3\delta^2 m} \tag{5.90}$$

provided that

$$\begin{aligned}&k,k_1,k_2\in[-3.5m/N_0', 3.2m/N_0'], \quad j\leq m+2\mathcal{D}+\max(|k|,|k_1|,|k_2|)/2,\\ &l\geq -m/14, \quad m\geq \mathcal{D}^2/8, \quad k_2,k_3,k_4\leq m/N_0', \quad [(k_3,j_3),(k_4,j_4)]\in X_{m,k_2}.\end{aligned} \tag{5.91}$$

The bound (5.42) and the same argument as in the proof of Lemma 3.5 show that

$$\begin{aligned}\big\|P_k\mathcal{J}_{l,\leq p}[f,g,h](s)\big\|_{L^2} \lesssim & 2^{(k+k_1+k_2)/2}2^{(k_2+k_3+k_4)/2}2^{-l}\min\big\{|f|_\infty|g|_2|h|_\infty, |f|_\infty|g|_\infty|h|_2,\\ &(1+2^{-l+2\delta^2 m+3\max(k_2,0)/2})|f|_2|g|_\infty|h|_\infty\big\} + 2^{-10m}|f|_2|g|_2|h|_2,\end{aligned} \tag{5.92}$$

provided that $s \in I_m$, $2^{-p}+2^{-l} \leq 2^{m-2\delta^2 m}$, $f = P_{[k_1-8,k_1+8]}f$, $g = P_{[k_3-8,k_3+8]}g$, $h = P_{[k_4-8,k_4+8]}h$, and, for $F \in \{f,g,h\}$,

$$|F|_q := \sup_{|t|\in[2^{m-4},2^{m+4}]} \|e^{it\Lambda}F\|_{L^q}. \tag{5.93}$$

In particular, the bounds (5.92) and (3.33) show that

$$2^{(1-50\delta)j}\big\|Q_{jk}\mathcal{C}_l[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}]\big\|_{L^2} \lesssim 2^{-\delta m}$$

provided that $\max(j_1,j_3,j_4) \geq 20m/21$. Therefore, it remains to prove (5.90) when

$$\max(j_1,j_3,j_4) \leq 20m/21. \tag{5.94}$$

Step 1. We consider first the contributions of $\mathcal{C}_{l,p}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}]$ for $p \geq -11m/21$. In this case we integrate by parts in s and rewrite

$$\begin{aligned}\mathcal{C}_{l,p}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}] = i2^{-p}\Big\{ &\int_{\mathbb{R}} q_m'(s)\widetilde{\mathcal{J}}_{l,p}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s)\,ds \\ &+ \widetilde{\mathcal{C}}_{l,p}[\partial_s f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}] + \widetilde{\mathcal{C}}_{l,p}[f^{\mu}_{j_1,k_1}, \partial_s f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}] + \widetilde{\mathcal{C}}_{l,p}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, \partial_s f^{\gamma}_{j_4,k_4}]\Big\},\end{aligned}$$

where the operators $\widetilde{\mathcal{J}}_{l,p}$ and $\widetilde{\mathcal{C}}_{l,p}$ are defined in the same way as the operators $\mathcal{J}_{l,p}$ and $\mathcal{C}_{l,p}$, but with $\varphi_p(\widetilde{\Phi}(\xi,\eta,\sigma))$ replaced by $\widetilde{\varphi}_p(\widetilde{\Phi}(\xi,\eta,\sigma))$, $\widetilde{\varphi}_p(x) = 2^p x^{-1}\varphi_p(x)$, (see the formula (5.87)). The operator $\widetilde{\mathcal{J}}_{l,p}$ also satisfies the L^2 bound (5.92). Recall the L^2 bounds (4.21) on $\partial_s P_{k'}f_\sigma$. Using (5.92) (with $\partial_s P_{k'}f_\sigma$ always placed in L^2, notice that $2^{-2l} \leq 2^{m/7}$), it follows that

$$\sum_{p\geq -11m/21} 2^{(1-50\delta)j}\big\|P_k\mathcal{C}_{l,p}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}]\big\|_{L^2} \lesssim 2^{-3\delta^2 m}.$$

Step 2. For (5.90) it remains to prove that

$$2^{(1-50\delta)j}\big\|Q_{jk}\mathcal{C}_{l,\leq -11m/21}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}]\big\|_{L^2} \lesssim 2^{-3\delta^2 m}. \tag{5.95}$$

Since $\max(j_1,j_3,j_4) \leq 20m/21$, see (5.94), we have the pointwise approximate identity

$$\begin{aligned}&P_k\mathcal{C}_{l,\leq -11m/21}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}] \\ &\quad= P_k\mathcal{C}_{l,\leq -11m/21}[A_{\geq \mathcal{D}_1,\gamma_0}f^{\mu}_{j_1,k_1}, A_{\geq \mathcal{D}_1-10,\gamma_0}f^{\beta}_{j_3,k_3}, A_{\geq \mathcal{D}_1-20,\gamma_0}f^{\gamma}_{j_4,k_4}] \\ &\quad+ P_k\mathcal{C}_{l,\leq -11m/21}[A_{< \mathcal{D}_1,\gamma_0}f^{\mu}_{j_1,k_1}, A_{\leq \mathcal{D}_1+10,\gamma_0}f^{\beta}_{j_3,k_3}, A_{\leq \mathcal{D}_1+20,\gamma_0}f^{\gamma}_{j_4,k_4}] + O(2^{-4m}),\end{aligned} \tag{5.96}$$

where $\mathcal{D}_1$ is the large constant used in section 7. This is a consequence of Lemma 3.3 (i) and the observation that $|\nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma)| \gtrsim 1$ in the other cases. Letting $g_1 = A_{\geq\mathcal{D}_1,\gamma_0}f^{\mu}_{j_1,k_1}$, $g_3 = A_{\geq\mathcal{D}_1-10,\gamma_0}f^{\beta}_{j_3,k_3}$, $g_4 = A_{\geq\mathcal{D}_1-20,\gamma_0}f^{\gamma}_{j_4,k_4}$, $h_1 = A_{<\mathcal{D}_1,\gamma_0}f^{\mu}_{j_1,k_1}$, $h_3 = A_{\leq\mathcal{D}_1+10,\gamma_0}f^{\beta}_{j_3,k_3}$, $h_4 = A_{\leq\mathcal{D}_1+20,\gamma_0}f^{\gamma}_{j_4,k_4}$, it remains to prove that

$$2^{(1-50\delta)j}\big\|Q_{jk}\mathcal{C}_{l,\leq -11m/21}[g_1,g_3,g_4]\big\|_{L^2} \lesssim 2^{-3\delta^2 m}. \tag{5.97}$$

and

$$2^{(1-50\delta)j}\big\|Q_{jk}\mathcal{C}_{l,\leq -11m/21}[h_1,h_3,h_4]\big\|_{L^2} \lesssim 2^{-3\delta^2 m}. \tag{5.98}$$

Proof of (5.97)**.** We use Lemma 7.6 (i). If $l \leq -4m/N_0'$ then $|\nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma)| \gtrsim 1$ in the support of the integral (due to (7.66)) and the contribution is negligible (due to Lemma 3.3 (i) and (5.94)). On the other hand, if

$$l \geq -4m/N_0' \quad \text{and} \quad j \leq 2m/3 + \max(j_1,j_3,j_4) \tag{5.99}$$

then we apply (5.92). The left hand side of (5.97) is dominated by

$$C2^{(1-50\delta)j}2^m(1+2^{-2l})2^{-5m/3+8\delta^2 m}2^{-\max(j_1,j_3,j_4)(1-50\delta)}\lesssim 2^{-10\delta},$$

as we notice that $\max(k,k_1,k_2,k_3,k_4)\le 20$. This suffices to prove (5.97) in this case.

Finally, if

$$l\ge -4m/N_0' \quad\text{and}\quad j\ge 2m/3+\max(j_1,j_3,j_4) \tag{5.100}$$

then $\max(j_1,j_2,j_4)\le m/3+10\delta m$ and $j\ge 2m/3$. We define the localized trilinear operators

$$\begin{split}\mathcal{F}\{\mathcal{J}_{l,\le p,\kappa}[f,g,h]\}(\xi,s):=\int_{\mathbb{R}^2\times\mathbb{R}^2}e^{is\widetilde{\Phi}(\xi,\eta,\sigma)}\widehat{f}(\xi-\eta)2^{-l}\widetilde{\varphi}_l(\Phi_{+\mu\nu}(\xi,\eta))\varphi_{\le p}(\widetilde{\Phi}(\xi,\eta,\sigma))\\ \times\varphi(\kappa^{-1}\nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma))\varphi_{k_2}(\eta)\mathfrak{m}_{\mu\nu}(\xi,\eta)\mathfrak{m}_{\nu\beta\gamma}(\eta,\sigma)\widehat{g}(\eta-\sigma)\widehat{h}(\sigma)\,d\sigma d\eta,\end{split} \tag{5.101}$$

which are similar to the trilinear operators defined in (5.87) with the additional cutoff factor in $\nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma)$. Set $\kappa:=2^{-m/2+\delta^2 m}$ and notice that

$$\|\mathcal{F}\{\mathcal{J}_{l,\le -11m/21}[g_1,g_3,g_4]-\mathcal{J}_{l,\le -11m/21,\kappa}[g_1,g_3,g_4]\}\|_{L^\infty}\lesssim 2^{-6m},$$

as a consequence of Lemma 3.3 (i). Moreover, $|\nabla_\xi\widetilde{\Phi}(\xi,\eta,\sigma)|\lesssim 2^{2p/3}\approx 2^{-22m/63}$ in the support of the integral defining $\mathcal{J}_{l,\le -11m/21,\kappa}[g_1,g_3,g_4]$, due to Lemma 7.6 (i). Therefore, using the approximate finite speed of propagation of argument (integration by parts in ξ),

$$\|Q_{jk}\mathcal{J}_{l,\le -11m/21,\kappa}[g_1,g_3,g_4]\|_{L^\infty}\lesssim 2^{-6m}.$$

The desired bound (5.97) follows in this case as well (in fact, one has rapid decay if (5.100) holds).

Proof of (5.98)**.** The desired estimate follows from (5.92) and the dispersive bounds (3.31)–(3.32) if $\max(j_1,j_3,j_4)\ge m/3$ or if $j\le 2m/3$ or if $l\ge -10\delta m$. Assume that

$$\max(j_1,j_3,j_4)\le m/3,\qquad j\ge 2m/3,\qquad l\le -10\delta m. \tag{5.102}$$

As before, we may replace $\mathcal{J}_{l,\le -11m/21}[h_1,h_3,h_4]$ with $\mathcal{J}_{l,\le -11m/21,\kappa}[h_1,h_3,h_4]$, at the expense of a small error, where $\kappa=2^{-m/2+20\delta m}$. Moreover, $|\nabla_\xi\widetilde{\Phi}(\xi,\eta,\sigma)|\lesssim\kappa$ in the support of the integral defining $\mathcal{J}_{l,\le -11m/21,\kappa}[h_1,h_3,h_4]$, due to Lemma 7.6 (ii). The approximate finite speed of propagation of argument (integration by parts in ξ) then gives rapid decay in the case when (5.102) holds. This completes the proof.

5.7. **The case of strongly resonant interactions, II.** In this subsection we prove Lemma 5.8. Let $\overline{k}:=\max(k,k_1,k_2,0)$. It suffices to prove the lemma in the case

$$k,k_1,k_2\in[-\overline{k}-20,\overline{k}],\quad j\le m+3\mathcal{D}+\overline{k}/2,\quad \overline{k}\le 7m/(6N_0'),\quad l_0<l\le -m/14. \tag{5.103}$$

Indeed, we can assume that $k,k_1,k_2\ge -\overline{k}-20$, since otherwise the operator is trivial (due to (7.6)). Moreover, if $\max(k_1,k_2)\ge 7m/(6N_0')-10$ then the $L^2\times L^\infty$ argument (with Lemma 3.5) easily gives the desired conclusion due to the assumption (5.7).

We define (compare with the definition of the operators $T_{m,l}$ in (5.19))

$$\widehat{T^{\|}_{m,l}[f,g]}(\xi)=\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))\varphi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)d\eta ds,$$

where $\kappa_\theta:=2^{-m/2+6\overline{k}+\delta^2 m}$. Let $T^{\perp}_{m,l}=T_{m,l}-T^{\|}_{m,l}$, and define $\mathcal{A}^{\|}_{m,l}$ and $\mathcal{B}^{\|}_{m,l}$ similarly, by inserting the factor $\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))$ in the integrals in (5.20). We notice that

$$T^{\|}_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]=i\mathcal{A}^{\|}_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]+i\mathcal{B}^{\|}_{m,l}[P_{k_1}\partial_s f^\mu,P_{k_2}f^\nu]+i\mathcal{B}^{\|}_{m,l}[P_{k_1}f^\mu,P_{k_2}\partial_s f^\nu].$$

It remains to prove that for any j_1, j_2

$$2^{(1-50\delta)j}\|Q_{jk}T^{\perp}_{m,l}[f^{\mu}_{j_1,k_1}, f^{\nu}_{j_2,k_2}]\|_{L^2} \lesssim 2^{-3\delta^2 m}, \tag{5.104}$$

$$\|Q_{jk}\mathcal{A}^{\|}_{m,l}[f^{\mu}_{j_1,k_1}, f^{\nu}_{j_2,k_2}]\|_{B_j} \lesssim 2^{-3\delta^2 m}, \tag{5.105}$$

and

$$\|Q_{jk}\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1}, \partial_s P_{k_2} f^{\nu}]\|_{B_j} \lesssim 2^{-3\delta^2 m}. \tag{5.106}$$

Proof of (5.104)**.** We may assume that $\min(j_1, j_2) \geq m - 2\overline{k} - \delta^2 m$, otherwise the conclusion follows from Lemma 3.4. We decompose $f^{\mu}_{j_1,k_1} = \sum_{n_1=0}^{j_1+1} f_{j_1,k_1,n_1}$, $f^{\nu}_{j_2,k_2} = \sum_{n_2=0}^{j_j+1} f_{j_2,k_2,n_2}$ and estimate, using Lemma 7.5, and (3.25)

$$\begin{aligned}
&\big\|P_k T^{\perp}_{m,l}[f_{j_1,k_1,n_1}, f_{j_2,k_2,n_2}]\big\|_{L^2} \\
&\lesssim 2^{2\overline{k}} 2^m 2^{l/2-n_1/2-n_2/2} \big\| \sup_{\theta\in\mathbb{S}^1} |\widehat{f_{j_1,k_1,n_1}}(r\theta)| \big\|_{L^2(rdr)} \big\| \sup_{\theta\in\mathbb{S}^1} |\widehat{f_{j_2,k_2,n_2}}(r\theta)| \big\|_{L^2(rdr)} \\
&\lesssim 2^{2\overline{k}} 2^m 2^{l/2} 2^{6\delta^2 m} 2^{-j_1+51\delta j_1} 2^{-j_2+51\delta j_2}.
\end{aligned}$$

Therefore, using also (5.103), the left-hand side of (5.104) is dominated by

$$2^{(1-50\delta)j} \cdot 2^{6\delta^2 m} 2^{2\overline{k}} 2^m 2^{l/2} 2^{-j_1+51\delta j_1} 2^{-j_2+51\delta j_2} \lesssim 2^{8\overline{k}} 2^{l/2} 2^{54\delta m}.$$

This suffices to prove the desired bound, since $2^{l/2} \lesssim 2^{-m/28}$ and $2^{8\overline{k}} 2^{54\delta m} \lesssim 2^{64\delta m} \lesssim 2^{m/30}$.

Proof of (5.105)**.** In view of Lemma 5.6, it suffices to prove that

$$2^{(1-50\delta)j}\|Q_{jk}\mathcal{A}^{\perp}_{m,l}[f^{\mu}_{j_1,k_1}, f^{\nu}_{j_2,k_2}]\|_{L^2} \lesssim 2^{-3\delta^2 m}.$$

This is similar to the proof of (5.104) above, using Lemma 7.5 and (3.25).

Proof of (5.106)**.** This is the more difficult estimate, where we need to use the more precise information in Lemma 4.2. We may assume $j_1 \leq 3m$, since in the case $j_1 \geq 3m$ we can simply estimate $\|\widehat{f^{\mu}_{j_1,k_1}}\|_{L^1} \lesssim 2^{-j_1+51\delta j_1}$ (see (3.26)) and the desired estimate follows easily. We decompose $\partial_s P_{k_2} f^{\nu}$ as in (4.8), and then we decompose $A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4} = \sum_{i=1}^{3} A^{a_3,\alpha_3;a_4,\alpha_4;[i]}_{k_2;k_3,j_3;k_4,j_4}$ as in (4.35). Notice that since $k_2 \geq -3m/(2N'_0)$ (see (5.103)), it follows from Lemma 4.1 (ii) (2) that $\min(k_2, k_3, k_4) \geq -2m/N'_0$, so Lemma 4.2 applies. It remains to prove that

$$\|Q_{jk}\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1}, P_{k_2} E^{a_2,\alpha_2}_{\nu}]\|_{B_j} \lesssim 2^{-4\delta^2 m}, \tag{5.107}$$

and, for any $[(k_3, j_3), (k_4, j_4)] \in X_{m,k_2}$, $i \in \{1, 2, 3\}$,

$$\|Q_{jk}\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1}, A^{a_3,\alpha_3;a_4,\alpha_4;[i]}_{k_2;k_3,j_3;k_4,j_4}]\|_{B_j} \lesssim 2^{-4\delta^2 m}. \tag{5.108}$$

These bounds follow from Lemmas 5.10, 5.11, and 5.12 below. Recall the definition

$$\widehat{\mathcal{B}^{\|}_{m,l}[f,g]}(\xi) = \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \varphi(\kappa_\theta^{-1}\Theta(\xi,\eta)) 2^{-l} \widetilde{\varphi}_l(\Phi(\xi,\eta)) m_0(\xi,\eta) \widehat{f}(\xi-\eta,s) \widehat{g}(\eta,s) d\eta ds. \tag{5.109}$$

Lemma 5.10. *Assume that* (5.103) *holds and* $\kappa_\theta = 2^{-m/2+6\overline{k}+\delta^2 m}$. *Then*

$$\|Q_{jk}\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1}, h]\|_{B_j} \lesssim 2^{-4\delta^2 m}, \tag{5.110}$$

provided that, for any $s \in I_m$

$$h(s) = P_{[k_2-2,k_2+2]} h(s), \qquad \|h(s)\|_{L^2} \lesssim 2^{-3m/2+35\delta m-22\overline{k}}. \tag{5.111}$$

Proof. The lemma is slightly stronger (with a weaker assumption on h) than we need to prove (5.107), since we intend to apply it in some cases in the proof of (5.108) as well. We would like to use Schur's lemma and Proposition 7.4 (iii). For this we need to further decompose the operator $\mathcal{B}^{\|}_{m,l}$. For $p,q\in\mathbb{Z}$ we define the operators $\mathcal{B}'_{p,q}$ by

$$\begin{split}\widehat{\mathcal{B}'_{p,q}[f,g]}(\xi):=\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta))\\ \times\varphi_p(\nabla_\xi\Phi(\xi,\eta))\varphi_q(\nabla_\eta\Phi(\xi,\eta))m_0(\xi,\eta)\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,d\eta ds.\end{split}\tag{5.112}$$

Let $H_{p,q}:=P_k\mathcal{B}'_{p,q}[f^\mu_{j_1,k_1},h]$. Using the bounds $\|\widehat{f^\mu_{j_1,k_1}}\|_{L^\infty}\lesssim 2^{2\delta j_1}2^{5\delta^2 m}2^{51\delta\overline{k}}\lesssim 2^{7\delta m}$ (see (3.27)), Proposition 7.4 (iii), and (5.111), we estimate

$$\begin{split}\|H_{p,q}\|_{L^2}&\lesssim 2^{2\overline{k}}2^m(2^{10\overline{k}}2^l\kappa_\theta 2^{-p_-/2}2^{-q_-/2}2^{\delta^2 m})2^{-l}\sup_{s\in I_m}\|\widehat{f^\mu_{j_1,k_1}}(s)\|_{L^\infty}\|h(s)\|_{L^2}\\&\lesssim 2^{-4\overline{k}}2^{-p_-/2}2^{-q_-/2}2^{-m+43\delta m},\end{split}\tag{5.113}$$

where $x_-=\min(x,0)$. In particular

$$\sum_{p\geq -4\delta m,\,q\geq -4\delta m}2^{j-50\delta j}\|P_k\mathcal{B}'_{p,q}[f^\mu_{j_1,k_1},h]\|_{L^2}\lesssim 2^{-\delta m}.\tag{5.114}$$

We show now that

$$\sum_{p\leq -4\delta m,\,q\in\mathbb{Z}}2^{j-50\delta j}\|P_k\mathcal{B}'_{p,q}[f^\mu_{j_1,k_1},h]\|_{L^2}\lesssim 2^{-\delta m}.\tag{5.115}$$

For this we notice now that if $p\leq -4\delta m$ then $P_k\mathcal{B}'_{p,q}[f^\mu_{j_1,k_1},P_{k_2}E^{a_3}_\nu]$ is nontrivial only when $|\eta|$ is close to γ_1, and $|\xi|,|\xi-\eta|$ are close to $\gamma_1/2$ (as a consequence of Proposition 7.2 (iii)). In particular $2^{\overline{k}}\lesssim 1$, $2^q\approx 1$, and $|\widehat{f^\mu_{j_1,k_1}}(\xi-\eta,s)|\lesssim 2^{2\delta^2 m}2^{-j_1/2+51\delta j_1}$ in the support of the integral. Therefore we have the stronger estimate, using also (7.44) (compare with (5.113))

$$\begin{split}\|H_{p,q}\|_{L^2}&\lesssim 2^{m-l}2^l\kappa_\theta\min(2^{-p/2},2^{p/2-l/2})2^{\delta^2 m}\sup_{s\in I_m}\|\widehat{f^\mu_{j_1,k_1}}(s)\|_{L^\infty}\|h(s)\|_{L^2}\\&\lesssim 2^{-j_1/2+51\delta j_1}\min(2^{-p/2},2^{p/2-l/2})2^{-m+36\delta m}.\end{split}\tag{5.116}$$

The desired bound (5.115) follows if $j_1\geq j-\delta m$ or if $j\leq 3m/4-5\delta m$, since $\min(2^{-p/2},2^{p/2-l/2})\lesssim 2^{-l/4}\lesssim 2^{m/4}$. On the other hand, if

$$j_1\leq j-\delta m\quad\text{and}\quad j\geq 3m/4-5\delta m$$

then the sum over $p\geq (j-m)-10\delta m$ in (5.115) can also be estimated using (5.116). The remaining sum over $p\leq (j-m)-10\delta m$ is negligible using the approximate finite speed of propagation argument (integration by parts in ξ). This completes the proof of (5.115).

Finally we show that

$$\sum_{p\in\mathbb{Z},\,q\leq -4\delta m}\|Q_{jk}\mathcal{B}'_{p,q}[f^\mu_{j_1,k_1},h]\|_{B_j}\lesssim 2^{-\delta m}.\tag{5.117}$$

As before, we notice now that if $q\leq -4\delta m$ then $P_k\mathcal{B}'_{p,q}[f^\mu_{j_1,k_1},h]$ is nontrivial only when $|\xi|$ is close to γ_1, and $|\eta|,|\xi-\eta|$ are close to $\gamma_1/2$ (as a consequence of Proposition 7.2 (iii)). In particular $2^{\overline{k}} \lesssim 1$, $2^p \approx 1$ and we have the stronger estimate (compare with (5.116))

$$\|H_{p,q}\|_{L^2} \lesssim 2^{-j_1/2+51\delta j_1} \min(2^{-q/2}, 2^{q/2-l/2}) 2^{-m+36\delta m} \lesssim \frac{2^{q/2}}{2^q + 2^{l/2}} 2^{-m+36\delta m}. \tag{5.118}$$

Moreover, since $|\Phi(\xi,\eta)| \lesssim 2^l$ and $|\nabla_\eta \Phi(\xi,\eta)| \lesssim 2^q$, the function $\widehat{H_{p,q}}$ is supported in the set $\{||\xi| - \gamma_1| \lesssim 2^l + 2^{2q}\}$ (see (7.21)). The main observation is that the B_j norm for functions supported in such a set carries an additional small factor. More precisely, after localization to a 2^j ball in the physical space, the function $\mathcal{F}\{Q_{jk}\mathcal{B}'_{p,q}[f^\mu_{j_1,k_1}, h]\}(\xi)$ is supported in the set $\{||\xi| - \gamma_1| \lesssim 2^l + 2^{2q} + 2^{-j+2\delta m}\}$, up to a negligible error. Therefore, using (5.118),

$$\begin{aligned}\|Q_{jk}\mathcal{B}'_{p,q}[f^\mu_{j_1,k_1}, P_{k_2}E^{a_3}_\nu]\|_{B_j} &\lesssim 2^{j-50\delta j}(2^l + 2^{2q} + 2^{-j+2\delta m})^{1/2-49\delta}\|H_{p,q}\|_{L^2} \\ &\lesssim 2^{j-50\delta j} 2^{-m+36\delta m}(2^{l/2} + 2^q + 2^{-j/2+\delta m})\frac{2^{q/2-100\delta q}}{2^q + 2^{l/2}} \\ &\lesssim 2^{q/8} 2^{-4\delta m}.\end{aligned}$$

The bound (5.117) follows. The bound (5.110) follows from (5.114), (5.115), and (5.117). □

Lemma 5.11. *Assume that* (5.103) *holds and* $\kappa_\theta = 2^{-m/2+6\overline{k}+\delta^2 m}$. *Then*

$$\|Q_{jk}\mathcal{B}^{||}_{m,l}[f^\mu_{j_1,k_1}, A^{a_3,\alpha_3;a_4,\alpha_4;[1]}_{k_2;k_3,j_3;k_4,j_4}]\|_{B_j} \lesssim 2^{-4\delta^2 m}. \tag{5.119}$$

Proof. Notice that $A^{a_3,\alpha_3;a_4,\alpha_4;[1]}_{k_2;k_3,j_3;k_4,j_4}$ is supported in the set $||\eta| - \gamma_1| \le 2^{-\mathcal{D}}$. Using also the conditions $\Phi(\xi,\eta) \lesssim 2^l$ and $\Theta(\xi,\eta) \lesssim \kappa_\theta$, we have

$$||\eta| - \gamma_1| \le 2^{-\mathcal{D}}, \quad |\xi|, |\xi - \eta| \in [2^{-50}, 2^{50}], \quad \min(||\xi| - \gamma_1|, ||\xi - \eta| - \gamma_1|) \ge 2^{-50} \tag{5.120}$$

in the support of the integral defining $\mathcal{F}\{P_k\mathcal{B}^{||}_{m,l}[f^\mu_{j_1,k_1}, G^{[1]}](\xi)\}$, where $G^{[1]} = A^{a_3,\alpha_3;a_4,\alpha_4;[1]}_{k_2;k_3,j_3;k_4,j_4}$.

Case 1. Assume first that

$$\max(j_3, j_4) \ge m/2. \tag{5.121}$$

In this case $\|G^{[1]}\|_{L^2} \lesssim 2^{-3m/2+30\delta m}$ (see (4.37)), and the conclusion follows from Lemma 5.10.

Case 2. Assume now that

$$\max(j_3, j_4) \le m/2, \qquad j_1 \ge m/2. \tag{5.122}$$

The bound (5.119) follows again by the same argument as in the proof of (5.110) above. In this case $\|\widehat{G^{[1]}}(s)\|_{L^\infty} \lesssim 2^{-m+4\delta m}$ (due to (4.41)) and $\|\widehat{A_{\le 0,\gamma_1} f^\mu_{j_1,k_1}}(s)\|_{L^2} \lesssim 2^{2\delta^2 m} 2^{-j_1+50\delta j_1}$ (see (3.27)). We make the change of variables $\eta \to \xi - \eta$, define $\Phi'(\xi,\eta) = \Phi(\xi, \xi - \eta)$ and define the operators $\mathcal{B}''_{p,q}$ as in (5.112), by inserting cutoff factors $\varphi_p((\nabla_\xi \Phi')(\xi,\eta))$ and $\varphi_q((\nabla_\xi \Phi')(\xi,\eta))$. In this case we notice that we may assume both $p \ge -\mathcal{D}$ and $q \ge -\mathcal{D}$. Indeed we have $|\Phi'(\xi,\eta)| \le 2^{-\mathcal{D}}$ and $||\xi - \eta| - \gamma_1| \le 2^{-\mathcal{D}}$, so $|(\nabla_\xi \Phi')(\xi,\eta)| \gtrsim 1$ and $|(\nabla_\eta \Phi')(\xi,\eta)| \gtrsim 1$ in the support of the integral (in view of Proposition 7.2 (iii)). Then we estimate, using (7.42),

$$\|P_k\mathcal{B}''_{p,q}[A_{\le 0,\gamma_1} f^\mu_{j_1,k_1}, G^{[1]}]\|_{L^2} \lesssim 2^{-j_1+50\delta j_1} 2^{-m/2+5\delta m}.$$

The bound (5.119) follows by summation over p and q.

Case 3. Assume now that

$$\max(j_1, j_3, j_4) \le m/2, \qquad j \le m/2 + 10\delta m. \tag{5.123}$$

We use the bounds $\|\widehat{G^{[1]}}(s)\|_{L^\infty} \lesssim 2^{-m+4\delta m}$ (see (4.37)) and $\|\widehat{f^\mu_{j_1,k_1}}(s)\|_{L^\infty} \lesssim 2^{3\delta m}$. Moreover, $|\nabla_\eta\Phi(\xi,\eta)| \gtrsim 1$ in the support of the integral. Therefore, using the first bound in (7.42),

$$\big\|\mathcal{F}\{P_k\mathcal{B}^{\|}_{m,l}[f^\mu_{j_1,k_1},G^{[1]}]\}\big\|_{L^\infty} \lesssim 2^{m-l}\kappa_\theta 2^l 2^{\delta^2 m}\sup_{s\in I_m}\|\widehat{G^{[1]}}(s)\|_{L^\infty}\|\widehat{f^\mu_{j_1,k_1}}(s)\|_{L^\infty} \lesssim 2^{-m/2+8\delta m}.$$

The desired bound (5.119) follows when $j \le m/2 + 10\delta m$.

Case 4. Finally, assume that

$$\max(j_1,j_3,j_4) \le m/2, \qquad j \ge m/2 + 10\delta m. \tag{5.124}$$

We examine the formula (5.109), decompose $G^{[1]}$ as in (4.41) and notice that the contribution of the error term is easy to estimate. To estimate the main term, we define the modified phase

$$\mathfrak{p}(\xi,\eta) := \Phi_{+\mu\nu}(\xi,\eta) + \Lambda_\nu(\eta) - 2\Lambda_\nu(\eta/2) = \Lambda(\xi) - \Lambda_\mu(\xi-\eta) - 2\Lambda_\nu(\eta/2). \tag{5.125}$$

For $r \in \mathbb{Z}$ we define the functions $\mathcal{G}_r = \mathcal{G}_{r,m,l,j,j_1}$ by

$$\begin{aligned}\widehat{\mathcal{G}_r}(\xi) := \int_{\mathbb{R}} q_m(s)\int_{\mathbb{R}^2} e^{is\mathfrak{p}(\xi,\eta)}\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta))m_0(\xi,\eta)\\ \times \varphi_r(\nabla_\eta\mathfrak{p}(\xi,\eta))\widehat{f^\mu_{j_1,k_1}}(\xi-\eta,s)g^{[1]}(\eta,s)\varphi(2^{3\delta m}(|\eta|-\gamma_1))d\eta ds.\end{aligned} \tag{5.126}$$

Notice that the functions $\mathcal{G}_r$ are negligible for, say, $r \le -10m$. It suffices to prove that

$$2^{j-50\delta j}\|Q_{jk}\mathcal{G}_r\|_{L^2} \lesssim 2^{-5\delta^2 m} \qquad \text{for any } r\in\mathbb{Z}. \tag{5.127}$$

We notice first that $\|P_k\mathcal{G}_r\|_{L^2} \lesssim 2^{-4m}$ if $r \ge \delta^2 m + \max(-l-m,-m/2)$, in view of Lemma 3.3 (i). In particular, we may assume that $r \le -\mathcal{D}$. In this case, the functions $\mathcal{G}_r$ are nontrivial only when $-\mu=\nu=+$ and ξ is close to $\eta/2$. Therefore $\mathfrak{p}(\xi,\eta) = \Lambda(\xi)+\Lambda(\eta-\xi)-2\Lambda(\eta/2)$, and we have, in the support of the integral defining $\widehat{\mathcal{G}_r}(\xi)$

$$\begin{aligned}&|\nabla_\eta\mathfrak{p}(\xi,\eta)| \approx |\xi-\eta/2| \approx |\nabla_\xi\mathfrak{p}(\xi,\eta)| \approx |\nabla_\xi\Phi(\xi,\eta)| \approx 2^r,\\ &|\mathfrak{p}(\xi,\eta)| \approx |\xi-\eta/2|^2 \approx 2^{2r},\\ &||\eta|-\gamma_1| \approx |\Lambda(\eta)-2\Lambda(\eta/2)| \lesssim |\Phi(\xi,\eta)|+|\mathfrak{p}(\xi,\eta)| \lesssim 2^l+2^{2r},\\ &||\xi|-\gamma_1/2| \lesssim 2^l+2^r.\end{aligned} \tag{5.128}$$

The finite speed of propagation argument (integration by parts in ξ) shows that $\|Q_{jk}\mathcal{G}_r\|_{L^2} \lesssim 2^{-4m}$ if $j \ge \delta^2 m + \max(m+r,-r)$. To summarize, it remains to prove that

$$\big(2^{m+r}+2^{-r}\big)^{1-50\delta}\|P_k\mathcal{G}_r\|_{L^2} \lesssim 2^{-\delta m} \qquad \text{if} \qquad r \le \delta^2 m + \max(-l-m,-m/2). \tag{5.129}$$

For ξ fixed, the variable η satisfies three restrictions: $\eta\cdot\xi^\perp \lesssim \kappa_\theta$, $\Phi(\xi,\eta) \lesssim 2^l$, and $|\eta-\xi/2| \lesssim 2^r$. Therefore, using also (4.41), we have the pointwise bound

$$\begin{aligned}|\widehat{\mathcal{G}_r}(\xi)| &\lesssim 2^{5\delta^2 m}2^{m-l}\min(2^r,2^{-m/2})\min(2^r,2^l)\sup_{s\in I_m}\|\widehat{f^\mu_{j_1,k_1}}(s)\|_{L^\infty}\|g^{[1]}(s)\|_{L^\infty}\\ &\lesssim 2^{8\delta m}\min(2^r,2^{-m/2})\min(2^{r-l},1).\end{aligned} \tag{5.130}$$

The desired bound (5.129) follows, using also the support assumption $||\xi|-\gamma_1/2| \lesssim 2^l+2^r$ in (5.128), if $r \le -m/2$ or if $r \in [-m/2,-m/3]$.

It remains to prove (5.129) when $-m/3 \le r \le -l - m + \delta^2 m$. The main observation in this case is that $|\mathfrak{p}(\xi,\eta)| \approx 2^{2r}$ is large enough to be able to integrate by parts is s. It follows that

$$|\widehat{\mathcal{G}_r}(\xi)| \lesssim \int_{\mathbb{R}}\int_{\mathbb{R}^2} 2^{-2r}\big|\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta))\varphi_r(\nabla_\eta \mathfrak{p}(\xi,\eta))\varphi(2^{3\delta m}(|\eta|-\gamma_1))\big| \\ \times \big|\partial_s[\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s)g^{[1]}(\eta,s)q_m(s)]\big|d\eta ds.$$

For ξ fixed, the integral in supported in a $O(\kappa_\theta \times 2^l)$ rectangle centered at $\eta = 2\xi$. In this support, we have the bounds, see Lemma 4.2 (ii) and (iii),

$$\|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^\infty} \lesssim 2^{\delta^2 m}, \qquad \|g^{[1]}(s)\|_{L^\infty} \lesssim 2^{-m+4\delta m} \qquad \|\partial_s g^{[1]}(s)\|_{L^\infty} \lesssim 2^{-2m+18\delta m}, \\ \partial_s f^{\mu}_{j_1,k_1} = h_2 + h_\infty, \quad \|h_2(s)\|_{L^2} \lesssim 2^{-3m/2+5\delta m}, \quad \|\widehat{h_\infty}(s)\|_{L^\infty} \lesssim 2^{-m+15\delta m}.$$

The integrals that do not contain the function h_2 can all be estimated pointwise, as in (5.130) by $C2^{-2r}2^{-l}2^{-m+20\delta m}(2^l\kappa_\theta) \lesssim 2^{-2r}2^{-3m/2+21\delta m}$. The integral that contains the function h_2 can be estimated pointwise, using Hölder's inequality, by

$$C2^{-2r}2^{-l}2^{-3m/2+10\delta m}(2^l\kappa_\theta)^{1/2} \lesssim 2^{-2r}2^{-l/2}2^{-7m/4+11\delta m} \lesssim 2^{-2r}2^{-5m/4+11\delta m}.$$

Therefore, using also the support assumption $||\xi| - \gamma_1/2| \lesssim 2^r$ in (5.128), and recalling that $r \ge -m/3$, $l \le -m/2$, we have

$$2^{m+r}\|P_k\mathcal{G}_r\|_{L^2} \lesssim 2^{-r/2}2^{-m/4+11\delta m}.$$

This suffices to prove (5.129), which completes the proof of the lemma. □

Lemma 5.12. *With the same notation as in Lemma 5.11, and assuming* (5.103)*, we have*

$$\|Q_{jk}\mathcal{B}^{||}_{m,l}[f^{\mu}_{j_1,k_1}, A^{a_3,\alpha_3;a_4,\alpha_4;[2]}_{k_2;k_3,j_3;k_4,j_4}]\|_{B_j} \lesssim 2^{-4\delta^2 m}. \tag{5.131}$$

Proof. The main observation here is that, since $|\Phi_{+\mu\nu}(\xi,\eta)| \lesssim 2^l$ and $|\Phi_{\nu\beta\gamma}(\eta,\sigma)| \gtrsim 2^{-10\delta m}$, we have $|\widetilde{\Phi}(\xi,\eta,\sigma)| \gtrsim 2^{-10\delta m}$, thus we can integrate by parts in s once more. Before this, however, we notice that we may assume that

$$k_3, k_4 \in [-2m/N_0', 2m/N_0'], \qquad \min(j_3,j_4) \le m - 4\delta m. \tag{5.132}$$

Indeed, the first claim follows from Lemma 4.1 (ii) (2), (3). For the second claim, we notice that if $\min(j_3,j_4) \ge m - 4\delta m$ then we would have $\|A^{a_3,\alpha_3;a_4,\alpha_4;[2]}_{k_2;k_3,j_3;k_4,j_4}\|_{L^2} \lesssim 2^{-3m/2+8\delta m}$ (by the same argument as in the proof of (4.31)), and the desired bound would follow from Lemma 5.10.

Step 1. For $r \in \mathbb{Z}$ we define (compare with (5.87)) the trilinear operators $\mathcal{J}^{[2]}_{l,r}$ by

$$\widehat{\mathcal{J}^{[2]}_{l,r}[f,g,h]}(\xi,s) := \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\widetilde{\Phi}(\xi,\eta,\sigma)}\widehat{f}(\xi-\eta)\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))2^{-l}\widetilde{\varphi}_l(\Phi_{+\mu\nu}(\xi,\eta)) \\ \times \varphi_r(\widetilde{\Phi}(\xi,\eta,\sigma))\chi^{[2]}(\eta,\sigma)\varphi_{k_2}(\eta)\mathfrak{m}^b_{\mu\nu}(\xi,\eta)\mathfrak{m}^{b_2}_{\nu\beta\gamma}(\eta,\sigma)\widehat{g}(\eta-\sigma)\widehat{h}(\sigma)\,d\sigma d\eta. \tag{5.133}$$

Let

$$\mathcal{C}^{[2]}_{l,r}[f,g,h] := \int_{\mathbb{R}} q_m(s)\mathcal{J}^{[2]}_{l,r}[f,g,h](s)\,ds, \tag{5.134}$$

and notice that

$$\mathcal{B}^{||}_{m,l}[f^{\mu}_{j_1,k_1}, A^{b_1,b_2,b_3,[2]}_{k_2;k_3,j_3;k_4,j_4}] = \sum_{r\ge -11\delta m}\mathcal{C}^{[2]}_{l,r}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}].$$

We integrate by parts in s to rewrite

$$\mathcal{C}^{[2]}_{l,r}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}] = i2^{-r}\Big\{ \int_{\mathbb{R}} q'_m(s)\widetilde{\mathcal{J}^{[2]}}_{l,r}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s)\,ds \\ + \widetilde{\mathcal{C}^{[2]}}_{l,r}[\partial_s f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}] + \widetilde{\mathcal{C}^{[2]}}_{l,r}[f^{\mu}_{j_1,k_1}, \partial_s f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}] + \widetilde{\mathcal{C}^{[2]}}_{l,r}[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, \partial_s f^{\gamma}_{j_4,k_4}]\Big\},$$

where the operators $\widetilde{\mathcal{J}^{[2]}}_{l,r}$ and $\widetilde{\mathcal{C}^{[2]}}_{l,r}$ are defined in the same way as the operators $\mathcal{J}^{[2]}_{l,r}$ and $\mathcal{C}^{[2]}_{l,r}$, but with $\varphi_p(\widetilde{\Phi}(\xi,\eta,\sigma))$ replaced by $\widetilde{\varphi}_p(\widetilde{\Phi}(\xi,\eta,\sigma))$, $\widetilde{\varphi}_p(x) = 2^p x^{-1}\varphi_p(x)$, (see the formula (5.133)). It suffices to prove that for any $s \in I_m$ and $r \geq -11\delta m$,

$$2^{j-50\delta j}\|Q_{jk}\widetilde{\mathcal{J}^{[2]}}_{l,r}[f,g,h]\|_{L^2} \lesssim 2^{-12\delta m}, \tag{5.135}$$

where $[f,g,h] = [f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s)$ or $[f,g,h] = [2^m\partial_s f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s)$ or $[f,g,h] = [f^{\mu}_{j_1,k_1}, 2^m\partial_s f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s)$ or $[f,g,h] = [f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, 2^m\partial_s f^{\gamma}_{j_4,k_4}](s)$.

Step 2. As in the proof of Lemma 3.5, the function $\widetilde{\varphi}_r(\widetilde{\Phi}(\xi,\eta,\sigma))$ can be incorporated with the phase $e^{is\widetilde{\Phi}(\xi,\eta,\sigma)}$, using the formula (3.20) and the fact that $2^{-r} \leq 2^{11\delta m}$. Then we integrate the variable σ and denote by H_1, H_2, and H_3 the resulting functions,

$$H_1 := I^{[2]}[f^{\beta}_{j_3,k_3}(s), f^{\gamma}_{j_4,k_4}(s)],\ \ H_2 := I^{[2]}[\partial_s f^{\beta}_{j_3,k_3}(s), f^{\gamma}_{j_4,k_4}(s)],\ \ H_3 := I^{[2]}[f^{\beta}_{j_3,k_3}(s), \partial_s f^{\gamma}_{j_4,k_4}(s)],$$
$$\mathcal{F}\{I^{[2]}[g,h]\}(\eta) := \int_{\mathbb{R}^2} e^{i(s+\lambda)\Phi_{\nu\beta\gamma}(\eta,\sigma)}\chi^{[2]}(\eta,\sigma)\varphi_{k_2}(\eta)\mathfrak{m}^{b_2}_{\nu\beta\gamma}(\eta,\sigma)\widehat{g}(\eta-\sigma)\widehat{h}(\sigma)\,d\sigma.$$

We claim that

$$\|H_1\|_{L^2} + 2^m\|H_2\|_{L^2} + 2^m\|H_3\|_{L^2} \lesssim 2^{-5m/6+10\delta m}. \tag{5.136}$$

Notice that the bound on H_1 is already proved (in a stronger form) in the proof of (4.38). The bounds on H_2 and H_3 follow in the same way from the $L^2\times L^\infty$ argument: indeed, we have $\|\partial_s f^{\beta}_{j_3,k_3}(s)\|_{L^2} + \|\partial_s f^{\gamma}_{j_4,k_4}(s)\|_{L^2} \lesssim 2^{-m+7\delta m}$ (due to (4.21)). Then we notice that we can remove the factor $\varphi(2^{20\delta m}\Theta_\beta(\eta,\sigma))$ from the multiplier $\chi^{[2]}(\eta,\sigma)$, at the expenses of a small error (due to Lemma 3.4 and (5.132)). The desired bounds in (5.136) follow using the $L^2\times L^\infty$ argument with Lemma 3.5.

Step 3. We prove now (5.135) for $[f,g,h] = [f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s)$. It suffices to show that

$$2^{4\overline{k}}2^{m-30\delta m}\big\|S[f^{\mu}_{j_1,k_1}(s), H_1]\big\|_{L^2} \lesssim 1 \tag{5.137}$$

for any $s \in I_m$, where

$$\mathcal{F}\{S[f,g]\}(\xi) := |\varphi_k(\xi)|\int_{\mathbb{R}^2}\big|\widehat{f}(\xi-\eta)\varphi(\kappa_\theta^{-1}\Theta(\xi,\eta))2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta))\varphi_{k_2}(\eta)\widehat{g}(\eta)\big|\,d\eta. \tag{5.138}$$

This follows using Schur's lemma, the bound (5.136), and Proposition 7.4 (iii). Indeed, we have $|\nabla_\eta\Phi(\xi,\eta)|+|\nabla_\xi\Phi(\xi,\eta)| \gtrsim 2^{-4\delta m}$ in the support of the integral (due to the location of space-time resonances), therefore the left-hand side of (5.137) is dominated by

$$C2^{4\overline{k}}2^{m-30\delta m}2^{-l}(2^{10\overline{k}}\kappa_\theta 2^{3l/4}2^{4\delta m})\|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^\infty}\|\widehat{H_1}\|_{L^2} \lesssim 2^{30\overline{k}}2^{-l/4}2^{-m/3}.$$

This suffices to prove (5.137) since $2^{-l} \leq 2^m$. Moreover, (5.135) follows in the same way for $[f,g,h] = [f^{\mu}_{j_1,k_1}, 2^m\partial_s f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s)$ or $[f,g,h] = [f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, 2^m\partial_s f^{\gamma}_{j_4,k_4}](s)$, since the L^2 bounds on 2^mH_2 and 2^mH_3 are the same as for H_1.

It remains to prove (5.135) for $[f,g,h]=[2^m\partial_s f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s)$. It suffices to prove that

$$2^{4\overline{k}}2^{m-30\delta m}\big\|S[2^m\partial_s f^{\mu}_{j_1,k_1}(s),H_1]\big\|_{L^2}\lesssim 1, \tag{5.139}$$

for any $s\in I_m$. Let $f=2^m\partial_s f^{\mu}_{j_1,k_1}(s)$ and $f_{2\gamma_0}:=A_{\geq\mathcal{D}-11,2\gamma_0}f$. We decompose, using (4.40),

$$f=f_{2\gamma_0}+f_2+f_\infty,\quad \|f_{2\gamma_0}\|_{L^2}\lesssim 2^{7\delta m},\quad \|f_2\|_{L^2}\lesssim 2^{-m/2+5\delta m},\quad \|\widehat{f_\infty}\|_{L^\infty}\lesssim 2^{3\overline{k}+15\delta m}.$$

The contribution of f_∞ can be estimated as before, using Schur's lemma, (5.136), and Proposition 7.4 (iii). To estimate the other contributions, we also use the bound (see (4.39))

$$\|\widehat{H_{1,\infty}}\|_{L^\infty}\lesssim 2^{3\overline{k}}2^{-m+14\delta m}\qquad\text{where}\qquad H_1=H_{1,2\gamma_0}+H_{1,\infty}=A_{\geq\mathcal{D}+1,2\gamma_0}H_1+A_{\leq\mathcal{D},2\gamma_0}H_1.$$

As before, we use Schur's test and Proposition 7.4 (iii), together with the fact that space-time resonances are possible only when $|\xi|,|\eta|,|\xi-\eta|$ are all close to either γ_1 or $\gamma_1/2$. We estimate

$$\begin{aligned}
\big\|S[f_2,H_{1,\infty}]\big\|_{L^2}&\lesssim 2^{-l}(2^{10\overline{k}}\kappa_\theta 2^{3l/4}2^{4\delta m})\|\widehat{f_2}\|_{L^2}\|\widehat{H_{1,\infty}}\|_{L^\infty}\lesssim 2^{20\overline{k}}2^{-l/4}2^{-2m+40\delta m},\\
\big\|S[f_{2\gamma_0},H_{1,\infty}]\big\|_{L^2}&\lesssim 2^{-l}(2^{10\overline{k}}\kappa_\theta 2^{3l/4}2^{4\delta m})\|\widehat{f_{2\gamma_0}}\|_{L^2}\|\widehat{H_{1,\infty}}\|_{L^\infty}\lesssim 2^{20\overline{k}}2^{-l/4}2^{-3m/2+40\delta m},\\
\big\|S[f_2,H_{1,2\gamma_0}]\big\|_{L^2}&\lesssim 2^{-l}(2^{10\overline{k}}\kappa_\theta 2^{l}2^{4\delta m})^{1/2}\|\widehat{f_2}\|_{L^2}\|\widehat{H_{1,2\gamma_0}}\|_{L^2}\lesssim 2^{15\overline{k}}2^{-l/2}2^{-19m/12+20\delta m},\\
S[f_{2\gamma_0},H_{1,2\gamma_0}]&\equiv 0.
\end{aligned}$$

These bounds suffice to prove (5.139), which completes the proof of the lemma. □

5.8. **Higher order terms.** In this subsection we consider the higher order components in the Duhamel formula (2.15) and show how to control their Z norms.

Proposition 5.13. *With the hypothesis in Proposition 2.2, for any $t\in[0,T]$ we have*

$$\|W_3(t)\|_Z+\Big\|\int_0^t e^{is\Lambda}\mathcal{N}_{\geq 4}(s)\,ds\Big\|_Z\lesssim\varepsilon_1^2. \tag{5.140}$$

The rest of this section is concerned with the proof of Proposition 5.13. The bound on $\mathcal{N}_{\geq 4}$ follows directly from the hypothesis $\|e^{is\Lambda}\mathcal{N}_{\geq 4}(s)\|_Z\leq\varepsilon_1^2(1+s)^{-1-\delta^2}$, see (2.25). To prove the bound on W_3 we start from the formula

$$\begin{aligned}
\Omega^a_\xi\widehat{W_3}(\xi,t)=\sum_{\mu,\nu,\beta\in\{+.-\}}\sum_{a_1+a_2+a_3=a}\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}e^{is\widetilde{\Phi}_{+\mu\nu\beta}(\xi,\eta,\sigma)}\mathfrak{n}_{\mu\nu\beta}(\xi,\eta,\sigma)\\
\times(\Omega^{a_1}\widehat{\mathcal{V}_\mu})(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}_\nu})(\eta-\sigma,s)(\Omega^{a_3}\widehat{\mathcal{V}_\beta})(\sigma,s)\,d\eta d\sigma ds.
\end{aligned}\tag{5.141}$$

We define the functions q_m as in (5.3) and the trilinear operators $C_m=C^{\mu\nu\beta}_{m,b}$

$$\mathcal{F}\{C_m[f,g,h]\}(\xi):=\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2\times\mathbb{R}^2}e^{is\widetilde{\Phi}(\xi,\eta,\sigma)}n_0(\xi,\eta,\sigma)\widehat{f}(\xi-\eta,s)\widehat{g}(\eta-\sigma,s)\widehat{h}(\sigma,s)\,d\eta d\sigma ds, \tag{5.142}$$

where $\widetilde{\Phi}:=\widetilde{\Phi}_{+\mu\nu\beta}$ and $n_0:=\mathfrak{n}_{\mu\nu\beta}$. It remains to prove that, for any $(k,j)\in\mathcal{J}$ and $m\in[0,L+1]$,

$$\sum_{k_1,k_2,k_3\in\mathbb{Z}}2^{j-50\delta j}\big\|Q_{jk}C_m[P_{k_1}D^{\alpha_1}\Omega^{a_1}\mathcal{V}_\mu,P_{k_2}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\nu,P_{k_3}D^{\alpha_3}\Omega^{a_3}\mathcal{V}_\beta]\big\|_{L^2}\lesssim 2^{-\delta^2 m}\varepsilon_1^3 \tag{5.143}$$

for any $\mu,\nu,\beta\in\{+,-\}$, provided that $a_1+a_2+a_3=a$ and $\alpha_1+\alpha_2+\alpha_3=\alpha$. Let

$$f^\mu:=\varepsilon^{-1}D^{\alpha_1}\Omega^{a_1}\mathcal{V}_\mu,\quad f^\nu:=\varepsilon^{-1}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\nu,\quad f^\beta:=\varepsilon^{-1}D^{\alpha_3}\Omega^{a_3}\mathcal{V}_\beta. \tag{5.144}$$

The bootstrap assumption (2.25) gives, for any $s\in[0,t]$ and $\gamma\in\{\mu,\nu,\beta\}$,

$$\|f^{\gamma}(s)\|_{H^{N_0'}\cap Z_1\cap H_\Omega^{N_1'}}\lesssim(1+s)^{\delta^2}. \tag{5.145}$$

Simple estimates, as in the proof of Lemma 5.3, show that the parts of the sum in (5.143) over $\max(k_1,k_2,k_3)\geq 2(j+m)/N_0'-\mathcal{D}^2$ or over $\min(k_1,k_2,k_3)\leq -(j+m)/2$ are bounded as claimed. For (5.143) it remains to prove that

$$2^{j-50\delta j}\big\|Q_{jk}C_m[P_{k_1}f^{\mu},P_{k_2}f^{\nu},P_{k_3}f^{\beta}]\big\|_{L^2}\lesssim 2^{-2\delta^2m-\delta^2j} \tag{5.146}$$

for any fixed $m\in[0,L+1]$, $(k,j)\in\mathcal{J}$, and $k_1,k_2,k_3\in\mathbb{Z}$ satisfying

$$k_1,k_2,k_3\in[-(j+m)/2,2(j+m)/N_0'-\mathcal{D}^2]. \tag{5.147}$$

Let $\overline{k}:=\max(k,k_1,k_2,k_3,0)$, $\underline{k}:=\min(k,k_1,k_2,k_3)$ and $[k]:=\max(|k|,|k_1|,|k_2|,|k_3|)$. The S^∞ bound in (2.22) and Lemma 3.1 (ii) show that

$$\begin{aligned}&\big\|C_m[P_{k_1}f^{\mu},P_{k_2}f^{\nu},P_{k_3}f^{\beta}]\big\|_{L^2}\\&\lesssim 2^{\underline{k}/2}2^{3\overline{k}}2^m\sup_{s\in I_m}\|e^{-is\Lambda_\mu}P_{k_1}f^{\mu}\|_{L^{p_1}}\|e^{-is\Lambda_\nu}P_{k_2}f^{\nu}\|_{L^{p_2}}\|e^{-is\Lambda_\beta}P_{k_3}f^{\beta}\|_{L^{p_3}},\end{aligned} \tag{5.148}$$

if $p_1,p_2,p_3\in\{2,\infty\}$ and $1/p_1+1/p_2+1/p_3=1/2$. The desired bound (5.146) follows unless

$$j\geq 2m/3+[k]/2+\mathcal{D}^2, \tag{5.149}$$

using the pointwise bounds in (3.34). Also, by estimating $\|P_kH\|_{L^2}\lesssim 2^k\|P_kH\|_{L^1}$, and using a bound similar to (5.148), the desired bound (5.146) follows unless

$$k\geq-(2/3)(j+m/6+\delta m). \tag{5.150}$$

Next, we notice that if $j\geq m+\mathcal{D}+[k]/2$, and (5.150) holds then the desired bound (5.146) follows. Indeed, we use the approximate finite speed of propagation argument as in the proof of (5.13). First we define $f^{\mu}_{j_1,k_1},f^{\nu}_{j_2,k_2},f^{\beta}_{j_3,k_3}$ as in (5.15). Then we notice that the contribution in the case $\min(j_1,j_2,j_3)\geq 9j/10$ is suitably controlled, due to (5.148). On the other and, if

$$\min(j_1,j_2,j_3)\leq 9j/10,$$

then we insert the cutoff functions $\varphi_{\leq l}(\eta)$ and $\varphi_{>l}(\eta)$ in the definition (5.142) of the operator C_m, where $l=-j+\delta j$. The contribution of the integral containing $\varphi_{>l}(\eta)$ is negligible, using integration by parts in ξ as before. On the other and, the contribution of the operator C_m containing $\varphi_{\leq l}(\eta)$ is bounded by $2^m2^{\delta m}2^{2l}\lesssim 2^{-2j+2\delta j}2^{m+\delta m}$ in L^2, which again suffices to prove (5.146). To summarize, in proving (5.146) we may assume that

$$2m/3+[k]/2+\mathcal{D}^2\leq j\leq m+\mathcal{D}+[k]/2,\quad \max(j,[k])\leq 2m+2\mathcal{D},\quad \overline{k}\leq 6m/N_0'. \tag{5.151}$$

We define now the functions $f^{\mu}_{j_1,k_1},f^{\nu}_{j_2,k_2},f^{\beta}_{j_3,k_3}$ as in (5.15). The contribution in the case $\max(j_1,j_2,j_3)\geq 2m/3$ can be bounded using (5.148). On the other hand, if $\max(j_1,j_2,j_3)\leq 2m/3$ then we can argue as in the proof of Lemma 5.7 when $2^l\approx 1$. More precisely, we define

$$g_1:=A_{\geq\mathcal{D}_1,\gamma_0}f^{\mu}_{j_1,k_1},\qquad g_2:=A_{\geq\mathcal{D}_1-10,\gamma_0}f^{\nu}_{j_2,k_2},\qquad A_{\geq\mathcal{D}_1-20,\gamma_0}f^{\beta}_{j_3,k_3}. \tag{5.152}$$

As in the proof of Lemma 5.7, see (5.96)–(5.89), (and after inserting cutoff functions of the form $\varphi_{\leq l}(\eta)$ and $\varphi_{>l}(\eta)$, $l=m-\delta m$, to bound the other terms) for (5.146) it suffices to prove that

$$2^{j-50\delta j}\big\|Q_{jk}C_m[g_1,g_2,g_3]\big\|_{L^2}\lesssim 2^{-\delta m}. \tag{5.153}$$

In proving (5.153), we may assume that $\max(j_1, j_2, j_3) \le m/3$ and $m \le L$ (otherwise we could use directly (5.148)) and that $k \ge -100$ (otherwise the contribution is negligible, by integrating by parts in η and σ). Therefore, using (5.151), we may assume that

$$[k] \le 100, \qquad m \le L, \qquad 2m/3 + \mathcal{D}^2 \le j \le m + 2\mathcal{D}, \qquad j_1, j_2, j_3 \in [0, m/3]. \tag{5.154}$$

As in the proof of Lemma 5.7, we decompose the operator C_m in dyadic pieces depending on the size of the modulation. More precisely, let

$$\widehat{\mathcal{J}_p[f,g,h]}(\xi, s) := \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\widetilde{\Phi}(\xi,\eta,\sigma)} \varphi_p(\widetilde{\Phi}(\xi,\eta,\sigma)) n_0(\xi,\eta,\sigma) \widehat{f}(\xi-\eta, s)\widehat{g}(\eta-\sigma, s)\widehat{h}(\sigma, s)\, d\sigma d\eta.$$

Let $\mathcal{J}_{\le p} = \sum_{q\le p} \mathcal{J}_{l,q}$ and

$$C_{m,p}[f,g,h] := \int_{\mathbb{R}} q_m(s)\mathcal{J}_{l,p}[f,g,h](s)\, ds.$$

For $p \ge -2m/3$ we integrate by parts in s. As in **Step 1** in the proof of Lemma 5.7, using also the L^2 bound (4.21), it follows easily that

$$2^{j-50\delta j} \sum_{p\ge -2m/3} \big\|P_k C_{m,p}[g_1, g_2, g_3]\big\|_{L^2} \lesssim 2^{-\delta m}.$$

To complete the proof of (5.153), it suffices to show that

$$2^{j-50\delta j} 2^m \sup_{s\in I_m} \big\|Q_{jk}\mathcal{J}_{\le -m/2}[g_1, g_2, g_3](s)\big\|_{L^2} \lesssim 2^{-\delta m}. \tag{5.155}$$

Let $\kappa = 2^{-m/3}$ and define the operators $\mathcal{J}_{\le -m/2,\le 0}$ and $\mathcal{J}_{\le -m/2,l}$ by inserting the factors $\varphi(\kappa^{-1}\nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma))$ and $\varphi_l(\kappa^{-1}\nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma))$, $l \ge 1$, in the definition of the operators $\mathcal{J}_p$ above. The point is to observe that $|\nabla_\xi\widetilde{\Phi}(\xi,\eta,\sigma)| \le 2^{-m/3+\mathcal{D}}$ in the support of the integral defining the operator $\mathcal{J}_{\le -m/2,\ge 0}$, due to Lemma 7.6 (i). Since $j \ge 2m/3 + \mathcal{D}^2$, see (5.154), the contribution of this operator is negligible, using integration by parts in ξ.

To estimate the operators $\mathcal{J}_{\le -m/2,l}$ notice that we may insert a factor of $\varphi(2^{2m/3+l-\delta m}\eta)$, at the expense of a negligible error (due to Lemma 3.3 (i)). To summarize, we define

$$\begin{aligned}\widehat{\mathcal{J}'_{\le -m/2,l}[f,g,h]}(\xi, s) := \int_{\mathbb{R}^2\times\mathbb{R}^2} & e^{is\widetilde{\Phi}(\xi,\eta,\sigma)} \varphi_l(\kappa^{-1}\nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma))\varphi_{\le -m/2}(\widetilde{\Phi}(\xi,\eta,\sigma)) \\ &\times \varphi(2^{2m/3+l-\delta m}\eta) n_0(\xi,\eta,\sigma)\widehat{f}(\xi-\eta, s)\widehat{g}(\eta-\sigma, s)\widehat{h}(\sigma, s)\, d\sigma d\eta,\end{aligned}$$

and it remains to show that, for $l \ge 1$ and $s \in I_m$,

$$2^{j-50\delta j} 2^m \big\|Q_{jk}\mathcal{J}'_{\le -m/2,l}[g_1, g_2, g_3](s)\big\|_{L^2} \lesssim 2^{-2\delta m}. \tag{5.156}$$

The bound (5.156) is clear when $l \ge m/3 - \delta m$, since $2^j \lesssim 2^m$ (see (5.154)). On the other hand, if $l \le m/3 - \delta m$ then the operator is nontrivial only if

$$\widetilde{\Phi}(\xi,\eta,\sigma) = \Lambda(\xi) - \Lambda(\xi-\eta) - \Lambda_\nu(\eta-\sigma) + \Lambda_\nu(\sigma), \qquad \nu \in \{+,-\},$$

due to the smallness of $|\eta|$, $|\nabla_\sigma\widetilde{\Phi}(\xi,\eta,\sigma)|$, and $|\widetilde{\Phi}(\xi,\eta,\sigma)|$ (recall the support restrictions in (5.152)). In this case $|\nabla_\xi\widetilde{\Phi}(\xi,\eta,\sigma)| \le 2^{-m/2}$ in the support of the integral, and the contribution is again negligible using integration by parts in ξ. This completes the proof of Proposition 5.13.

6. Proof of Proposition 1.3

We show now that Proposition 1.3 follows from Proposition 2.2. The starting point is the system (1.4). We need to verify that it can be rewritten in the form stated in Proposition 2.2. For this we need to expand the Dirichlet–Neumann operator

$$G(h)\phi = |\nabla|\phi + N_2[h,\phi] + N_3[h,h,\phi] + \text{Quartic Remainder},$$

and then prove the required claims. To justify this rigorously and estimate the remainder, the main issue is to prove space localization. We prefer not to work with the Z norm itself, which is too complicated, but define instead certain auxiliary spaces which are used only in this section.

We need some results about the Dirichlet–Neumann operator, which are proved in section 9 in [32]. We recall that potential loss of derivatives is not an issue in this paper, so we do not need the results concerning paralinearization in subsection 9.2 in [32]. Assume (h,ϕ) are as in Proposition 1.3 and let $\Omega := \{(x,z) \in \mathbb{R}^3 : z \le h(x)\}$. Let Φ denote the unique harmonic function in Ω satisfying $\Phi(x,h(x)) = \phi(x)$. We define the Dirichlet-Neumann map as

$$G(h)\phi = \sqrt{1+|\nabla h|^2}(\nu\cdot\nabla\Phi) \tag{6.1}$$

where ν denotes the outward pointing unit normal to the domain Ω.

We use a change of variable to flatten the surface. We thus define

$$\begin{aligned} &u(x,y) := \Phi(x,h(x)+y), \qquad (x,y)\in\mathbb{R}^2\times(-\infty,0],\\ &\Phi(x,z) = u(x,z-h(x)). \end{aligned} \tag{6.2}$$

In particular $u_{|y=0} = \phi$, $\partial_y u_{|y=0} = B$, and the Dirichlet-Neumann operator is given by

$$G(h)\phi = (1+|\nabla h|^2)\partial_y u_{|y=0} - \nabla_x h\cdot\nabla_x u_{|y=0}. \tag{6.3}$$

The main formulas we need in this section, see Lemma 9.4 in [32], are

$$\begin{aligned} &u = e^{y|\nabla|}\phi + L(u),\\ &L(u) := -\frac{1}{2}e^{y|\nabla|}\int_{-\infty}^0 e^{s|\nabla|}(Q_a(s)-Q_b(s))ds + \frac{1}{2}\int_{-\infty}^0 e^{-|y-s||\nabla|}(\mathrm{sgn}(y-s)Q_a(s)-Q_b(s))ds, \end{aligned} \tag{6.4}$$

where $Q_a[u] = \nabla u\cdot\nabla h - |\nabla h|^2\partial_y u$ and $Q_b[u] = \mathcal{R}(\partial_y u\nabla h)$, and

$$\partial_y u(y) - |\nabla|u(y) = Q_a(y) + \int_{-\infty}^y |\nabla|e^{-|s-y||\nabla|}(Q_b(s)-Q_a(s))ds. \tag{6.5}$$

Step 1. We assume that the bootstrap assumption (1.13) holds. Notice first that

$$\sup_{2a+|\alpha|\le N_1+N_4,\, a\le N_1/2+20}\ \sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{-\theta|k|/2}\|Q_{j,k}D^\alpha\Omega^a\mathcal{U}(t)\|_{L^2} \lesssim \varepsilon_1(1+t)^{\theta+6\delta^2}, \tag{6.6}$$

$$\sup_{2a+|\alpha|\le N_1+N_4,\, a\le N_1/2+20}\ \sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{-\theta|k|/2}\|Q_{j,k}D^\alpha\Omega^a\mathcal{U}(t)\|_{L^\infty} \lesssim \varepsilon_1(1+t)^{-5/6+\theta+6\delta^2}, \tag{6.7}$$

for $\theta\in[0,1/3]$, where the operators Q_{jk} are defined as in (2.2). Indeed, let $f = e^{it\Lambda}\Omega^a D^\alpha\mathcal{U}(t)$ and assume that $t\in[2^m-1,2^{m+1}]$, $m\ge 0$. We have

$$\|f\|_{H^{N'_0}\cap H^{N'_1}_\Omega} + \|f\|_{Z_1} \lesssim \varepsilon_1 2^{\delta^2 m}, \tag{6.8}$$

as a consequence (1.13), where, as in (4.27), $N_1' := (N_1 - N_4)/2 = 1/(2\delta)$ and $N_0' := (N_0 - N_3)/2 - N_4 = 1/\delta$. To prove (6.6) we need to show that

$$\sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{-\theta|k|/2}\|Q_{jk}e^{-it\Lambda}f\|_{L^2} \lesssim \varepsilon_1 2^{\theta m+6\delta^2 m}. \tag{6.9}$$

The sum over $j \leq m + \delta^2 m + |k|/2$ or over $j \leq |k| + \mathcal{D}$ is easy to control. On the other hand, if $j \geq \max(m + \delta^2 m + |k|/2, |k| + \mathcal{D})$ then we decompose $f = \sum_{(k',j')\in\mathcal{J}} f_{j',k'}$ as in (3.23). We may assume that $|k' - k| \leq 10$; the contribution of $j' \leq j - \delta^2 j$ is negligible, using integration by parts, while for $j' \geq j - \delta^2 j - 10$ we have

$$\|Q_{jk}e^{-it\Lambda}f_{j',k'}\|_{L^2} \lesssim \varepsilon_1 2^{\delta^2 m}\min(2^{-2j'/5}, 2^{-N_0' k_+}).$$

The desired bound (6.9) follows, which completes the proof of (6.6). The proof of (6.7) is similar, using also the decay bound (3.34). As a consequence, it follows that

$$\begin{aligned}&\sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{-\theta|k|/2}\|Q_{j,k}g(t)\|_{L^2} \lesssim \varepsilon_1 2^{\theta m+6\delta^2 m},\\ &\sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{-\theta|k|/2}\|Q_{j,k}g(t)\|_{L^\infty} \lesssim \varepsilon_1 2^{-5m/6+\theta m+6\delta^2 m}.\end{aligned} \tag{6.10}$$

for $g \in \{D^\alpha\Omega^a\langle\nabla\rangle h, D^\alpha\Omega^a|\nabla|^{1/2}\phi : 2a + |\alpha| \leq N_1 + N_4,\ a \leq N_1/2 + 20\}$ and $\theta \in [0, 1/3]$.

Step 2. We need to define now certain norms that allow us to extend our estimates to the region $\{y \leq 0\}$.

Lemma 6.1. *For $q \geq 0$ and $\theta \in [0,1]$, $p, r \in [1,\infty]$, define the norms*

$$\|f\|_{Y^p_{\theta,q}(\mathbb{R}^2)} := \sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{qk^+}\|Q_{jk}f\|_{L^p}, \quad \|f\|_{L^r_y Y^p_{\theta,q}(\mathbb{R}^2\times(-\infty,0])} := \sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{qk^+}\|Q_{jk}f\|_{L^r_y L^p_x}.$$

(i) Then, for any $p \in [2,\infty]$ and $\theta \in [0,1]$,

$$\|e^{y|\nabla|}f\|_{L^\infty_y Y^p_{\theta,q}} + \||\nabla|^{1/2}e^{y|\nabla|}f\|_{L^2_y Y^p_{\theta,q}} \lesssim \|f\|_{Y^p_{\theta,q}} \tag{6.11}$$

and

$$\begin{aligned}&\Big\|\int_{-\infty}^0 |\nabla|^{1/2}e^{-|s-y||\nabla|}\mathbf{1}_\pm(y-s)f(s)ds\Big\|_{L^\infty_y Y^2_{\theta,q}}\\ &\qquad + \Big\|\int_{-\infty}^0 |\nabla|e^{-|s-y||\nabla|}\mathbf{1}_\pm(y-s)f(s)ds\Big\|_{L^2_y Y^2_{\theta,q}} \lesssim \|f\|_{L^2_y Y^2_{\theta,q}}.\end{aligned} \tag{6.12}$$

(ii) If $p_1, p_2, p, r_1, r_2, r \in \{2,\infty\}$, $1/p = 1/p_1 + 1/p_2$, $1/r = 1/r_1 + 1/r_2$ then

$$\|(fg)\|_{L^r_y Y^p_{\theta_1+\theta_2-\delta^2,q-\delta^2}} \lesssim \|f\|_{L^{r_1}_y Y^{p_1}_{\theta_1,q}}\|g\|_{L^{r_2}_y Y^{p_2}_{\theta_2,q}} \tag{6.13}$$

provided that $\theta_1, \theta_2 \in [0,1]$, $\theta_1 + \theta_2 \in [\delta^2, 1]$, $q \geq \delta^2$. Moreover

$$\|(fg)\|_{L^2_y Y^2_{\theta_1-\delta^2,q-\delta^2}} \lesssim \|f\|_{L^\infty_y Y^\infty_{\theta_1,q}}\|g\|_{L^2_y H^q_x}. \tag{6.14}$$

Proof. The linear bounds in part (i) follow by parabolic estimates, once we notice that the kernel of the operator $e^{y|\nabla|}P_k$ is essentially localized in a ball of radius $\lesssim 2^{-k}$ and is bounded by $C2^{2k}(1 + 2^k|y|)^{-4}$.

The bilinear estimates in part (ii) follow by unfolding the definitions. The implicit factors $2^{-\delta^2 j}2^{-\delta^2 k^+}$ in the left-hand side allow one to prove the estimate for (k,j) fixed. Then one can decompose $f = \sum f_{j_1,k_1}$, $g = \sum g_{j_2,k_2}$ as in (3.23) and estimate $\|Q_{jk}(f_{j_1,k_1} \cdot g_{j_2,k_2})\|_{L^r_y L^p_x}$ using simple product estimates. The case $j = -k \gg \min(j_1, j_2)$ requires some additional attention; in this case one can use first Sobolev imbedding and the hypothesis $\theta_1 + \theta_2 \leq 1$. □

Step 3. Recall now the formula (6.4). Let

$$u^{(1)} = e^{y|\nabla|}\phi, \qquad u^{(n+1)} = e^{y|\nabla|}\phi + L(u^{(n)}),\ n \geq 1. \tag{6.15}$$

We can prove now a precise asymptotic expansion on the Dirichlet–Neumann operator.

Lemma 6.2. *We have*

$$G(h)\phi = |\nabla|\phi + N_2[h,\phi] + N_3[h,\phi] + |\nabla|^{1/2}N_4[h,\phi], \tag{6.16}$$

where

$$\mathcal{F}\{N_2[h,\phi]\}(\xi) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} n_2(\xi,\eta)\widehat{h}(\xi-\eta)\widehat{\phi}(\eta)\,d\eta, \qquad n_2(\xi,\eta) := \xi\cdot\eta - |\xi||\eta|, \tag{6.17}$$

$$\begin{aligned}\mathcal{F}\{N_3[h,\phi]\}(\xi) &= \frac{1}{(4\pi^2)^2}\int_{(\mathbb{R}^2)^2} n_3(\xi,\eta,\sigma)\widehat{h}(\xi-\eta)\widehat{h}(\eta-\sigma)\widehat{\phi}(\sigma)\,d\eta d\sigma,\\ n_3(\xi,\eta,\sigma) &:= \frac{|\xi||\sigma|}{|\xi|+|\sigma|}\big[(|\xi|-|\eta|)(|\eta|-|\sigma|) - (\xi-\eta)(\eta-\sigma)\big],\end{aligned} \tag{6.18}$$

and, for $\theta \in [\delta^2, 1/3]$ and $V \in \{D^\alpha\Omega^a : a \leq N_1/2 + 20,\ 2a + |\alpha| \leq N_1 + N_4 - 2\}$,

$$\|VN_4[h,\phi]\|_{Y^2_{3\theta-3\delta^2,1-3\delta^2}} \lesssim \varepsilon_1^4 2^{3\theta m - 5m/2 + 24\delta^2 m}. \tag{6.19}$$

Proof. Recall that h is constant in y. In view of (6.10) we have, for $t \in [2^m - 1, 2^{m+1}]$,

$$\||\nabla|^{1/6}\langle\nabla\rangle^{5/6}Vh(t)\|_{L^\infty_y Y^2_{\theta,1}} \lesssim \varepsilon_1 2^{\theta m + 6\delta^2 m}, \qquad \theta \in [0, 1/3], \tag{6.20}$$

and

$$\||\nabla|^{1/6}\langle\nabla\rangle^{5/6}Vh(t)\|_{L^\infty_y Y^\infty_{\theta,1}} \lesssim \varepsilon_1 2^{\theta m - 5m/6 + 6\delta^2 m}, \qquad \theta \in [0, 1/3], \tag{6.21}$$

for $V \in \{D^\alpha\Omega^a : a \leq N_1/2 + 20,\ 2a + |\alpha| \leq N_1 + N_4 - 2\}$. Moreover, using Lemma 9.4 in [32],

$$\||\nabla|Vu(t)\|_{L^2_y H^1_x} + \|(\partial_y Vu)(t)\|_{L^2_y H^1_x} \lesssim \varepsilon_1 2^{6\delta^2 m}, \tag{6.22}$$

for operators V as before. Therefore, using (6.14),

$$\|V[Q[u]]\|_{L^2_y Y^2_{\theta-\delta^2,1-\delta^2}} \lesssim \varepsilon_1^2 2^{\theta m - 5m/6 + 12\delta^2 m},$$

for $Q \in \{Q_a, Q_b\}$ and $\theta \in [\delta^2, 1/3]$. Therefore

$$\||\nabla|VL(u)\|_{L^2_y Y^2_{\theta-\delta^2,1-\delta^2}} + \|\partial_y VL(u)\|_{L^2_y Y^2_{\theta-\delta^2,1-\delta^2}} \lesssim \varepsilon_1^2 2^{\theta m - 5m/6 + 12\delta^2 m}, \tag{6.23}$$

using (6.11)–(6.12). Therefore, using the definition,

$$\||\nabla|V[u - u^{(1)}]\|_{L^2_y Y^2_{\theta-\delta^2,1-\delta^2}} + \|\partial_y V[u - u^{(1)}]\|_{L^2_y Y^2_{\theta-\delta^2,1-\delta^2}} \lesssim \varepsilon_1^2 2^{\theta m - 5m/6 + 12\delta^2 m}. \tag{6.24}$$

Since $u - u^{(2)} = L(u - u^{(1)})$, we can repeat this argument to prove that for $\theta \in [\delta^2, 1/3]$ and $V \in \{D^\alpha\Omega^a : a \leq N_1/2 + 20,\ 2a + |\alpha| \leq N_1 + N_4 - 2\}$,

$$\||\nabla|V[u - u^{(2)}]\|_{L^2_y Y^2_{2\theta-2\delta^2,1-2\delta^2}} + \|\partial_y V[u - u^{(2)}]\|_{L^2_y Y^2_{2\theta-2\delta^2,1-2\delta^2}} \lesssim \varepsilon_1^3 2^{2\theta m - 5m/3 + 18\delta^2 m}. \tag{6.25}$$

To prove the decomposition (6.16) we start from the identities (6.5) and (6.3), which gives $G(h)\phi = \partial_y u - Q_a$. Letting $Q_a^{(n)} = Q_a[u^{(n)}]$, $Q_b^{(n)} = Q_b[u^{(n)}]$, $n \in \{1,2\}$, it follows that

$$\begin{aligned} G(h)\phi &= |\nabla|\phi + \int_{-\infty}^{0} |\nabla| e^{-|s||\nabla|}(Q_b^{(2)}(s) - Q_a^{(2)}(s))\,ds + N_{4,1}, \\ N_{4,1} &:= \int_{-\infty}^{0} |\nabla| e^{-|s||\nabla|}[(Q_b - Q_b^{(2)})(s) - (Q_a - Q_a^{(2)})(s)]\,ds. \end{aligned} \tag{6.26}$$

In view of (6.25), (6.21), and the algebra rule (6.14), we have

$$\|V(Q - Q^{(2)})\|_{L^2_y Y^2_{3\theta-3\delta^2,1-3\delta^2}} \lesssim \varepsilon_1^4 2^{3\theta m - 5m/2 + 24\delta^2 m},$$

for $Q \in \{Q_a, Q_b\}$. Therefore, using (6.12), $|\nabla|^{-1/2} N_{4,1}$ satisfies the desired bound (6.19).

It remains to calculate the integral in the first line of (6.26). Letting $\alpha = |\nabla h|^2$ we have

$$\begin{aligned} &\mathcal{F}\{u^{(1)}\}(\xi,y) = e^{y|\xi|}\widehat{\phi}(\xi), \\ &\mathcal{F}\{Q_a^{(1)}\}(\xi,y) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} (\xi-\eta)\cdot\eta e^{y|\eta|}\widehat{h}(\xi-\eta)\widehat{\phi}(\eta)\,d\eta - \frac{1}{4\pi^2}\int_{\mathbb{R}^2} |\eta| e^{y|\eta|}\widehat{\alpha}(\xi-\eta)\widehat{\phi}(\eta)\,d\eta, \\ &\mathcal{F}\{Q_b^{(1)}\}(\xi,y) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{(\xi-\eta)\cdot\xi}{|\xi|}|\eta| e^{y|\eta|}\widehat{h}(\xi-\eta)\widehat{\phi}(\eta)\,d\eta. \end{aligned} \tag{6.27}$$

Therefore

$$\begin{aligned} \mathcal{F}\{L(u^{(1)})\}(\xi,y) &= \frac{1}{8\pi^2}\int_{\mathbb{R}^2} (e^{y|\xi|} - e^{y|\eta|})\Big[\frac{(\xi-\eta)\cdot\eta}{|\xi|+|\eta|} - \frac{|\eta|(\xi-\eta)\cdot\xi}{|\xi|(|\xi|+|\eta|)}\Big]\widehat{h}(\xi-\eta)\widehat{\phi}(\eta)\,d\eta \\ &+ \frac{1}{8\pi^2}\int_{\mathbb{R}^2} (e^{y|\xi|} - e^{y|\eta|})\Big[\frac{(\xi-\eta)\cdot\eta}{-|\xi|+|\eta|} + \frac{|\eta|(\xi-\eta)\cdot\xi}{|\xi|(-|\xi|+|\eta|)}\Big]\widehat{h}(\xi-\eta)\widehat{\phi}(\eta)\,d\eta \\ &+ \widehat{E_1}(\xi,y), \end{aligned}$$

where

$$\||\nabla| V E_1\|_{L^2_y Y^2_{2\theta-2\delta^2,1-2\delta^2}} + \|\partial_y V E_1\|_{L^2_y Y^2_{2\theta-2\delta^2,1-2\delta^2}} \lesssim \varepsilon_1^3 2^{2\theta m - 5m/3 + 18\delta^2 m}.$$

After algebraic simplifications, this gives

$$\mathcal{F}\{L(u^{(1)})\}(\xi,y) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} (e^{y|\xi|} - e^{y|\eta|})|\eta|\widehat{h}(\xi-\eta)\widehat{\phi}(\eta)\,d\eta + \widehat{E_1}(\xi,y).$$

Since $u^{(2)} - u^{(1)} = L(u^{(1)})$ we calculate

$$\begin{aligned} &\mathcal{F}\{Q_a^{(2)} - Q_a^{(1)}\}(\xi,y) \\ &\quad = \frac{1}{16\pi^4}\int_{(\mathbb{R}^2)^2} |\sigma|(\xi-\eta)\cdot\eta(e^{y|\eta|} - e^{y|\sigma|})\widehat{h}(\xi-\eta)\widehat{h}(\eta-\sigma)\widehat{\phi}(\sigma)\,d\eta d\sigma + \widehat{E_2}(\xi,y) \end{aligned} \tag{6.28}$$

and

$$\begin{aligned} &\mathcal{F}\{Q_b^{(2)} - Q_b^{(1)}\}(\xi,y) \\ &\quad = \frac{1}{16\pi^4}\int_{(\mathbb{R}^2)^2} |\sigma|\frac{(\xi-\eta)\cdot\xi}{|\xi|}(|\eta|e^{y|\eta|} - |\sigma|e^{y|\sigma|})\widehat{h}(\xi-\eta)\widehat{h}(\eta-\sigma)\widehat{\phi}(\sigma)\,d\eta d\sigma + \widehat{E_3}(\xi,y) \end{aligned} \tag{6.29}$$

where

$$\|V E_2\|_{L^2_y Y^2_{3\theta-3\delta^2,1-3\delta^2}} + \|V E_3\|_{L^2_y Y^2_{3\theta-3\delta^2,1-3\delta^2}} \lesssim \varepsilon_1^4 2^{3\theta m - 5m/2 + 24\delta^2 m}.$$

We examine now the formula in the first line of (6.26). The contributions of E_2 and E_3 can be estimated as part of the quartic error term, using also (6.12). The main contributions can be divided into quadratic terms (coming from $Q_a^{(1)}$ and $Q_b^{(1)}$ in (6.27)), and cubic terms coming from (6.28)–(6.29) and the cubic term in $Q_a^{(1)}$. The conclusion of the lemma follows. $\square$

Step 4. Finally, we can prove the desired expansion of the water-wave system.

Lemma 6.3. *Assume that (h,ϕ) satisfy (1.4) and (1.13). Then*

$$(\partial_t + i\Lambda)\mathcal{U} = \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_{\geq 4}, \tag{6.30}$$

where $\mathcal{U} = \langle\nabla\rangle h + i|\nabla|^{1/2}\phi$ and $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_{\geq 4}$ are as in subsection 2.2.

Proof. We rewrite (1.4) in the form

$$\partial_t\mathcal{U} = \langle\nabla\rangle G(h)\phi + i|\nabla|^{1/2}\Big[-h + \operatorname{div}\Big[\frac{\nabla h}{(1+|\nabla h|^2)^{1/2}}\Big] - \frac{1}{2}|\nabla\phi|^2 + \frac{(G(h)\phi + \nabla h\cdot\nabla\phi)^2}{2(1+|\nabla h|^2)}\Big]. \tag{6.31}$$

We use now the formula (6.16) to extract the linear, the quadratic, and the cubic terms in the right-hand side of this formula. More precisely, we set

$$\begin{aligned}
&\mathcal{N}_1 := \langle\nabla\rangle|\nabla|\phi + i|\nabla|^{1/2}(-h+\Delta h) = -i\Lambda\mathcal{U},\\
&\mathcal{N}_2 := \langle\nabla\rangle N_2[h,\phi] + i|\nabla|^{1/2}\big[-\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}(|\nabla|\phi)^2\big],\\
&\mathcal{N}_3 := \langle\nabla\rangle N_3[h,h,\phi] + i|\nabla|^{1/2}\big[-\frac{1}{2}\operatorname{div}(\nabla h|\nabla h|^2) + |\nabla|\phi\cdot(N_2[h,\phi] + \nabla h\cdot\nabla\phi)\big].
\end{aligned} \tag{6.32}$$

Then we substitute $h = \langle\nabla\rangle^{-1}(\mathcal{U}+\overline{\mathcal{U}})/2$ and $|\nabla|^{1/2}\phi = (\mathcal{U}-\overline{\mathcal{U}})/(2i)$. The symbols that define the quadratic component $\mathcal{N}_2$ are linear combinations of the symbols

$$n_{2,1}(\xi,\eta) = \sqrt{1+|\xi|^2}\frac{\xi\cdot\eta - |\xi||\eta|}{|\eta|^{1/2}\sqrt{1+|\xi-\eta|^2}}, \qquad n_{2,2}(\xi,\eta) = |\xi|^{1/2}\frac{(\xi-\eta)\cdot\eta + |\xi-\eta||\eta|}{|\xi-\eta|^{1/2}|\eta|^{1/2}}.$$

It is easy to see that these symbols verify the properties (2.21). A slightly nontrivial argument is needed for $n_{2,1}$ in the case $k_1 = \min(k,k_1,k_2) \ll k$.

The cubic terms in $\mathcal{N}_3$ in (6.32) are defined by finite linear combinations of the symbols

$$\begin{aligned}
n_{3,1}(\xi,\eta,\sigma) &= \sqrt{\frac{1+|\xi|^2}{(1+|\xi-\eta|^2)(1+|\eta-\sigma|^2)}}\frac{|\xi||\sigma|^{1/2}}{|\xi|+|\sigma|}\big[(|\xi|-|\eta|)(|\eta|-|\sigma|) - (\xi-\eta)(\eta-\sigma)\big],\\
n_{3,2}(\xi,\eta,\sigma) &= |\xi|^{1/2}\frac{(\xi\cdot(\xi-\eta))((\eta-\sigma)\cdot\sigma)}{\sqrt{(1+|\xi-\eta|^2)(1+|\eta-\sigma|^2)(1+|\sigma|^2)}},\\
n_{3,3}(\xi,\eta,\sigma) &= |\xi|^{1/2}|\xi-\eta|^{1/2}|\sigma|^{1/2}\frac{|\sigma|-|\eta|}{\sqrt{1+|\eta-\sigma|^2}}.
\end{aligned}$$

It is easy to verify the properties (2.22) for these explicit symbols.

The higher order remainder in the right-hand of (6.31) can be written in the form

$$\mathcal{N}_{\geq 4} = |\nabla|^{1/2}N_4', \qquad \sup_{a\leq N_1/2+20,\, 2a+|\alpha|\leq N_1+N_4-4}\|D^\alpha\Omega^a N_4'\|_{Y^2_{1-\delta,1-\delta}} \lesssim \varepsilon_1^4 2^{-3m/2+\delta m}, \tag{6.33}$$

using (6.19), (6.10), and the algebra property (6.13). Moreover, using only the $\mathcal{O}$ hierarchy as in the proof of Corollary 9.7 in [32], we have $\|\mathcal{N}_{\geq 4}\|_{\mathcal{O}_{4,-4}} \lesssim \varepsilon_1^4$, i.e.

$$\|\mathcal{N}_{\geq 4}\|_{H^{N_0-4}} + \|\mathcal{N}_{\geq 4}\|_{H^{N_1,N_3-4}} \lesssim \varepsilon_1^4 2^{-5m/2+\delta m}. \tag{6.34}$$

These two bounds suffice to prove the desired claims on $\mathcal{N}_{\geq 4}$ in (2.25). Indeed, the L^2 bound follows directly from (6.34). For the Z norm bound it suffices to prove that, for any $(k,j)\in\mathcal{J}$,

$$\sup_{a\leq N_1/2+20,\, 2a+|\alpha|\leq N_1+N_4} 2^{j(1-50\delta)}\|Q_{jk}e^{it\Lambda}D^\alpha\Omega^a\mathcal{N}_{\geq 4}\|_{L^2}\lesssim \varepsilon_1^4 2^{-m-\delta m}. \tag{6.35}$$

This follows easily from (6.34) and (6.33), unless

$$j\geq 3m/2+(N_0/4)k^+ +\mathcal{D}\qquad\text{and}\qquad j\geq 3m/2-k/2+\mathcal{D}.$$

On the other hand, if these inequalities hold then let $f=D^\alpha\Omega^a\mathcal{N}_{\geq 4}$, $a\leq N_1/2+20$, $2a+|\alpha|\leq N_1+N_4$, and decompose $f=\sum_{(k',j')\in\mathcal{J}}f_{j',k'}$ as in (3.23). The bound (6.33) shows that

$$\sum_{(k',j')\in\mathcal{J}}2^{-4\max(k',0)}2^{j'(1-\delta)}\|f_{j',k'}\|_{L^2}\lesssim \varepsilon_1^4 2^{-3m/2+\delta m}. \tag{6.36}$$

The desired bound (6.34) follows by the usual approximate finite speed of propagation argument: we may assume $|k'-k|\leq 4$ and consider the cases $j'\leq j-\delta j$ (which gives negligible contributions) and $j'\geq j-\delta j$ (in which case (6.36) suffices). This completes the proof. □

7. Analysis of phase functions

In this section we collect and prove some important facts about the phase functions Φ.

7.1. **Basic properties.**

Recall that

$$\begin{aligned}&\Phi(\xi,\eta)=\Phi_{\sigma\mu\nu}(\xi,\eta)=\Lambda_\sigma(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta),\qquad \sigma,\mu,\nu\in\{+,-\},\\ &\Lambda_\kappa(\xi)=\lambda_\kappa(|\xi|)=\kappa\lambda(|\xi|)=\kappa\sqrt{|\xi|+|\xi|^3}.\end{aligned} \tag{7.1}$$

We have

$$\lambda'(x)=\frac{1+3x^2}{2\sqrt{x+x^3}},\qquad \lambda''(x)=\frac{3x^4+6x^2-1}{4(x+x^3)^{3/2}},\qquad \lambda'''(x)=\frac{3(1+5x^2-5x^4-x^6)}{8(x+x^3)^{5/2}}. \tag{7.2}$$

Therefore

$$\lambda''(x)\geq 0\ \text{ if }\ x\geq\gamma_0,\qquad \lambda''(x)\leq 0\ \text{ if }\ x\in[0,\gamma_0],\qquad \gamma_0:=\sqrt{\frac{2\sqrt{3}-3}{3}}\approx 0.393. \tag{7.3}$$

It follows that

$$\lambda(\gamma_0)\approx 0.674,\qquad \lambda'(\gamma_0)\approx 1.086,\qquad \lambda'''(\gamma_0)\approx 4.452,\qquad \lambda''''(\gamma_0)\approx -28.701. \tag{7.4}$$

Let $\gamma_1:=\sqrt{2}\approx 1.414$ denote the radius of the space-time resonant sphere, and notice that

$$\lambda(\gamma_1)=\sqrt{3\sqrt{2}}\approx 2.060,\quad \lambda'(\gamma_1)=\frac{7}{2\sqrt{3\sqrt{2}}}\approx 1.699,\quad \lambda''(\gamma_1)=\frac{23}{4\sqrt{54\sqrt{2}}}\approx 0.658. \tag{7.5}$$

The following simple observation will be used many times: if $U_2\geq 1$, $\xi,\eta\in\mathbb{R}^2$, $\max(|\xi|,|\eta|,|\xi-\eta|)\leq U_2$, $\min(|\xi|,|\eta|,|\xi-\eta|)=a\leq 2^{-10}U_2^{-1}$, then

$$|\Phi(\xi,\eta)|\geq\lambda(a)-\sup_{b\in[a,U_2]}(\lambda(a+b)-\lambda(b))\geq\lambda(a)-a\max\{\lambda'(a),\lambda'(U_2+1)\}\geq\lambda(a)/4. \tag{7.6}$$

Lemma 7.1. *(i) The function λ' is strictly decreasing on the interval $(0,\gamma_0]$ and strictly increasing on the interval $[\gamma_0,\infty)$, and*

$$\lim_{x\to\infty}\Big[\lambda'(x)-\frac{3\sqrt{x}}{2}\Big]=0,\qquad \lim_{x\to 0}\Big[\lambda'(x)-\frac{1}{2\sqrt{x}}\Big]=0. \tag{7.7}$$

The function λ' is concave up on the interval $(0,1]$ and concave down on the interval $[1,\infty)$. For any $y>\lambda'(\gamma_0)$ the equation $\lambda'(r)=y$ has two solutions $r_1(y)\in(0,\gamma_0)$ and $r_2(y)\in(\gamma_0,\infty)$.

(ii) If $a\neq b\in(0,\infty)$ then

$$\lambda'(a)=\lambda'(b)\quad \text{if and only if}\quad (a-b)^2=\frac{(3ab+1)(3a^2b^2+6ab-1)}{1-9ab}. \tag{7.8}$$

In particular, if $a\neq b\in(0,\infty)$ and $\lambda'(a)=\lambda'(b)$ then $ab\in(1/9,\gamma_0^2]$.

(iii) Let $b:[\gamma_0,\infty)\to(0,\gamma_0]$ be the implicit function defined by $\lambda'(a)=\lambda'(b(a))$. Then b is a smooth decreasing function and[3]

$$\begin{aligned}&b'(a)\in[-1,-b(a)/a],\qquad a+b(a)\ \text{is increasing on}\ [\gamma_0,\infty),\qquad b(a)\approx 1/a,\\ &-b'(a)\approx 1/a^2,\qquad b'(a)+1\approx(a-\gamma_0)/a.\end{aligned} \tag{7.9}$$

In particular,

$$a+b(a)-2\gamma_0\approx\frac{(a-\gamma_0)^2}{a}. \tag{7.10}$$

Moreover,

$$-\left[\lambda''(b(a))+\lambda''(a)\right]\approx a^{-1/2}(a-\gamma_0)^2. \tag{7.11}$$

(iv) If $a,b\in(0,\infty)$ then

$$\lambda(a+b)=\lambda(a)+\lambda(b)\quad \text{if and only if}\quad (a-b)^2=\frac{4+8ab-32a^2b^2}{9ab-4}. \tag{7.12}$$

In particular, if $a,b\in(0,\infty)$ and $\lambda(a+b)=\lambda(a)+\lambda(b)$ then $ab\in[4/9,1/2]$. Moreover,

$$\begin{aligned}&\text{if}\ \ ab>1/2\ \ \text{then}\ \ \lambda(a+b)-\lambda(a)-\lambda(b)>0,\\ &\text{if}\ \ ab<4/9\ \ \text{then}\ \ \lambda(a+b)-\lambda(a)-\lambda(b)<0.\end{aligned} \tag{7.13}$$

Proof. The conclusions (i) and (ii) follow from (7.2)–(7.4) by elementary arguments. For part (iii) we notice that, with $Y=ab$.

$$(a+b(a))^2=F(Y):=\frac{-9Y^3-21Y^2-3Y+1}{9Y-1}+4Y=\frac{32/81}{9Y-1}-Y^2+14Y/9-49/81,$$

as a consequence of (7.8). Taking the derivative with respect to a it follows that

$$2(a+b(a))(1+b'(a))=[ab'(a)+b(a)]F'(Y). \tag{7.14}$$

Since $F'(Y)\leq -1/10$ for all $Y\in(1/9,\gamma_0^2]$, it follows that $b'(a)\in[-1,-b(a)/a]$ for all $a\in[\gamma_0,\infty)$. The claims in the first line of (7.9) follow.

The claim $-b'(a)\approx 1/a^2$ follows from the identity $\lambda''(a)-\lambda''(b(a))b'(a)=0$. The last claim in (7.9) is clear if $a-\gamma_0\gtrsim 1$; on the other hand, if $a-\gamma_0=\rho\ll 1$ then (7.14) gives

$$-\frac{1+b'(a)}{b'(a)+b(a)/a}\approx 1,\qquad \gamma_0-b(a)\approx\rho.$$

In particular $1-b(a)/a\approx\rho$ and the last conclusion in (7.9) follows.

[3]In a neighborhood of γ_0, $\lambda'(x)$ behaves like $A+B(x-\gamma_0)^2-C(x-\gamma_0)^3$, where $A,B,C>0$. The asymptotics described in (7.9)–(7.11) are consistent with this behaviour.

The claim in (7.10) follows by integrating the approximate identity $b'(x)+1 \approx (x-\gamma_0)/x$ between γ_0 and a. To prove (7.11) we recall that $\lambda''(a)-\lambda''(b(a))b'(a)=0$. Therefore

$$-[\lambda''(b(a))+\lambda''(a)] = -\lambda''(b(a))(1+b'(a)) = \lambda''(a)\frac{1+b'(a)}{-b'(a)},$$

and the desired conclusion follows using also (7.9).

To prove (iv), we notice that (7.12) and the claim that $ab \in [4/9, 1/2]$ follow from (7.2)–(7.4) by elementary arguments. To prove (7.13), let $G(x) := \lambda(a+x)-\lambda(a)-\lambda(x)$. For $a \in (0,\infty)$ fixed we notice that $G(x)>0$ if x is sufficiently large and $G(x)<0$ if $x>0$ is sufficiently small. The desired conclusion follows from the continuity of G. □

7.2. **Resonant sets.** We prove now an important proposition describing the geometry of resonant sets.

Proposition 7.2. *(Structure of resonance sets) The following claims hold:*

(i) There are functions $p_{++1}=p_{--1}:(0,\infty)\to(0,\infty)$, $p_{++2}=p_{--2}:[2\gamma_0,\infty)\to(0,\gamma_0]$, $p_{+-1}=p_{-+1}:(0,\infty)\to(\gamma_0,\infty)$ such that, if $\sigma,\mu,\nu\in\{+,-\}$ and $\xi\neq 0$ then

$$(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)=0 \text{ if and only if } \eta\in P_{\mu\nu}(\xi):=\Big\{p_{\mu\nu k}(|\xi|)\frac{\xi}{|\xi|},\xi-p_{\mu\nu k}(|\xi|)\frac{\xi}{|\xi|}:k\in\{1,2\}\Big\}. \tag{7.15}$$

(ii) (Space resonances) With $\mathcal{D}_{k,k_1,k_2}$ as in (2.3), assume that

$$(\xi,\eta)\in\mathcal{D}_{k,k_1,k_2} \quad \text{and} \quad |(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)|\leq\epsilon_2\leq 2^{-\mathcal{D}_1}2^{k-\max(k_1,k_2)}, \tag{7.16}$$

for some constant $\mathcal{D}_1$ sufficiently large. Then $\big||k_1|-|k_2|\big|\leq 20$ and, for some $p\in P_{\mu\nu}(\xi)$[4]*,*

- *if $|k|\leq 100$ then $\max(|k_1|,|k_2|)\leq 200$ and*

$$\begin{cases} \text{either} & \mu=-\nu,\ \big|\eta-p\big|\lesssim\epsilon_2,\\ \text{or} & \mu=\nu,\ \big|(\eta-p)\cdot\xi^\perp/|\xi|\big|\lesssim\epsilon_2,\ \text{and } \big|(\eta-p)\cdot\xi/|\xi|\big|\lesssim\frac{\epsilon_2}{\epsilon_2^{2/3}+\big||\xi|-2\gamma_0\big|}; \end{cases} \tag{7.17}$$

- *if $k\leq -100$ then*

$$\begin{cases} \text{either} & \mu=-\nu,\ k_1,k_2\in[-10,10],\ \text{and } \big|\eta-p\big|\lesssim\epsilon_2 2^{|k|},\\ \text{or} & \mu=\nu,\ k_1,k_2\in[k-10,k+10],\ \text{and } |\eta-\xi/2|\lesssim 2^{-3|k|/2}\epsilon_2; \end{cases} \tag{7.18}$$

- *if $k\geq 100$ then*

$$\big|\eta-p\big|\lesssim\epsilon_2 2^{k/2}. \tag{7.19}$$

(iii) (Space-time resonances) Assume that $(\xi,\eta)\in\mathcal{D}_{k,k_1,k_2}$,

$$|\Phi_{\sigma\mu\nu}(\xi,\eta)|\leq\epsilon_1\leq 2^{-\mathcal{D}_1}2^{\min(k,k_1,k_2,0)/2}, \quad |(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)|\leq\epsilon_2\leq 2^{-\mathcal{D}_1}2^{k-\max(k_1,k_2)}2^{-2k_+}. \tag{7.20}$$

Then, with $\gamma_1:=\sqrt{2}$,

$$\pm(\sigma,\mu,\nu)=(+,+,+), \quad |\eta-p_{++1}(\xi)|=|\eta-\xi/2|\lesssim\epsilon_2, \quad \big||\xi|-\gamma_1\big|\lesssim\epsilon_1+\epsilon_2^2. \tag{7.21}$$

Proof. (i) We have

$$(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)=\mu\lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|}-\nu\lambda'(|\eta|)\frac{\eta}{|\eta|}. \tag{7.22}$$

[4]The set $P_{\mu\nu}(\xi)$ contains 2 points if $(\mu,\nu)\in\{(+.-),(-,+)\}$ and at most 3 points if $(\mu,\nu)\in\{(+.+),(-,-)\}$.

Assume that $\xi = \alpha e$ for some $\alpha \in (0,\infty)$ and $e \in \mathbb{S}^1$. In view of (7.22), $(\nabla_\eta \Phi_{\sigma\mu\nu})(\xi,\eta) = 0$ if and only if

$$\eta = \beta e, \quad \beta \in \mathbb{R} \setminus \{0,\alpha\}, \quad \mu\lambda'(|\alpha-\beta|)\,\mathrm{sgn}(\alpha-\beta) = \nu\lambda'(|\beta|)\,\mathrm{sgn}(\beta). \tag{7.23}$$

We observe that it suffices to define the functions p_{++1}, p_{++2}, and p_{+-1} satisfying (7.15), since clearly $p_{--1} = p_{++1}$, $p_{--2} = p_{++2}$, and $p_{-+1} = p_{+-1}$.

If $(\mu,\nu) = (+,+)$ then, as a consequence of (7.23), $\beta \in (0,\alpha)$ and $\lambda'(\alpha-\beta) = \lambda'(\beta)$. Therefore, according to Lemma 7.1 (i)–(iii), there are two possible solutions,

$$\begin{aligned} &\beta = p_{++1}(\alpha) := \alpha/2, \\ &\beta = p_{++2}(\alpha) \quad \text{ uniquely determined by } \lambda'(\beta) = \lambda'(\alpha-\beta) \text{ and } \beta \in (0,\gamma_0]. \end{aligned} \tag{7.24}$$

The uniqueness of the point $p_{++2}(\alpha)$ is due to the fact that the function $x \to x + b(x)$ is increasing on $[\gamma_0,\infty)$, see (7.9). On the other hand, if $(\mu,\nu) = (+,-)$ then, as a consequence of (7.23), $\beta < 0$ or $\beta > \alpha$ and $\lambda'(|\alpha-\beta|) = \lambda'(|\beta|)$. Therefore, according to Lemma 7.1, there is only one solution $\beta \geq \gamma_0$,

$$\beta = p_{+-1}(\alpha) \quad \text{ uniquely determined by } \lambda'(\beta) = \lambda'(\beta-\alpha) \text{ and } \beta \in [\max(\alpha,\gamma_0), \alpha+\gamma_0]. \tag{7.25}$$

The conclusions in part (i) follow.

(ii) Assume that (7.16) holds and that $(\mu,\nu) \in \{(+,+),(+,-)\}$. Let $\xi = \alpha e$, $|e| = 1$, $\alpha \in [2^{k-4}, 2^{k+4}]$, $\eta = \beta e + v$, $v\cdot e = 0$, $(\beta^2+|v|^2)^{1/2} \in [2^{k_2-4}, 2^{k_2+4}]$. The condition $|(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)| \leq \epsilon_2$ gives, using (7.22), $\big||k_1|-|k_2|\big| \leq 20$ and

$$\Big|\mu\lambda'(|\xi-\eta|)\frac{(\alpha-\beta)}{|\xi-\eta|} - \nu\lambda'(|\eta|)\frac{\beta}{|\eta|}\Big| \leq \epsilon_2, \qquad \Big| -\mu\frac{\lambda'(|\xi-\eta|)}{|\xi-\eta|} - \nu\frac{\lambda'(|\eta|)}{|\eta|}\Big||v| \leq \epsilon_2. \tag{7.26}$$

Since $\alpha \gtrsim 2^k$ and $|\xi-\eta|^{-1}\lambda'(|\xi-\eta|) \gtrsim 2^{|k_1|/2-k_1}$, the first inequality in (7.26) shows that

$$\Big|\mu\lambda'(|\xi-\eta|)\frac{-\beta}{|\xi-\eta|} - \nu\lambda'(|\eta|)\frac{\beta}{|\eta|}\Big| \gtrsim 2^{k+|k_1|/2-k_1}.$$

Since $1/|\beta| \geq 2^{-k_2-4}$, using also the second inequality in (7.26) it follows that

$$|v| \lesssim \epsilon_2 2^{-k-|k_1|/2+k_1+k_2} \tag{7.27}$$

and

$$\Big| -\mu\frac{\lambda'(|\xi-\eta|)}{|\xi-\eta|} - \nu\frac{\lambda'(|\eta|)}{|\eta|}\Big| \gtrsim 2^{k+|k_1|/2-k_1-k_2}.$$

In particular $|v| \leq 2^{-20}2^{\min(k_1,k_2)}$,

$$\big||\eta|-|\beta|\big| \lesssim \epsilon_2^2 2^{-2k-|k_1|+2k_1+k_2}, \qquad \big||\xi-\eta|-|\alpha-\beta|\big| \lesssim \epsilon_2^2 2^{-2k-|k_1|+k_1+2k_2}. \tag{7.28}$$

Using the first inequality in (7.26) it follows that

$$\big|\mu\lambda'(|\alpha-\beta|)\mathrm{sgn}(\alpha-\beta) - \nu\lambda'(|\beta|)\mathrm{sgn}(\beta)\big| \leq \epsilon_2 + C\epsilon_2^2 2^{-2k-|k_1|/2+2\max(k_1,k_2)}. \tag{7.29}$$

Proof of (7.17)**.** Assume first that $|k| \leq 100$. Then $\max(|k_1|,|k_2|) \leq 200$, since otherwise (7.29) cannot hold (so there are no points (ξ,η) satisfying (7.16)). The conclusion $\big|(\eta-p)\cdot \xi^\perp/|\xi|\big| \lesssim \epsilon_2$ in (7.17) follows from (7.27).

Case 1. If $(\mu,\nu) = (+,-)$ then (7.29) gives

$$\big|\lambda'(|\alpha-\beta|) - \lambda'(|\beta|)\big| \leq 2\epsilon_2, \qquad \mathrm{sgn}(\alpha-\beta) + \mathrm{sgn}(\beta) = 0.$$

Therefore either $\beta>\alpha$ and $\big|\lambda'(\beta-\alpha)-\lambda'(\beta)\big|\leq 2\epsilon_2$, in which case $\beta-\alpha<\gamma_0$, $\beta>\gamma_0$, and $|\beta-p_{+-1}(\alpha)|\lesssim\epsilon_2$, or $\beta<0$ and $\big|\lambda'(\alpha-\beta)-\lambda'(-\beta)\big|\leq 2\epsilon_2$, in which case $\alpha-\beta>\gamma_0$, $-\beta<\gamma_0$, and $|\alpha-\beta-p_{+-1}(\alpha)|\lesssim\epsilon_2$. The desired conclusion follows in the stronger form $|\eta-p|\lesssim\epsilon_2$.

Case 2. If $(\mu,\nu)=(+,+)$ then (7.29) gives

$$\big|\lambda'(|\alpha-\beta|)-\lambda'(|\beta|)\big|\leq 2\epsilon_2,\qquad \mathrm{sgn}(\alpha-\beta)=\mathrm{sgn}(\beta).$$

Therefore

$$\beta\in(0,\alpha)\qquad\text{and}\qquad \big|\lambda'(\alpha-\beta)-\lambda'(\beta)\big|\leq 2\epsilon_2. \tag{7.30}$$

Assume α fixed and let $G(\beta):=\lambda'(\beta)-\lambda'(\alpha-\beta)$. The function G vanishes when $\beta=\alpha/2$ or $\beta\in\{p_{++2}(\alpha),\alpha-p_{++2}(\alpha)\}$ (if $\alpha\geq 2\gamma_0$).

Assume that $\alpha=2\gamma_0+\rho\geq 2\gamma_0$, $\rho\in[0,2^{110}]$. Then, using Lemma 7.1 (iii),

$$p_{++2}(\alpha)\leq\gamma_0\leq\alpha/2\leq\alpha-p_{++2}(\alpha),\qquad \alpha/2-\gamma_0=\rho/2,\ \gamma_0-p_{++2}(\alpha)\approx\sqrt{\rho}, \tag{7.31}$$

where the last conclusion follows from (7.10) with $a=\alpha-p_{++2}(\alpha)$, $b(a)=p_{++2}(\alpha)$. Moreover, $|G'(\beta)|=|\lambda''(\beta)+\lambda''(\alpha-\beta)|\approx\rho$ if $\beta\in\{\alpha/2,p_{++2}(\alpha),\alpha-p_{++2}(\alpha)\}$, using (7.11) and (7.31). Also, $|G''(\beta)|=|\lambda'''(\beta)-\lambda'''(\alpha-\beta)|\lesssim\sqrt{\rho}$ if $|\beta-\alpha/2|\lesssim\sqrt{\rho}$, therefore

$$|G'(\beta)|\approx\rho\ \text{ if }\ \beta\in I_\alpha:=\{x:\ \min\big(|x-\alpha/2|,|x-p_{++2}(\alpha)|,|x-\alpha+p_{++2}(\alpha)|\big)\leq\sqrt{\rho}/C_0\}, \tag{7.32}$$

for some large constant C_0.

If $\rho\leq C_0^4\epsilon_2^{2/3}$ then the points $\alpha/2,p_{++2}(\alpha),\alpha-p_{++2}(\alpha)$ are within distance $\leq C_0^4\epsilon_2^{1/3}$. In this case it suffices to prove that $|G(\beta)|\geq 3\epsilon_2$ if $|\beta-\alpha/2|\geq 2C_0^4\epsilon_2^{1/3}$. Assume, for contradiction, that this is not true, so there is $\beta\leq\gamma_0-C_0^4\epsilon_2^{1/3}$ such that $|\lambda'(\beta)-\lambda'(\alpha-\beta)|\leq 3\epsilon_2$. So there is x close to β, $|x-\beta|\lesssim\epsilon_2^{2/3}$, such that $\lambda'(x)=\lambda'(\alpha-\beta)$. In particular, using (7.10) with $a=\alpha-\beta$, $b(a)=x$, we have $\alpha-\beta+x-2\gamma_0\geq C_0^7\epsilon_2^{2/3}$. Therefore $\alpha-2\gamma_0\geq C_0^6\epsilon_2^{2/3}$, in contradiction with the assumption $\alpha-2\gamma_0=\rho\leq C_0^4\epsilon_2^{2/3}$.

Assume now that $\rho\geq C_0^4\epsilon_2^{2/3}$. In view of (7.32), it suffices to prove that if $\beta\notin I_\alpha$ then $|G(\beta)|\geq 3\epsilon_2$. Assume, for contradiction, that this is not true, so there is $\beta\in(0,\alpha/2]\setminus I_\alpha$ such that $|\lambda'(\beta)-\lambda'(\alpha-\beta)|\leq 3\epsilon_2$. Since $\beta\leq\alpha/2-\sqrt{\rho}/C_0$, we may in fact assume that $\beta\leq\gamma_0-\sqrt{\rho}/(2C_0)$, provided that the constant $\mathcal{D}_1$ in (7.16) is sufficiently large. So there is x close to β, $|x-\beta|\lesssim\epsilon_2/\sqrt{\rho}$, such that $\lambda'(x)=\lambda'(\alpha-\beta)$. Using (7.9), it follows there is a point y close to x, $|y-x|\lesssim\epsilon_2/\rho$, such that $\lambda'(y)=\lambda'(\alpha-y)$. Therefore $y=p_{++2}(\alpha)$. In particular $|\beta-p_{++2}(\alpha)|\lesssim\epsilon_2/\rho$, in contradiction with the assumption $\beta\notin I_\alpha$, so $|\beta-p_{++2}(\alpha)|\geq\sqrt{\rho}/C_0$ (recall that $\rho\geq C_0^4\epsilon_2^{2/3}$).

The case $\alpha=2\gamma_0-\rho\leq 2\gamma_0$ is easier, since there is only one point to consider, namely $\alpha/2$. As in (7.32), $|G'(\beta)|\approx\rho$ if $|x-\alpha/2|\leq\sqrt{\rho}/C_0$. The proof then proceeds as before, by considering the two cases $\rho\leq C_0^4\epsilon_2^{2/3}$ and $\rho\geq C_0^4\epsilon_2^{2/3}$.

Proof of (7.18)**.** Assume now that $k\leq -100$, so $|k_1-k_2|\leq 20$, and consider two cases:

Case 1. Assume first that $(\mu,\nu)=(+,-)$. In view of (7.22) we have

$$\Big|\lambda'(|\eta|)\frac{\eta}{|\eta|}-\lambda'(|w|)\frac{w}{|w|}\Big|\leq\epsilon_2,\qquad\text{where } w=\eta-\xi. \tag{7.33}$$

If $\min(|\eta|,|w|)\leq\gamma_0-2^{-10}$ or $\max(|\eta|,|w|)\geq\gamma_0+2^{-10}$ it follows from (7.33) that $\big|\lambda'(|\eta|)-\lambda'(|w|)\big|\leq\epsilon_2$, therefore $\big||\eta|-|w|\big|\lesssim\epsilon_2 2^{-|k_1|/2+k_1}$. Therefore

$$\Big|\frac{\eta}{|\eta|}-\frac{w}{|w|}\Big|\lesssim\epsilon_2 2^{-|k_1|/2}\quad\text{ and }\quad\Big|\frac{1}{|\eta|}-\frac{1}{|w|}\Big|\lesssim\epsilon_2 2^{-|k_1|/2-k_1}.$$

As a consequence $|\eta-w| \lesssim \epsilon_2 2^{-|k_1|/2+k_1}$. On the other hand $|\eta - w| = |\xi| \gtrsim 2^k$, in contradiction with the assumption $\epsilon_2 \leq 2^{-\mathcal{D}_1}2^{k-k_1}$. Therefore either $|\eta|$ or $|w|$ has to belong to the interval $[\gamma_0 - 2^{-10}, \gamma_0 + 2^{-10}]$. Since $|\eta - w| \leq 2^{-90}$ it follows that

$$|\eta|, |\eta-\xi| \in [\gamma_0 - 2^{-9}, \gamma_0 + 2^{-9}]. \tag{7.34}$$

In particular $k_1, k_2 \in [-10, 10]$, as claimed. Moreover $|v| \lesssim \epsilon_2 2^{|k|}$ as desired, in view of (7.27). The condition (7.29) gives

$$\big|\lambda'(|\alpha-\beta|) - \lambda'(|\beta|)\big| \leq \epsilon_2 + C\epsilon_2^2 2^{-2k}, \qquad \mathrm{sgn}(\alpha-\beta) + \mathrm{sgn}(\beta) = 0.$$

Without loss of generality, we may assume that

$$\beta > \alpha, \qquad \big|\lambda'(\beta-\alpha) - \lambda'(\beta)\big| \leq \epsilon_2 + C\epsilon_2^2 2^{-2k}. \tag{7.35}$$

Notice that $p_{+-1}(\alpha) \in (\gamma_0, \alpha+\gamma_0)$. We have two cases: if $\epsilon_2 \geq 2^{-\mathcal{D}_1}2^{2k}$ then we need to prove that $|\beta - \gamma_0| \leq 2^{4\mathcal{D}_1}\epsilon_2 2^{|k|}$. This follows from (7.33): otherwise, if $|\beta - \gamma_0| = d \geq 2^{4\mathcal{D}_1}\epsilon_2 2^{|k|} \geq 2^{3\mathcal{D}_1}2^k$ then $\big||\eta| - \gamma_0\big| \approx d$ and $\big||w| - \gamma_0\big| \approx d$, using also (7.27). As a consequence of (7.33), we have $\big||\eta| - |w|\big| \lesssim \epsilon_2 d^{-1}$, so

$$\Big|\frac{\eta}{|\eta|} - \frac{w}{|w|}\Big| \lesssim \epsilon_2 \qquad \text{and} \qquad \Big|\frac{1}{|\eta|} - \frac{1}{|w|}\Big| \lesssim \epsilon_2 d^{-1}.$$

Thus $|\eta - w| \lesssim \epsilon_2 + \epsilon_2 d^{-1} \lesssim \epsilon_2 + 2^{k-4\mathcal{D}_1}$, in contradiction with the fact that $|\eta - w| = |\xi| \gtrsim 2^k$.

On the other hand, if $\epsilon_2 \leq 2^{-\mathcal{D}_1}2^{2k}$ then (7.35) gives $\big|\lambda'(\beta-\alpha) - \lambda'(\beta)\big| \leq 2\epsilon_2$ and $\beta \in (\gamma_0, \gamma_0+\alpha)$. Let $H(\beta) := \lambda'(\beta) - \lambda'(\beta-\alpha)$ and notice that

$$|H'(\beta)| \gtrsim |\beta - \gamma_0| + |\beta - \alpha - \gamma_0| \gtrsim 2^k$$

if β is in this set. The desired conclusion follows since $H(p_{+-1}(\alpha)) = 0$.

Case 2. If $(\mu,\nu) = (+,+)$ then (7.29) gives

$$\big|\lambda'(\alpha-\beta) - \lambda'(\beta)\big| \leq \epsilon_2 + C\epsilon_2^2 2^{-2k-|k_1|/2+2\max(k_1,k_2)}, \qquad \beta \in (0,\alpha).$$

This shows easily that $k_1, k_2 \in [k-10, k+10]$ and $|\alpha - 2\beta| \lesssim 2^{-3|k|/2}\epsilon_2$. The desired conclusion follows using also (7.27).

Proof of (7.19)**.** Assume now that $k \geq 100$ and consider two cases:

Case 1. If $(\mu,\nu) = (+,-)$ then (7.29) gives

$$\big|\lambda'(|\alpha-\beta|) - \lambda'(|\beta|)\big| \leq \epsilon_2 + C\epsilon_2^2 2^{-2k-|k_1|/2+2\max(k_1,k_2)}, \qquad \mathrm{sgn}(\alpha-\beta) + \mathrm{sgn}(\beta) = 0.$$

We may assume $\beta > \alpha$, $|\max(k_1,k_2) - k| \leq 20$, and $\big|\lambda'(\beta-\alpha) - \lambda'(\beta)\big| \leq 2\epsilon_2$. In particular $\beta \in (\alpha, \alpha+\gamma_0)$. Let $H(\beta) := \lambda'(\beta) - \lambda'(\beta-\alpha)$ as before and notice that $|H'(\beta)| \gtrsim 2^{3k/2}$ in this set. The desired conclusion follows since $H(p_{+-1}(\alpha)) = 0$, using also (7.27).

Case 2. If $(\mu,\nu) = (+,+)$ then (7.29) gives

$$\big|\lambda'(\alpha-\beta) - \lambda'(\beta)\big| \leq \epsilon_2 + C\epsilon_2^2 2^{-2k-|k_1|/2+2\max(k_1,k_2)}, \qquad \beta \in (0,\alpha). \tag{7.36}$$

If both β and $\alpha - \beta$ are in $[\gamma_0, \infty)$ then (7.36) gives $|\beta - \alpha/2| \lesssim \epsilon_2 2^{k/2}$, which suffices (using also (7.27)). Otherwise, assuming for example that $\beta \in (0, \gamma_0)$, it follows from (7.36) that $\beta \leq 2^{-k+20}$. Let, as before, $G(\beta) := \lambda'(\beta) - \lambda'(\alpha-\beta)$ and notice that $|G'(\beta)| \gtrsim 2^{3k/2}$ if $\beta \in (0, 2^{-k+20}]$. The desired conclusion follows since $G(p_{++2}(\alpha)) = 0$, using also (7.27).

(iii) If $k \leq -100$ then $\Phi_{\sigma\mu\nu}(\xi,\eta) \gtrsim 2^{k/2}$, in view of (7.6) and (7.18), which is not not allowed by the condition on ϵ_1.

If $k\geq 100$ and $(\mu,\nu)=(+,-)$ then $p_{+-1}(\alpha)-\alpha\leq 2^{-k+10}\leq 2^{k-10}\leq\alpha$ and

$$|\Phi(\xi,\eta)|\geq\big|\pm\lambda(\alpha)-\lambda(p_{+-1}(\alpha))+\lambda(p_{+-1}(\alpha)-\alpha)\big|-C\epsilon_2 2^k,$$

for some constant C sufficiently large. Moreover, in view of Lemma 7.1 (i), $\alpha(p_{+-1}(\alpha)-\alpha)\leq\gamma_0^2\leq 0.2$. In particular, using also Lemma 7.1 (iv), $|\Phi(\xi,\eta)|\gtrsim 2^{-k/2}$, which is impossible in view of the assumption on ϵ_1. A similar argument works also in the case $k\geq 100$ and $(\mu,\nu)=(+,+)$ to show that there are no points (ξ,η) satisfying (7.20).

Finally, assume that $|k|\leq 100$, so $|k_1|,|k_2|\in[0,200]$. If $(\mu,\nu)=(+,-)$ then there are still no solutions (ξ,η) of (7.20), using the same argument as before: in view of Lemma 7.1 (i), $\alpha(p_{+-1}(\alpha)-\alpha)\leq\gamma_0^2\leq 0.2$, so $|\Phi(\xi,\eta)|\gtrsim 1$ as a consequence of Lemma 7.1 (iv).

On the other hand, if $(\mu,\nu)=(+,+)$ then we may also assume that $\sigma=+$. If β is close to $p_{++2}(\alpha)$ or to $\alpha-p_{++2}(\alpha)$ then $\Phi(\xi,\eta)\gtrsim 1$, for the same reason as before. We are left with the case $|\beta-\alpha/2|\lesssim\epsilon_2$ and $\alpha\geq 1$. Therefore $|\eta-\xi/2|\lesssim\epsilon_2$. We notice now that the equation $\lambda(x)-2\lambda(x/2)=0$ has the unique solution $x=\sqrt{2}=:\gamma_1$, and the desired bound on $\big||\xi|-\gamma_1\big|$ follows since

$$\big||\xi|-\gamma_1\big|\lesssim|\Phi_{\sigma\mu\nu}(\xi,\xi/2)|\lesssim|\Phi_{\sigma\mu\nu}(\xi,\eta)|+|\Phi_{\sigma\mu\nu}(\xi,\xi/2)-\Phi_{\sigma\mu\nu}(\xi,\eta)|\lesssim\epsilon_1+\epsilon_2^2.$$

This completes the proof of the proposition. □

7.3. **Bounds on sublevel sets.** In this subsection we analyze the sublevel sets of the phase functions Φ, and the interaction of these sublevel sets with several other structures. We start with a general bound on the size of sublevel sets of functions, see [31, Lemma 8.5] for the proof.

Lemma 7.3. *Suppose $L,R,M\in\mathbb{R}$, $M\geq\max(1,L,L/R)$, and $Y:B_R:=\{x\in\mathbb{R}^n:|x|<R\}\to\mathbb{R}$ is a function satisfying $\|\nabla Y\|_{C^l(B_R)}\leq M$, for some $l\geq 1$. Then, for any $\epsilon>0$,*

$$\Big|\Big\{x\in B_R:|Y(x)|\leq\epsilon\ \text{and}\ \sum_{|\alpha|\leq l}|\partial_x^\alpha Y(x)|\geq L\Big\}\Big|\lesssim R^nML^{-1-1/l}\epsilon^{1/l}. \tag{7.37}$$

Moreover, if $n=l=1$, K is a union of at most A intervals, and $|Y'(x)|\geq L$ on K, then

$$|\{x\in K:|Y(x)|\leq\epsilon\}|\lesssim AL^{-1}\epsilon. \tag{7.38}$$

We prove now several important bounds on the sets of time resonances. Assume $\Phi=\Phi_{\sigma\mu\nu}$, for some choice of $\sigma,\mu,\nu\in\{+.-\}$, and $\mathcal{D}_1$ is the large constant fixed in Proposition 7.2.

Proposition 7.4 (Volume bounds of sublevel sets)**.** *Assume that $k,k_1,k_2\in\mathbb{Z}$, define $\mathcal{D}_{k,k_1,k_2}$ as in (2.3), let $\overline{k}:=\max(k,k_1,k_2)$, and assume that*

$$\min(k,k_1,k_2)+\max(k,k_1,k_2)\geq -100. \tag{7.39}$$

(i) Let

$$E_{k,k_1,k_2;\epsilon}:=\{(\xi,\eta)\in\mathcal{D}_{k,k_1,k_2}:\,|\Phi(\xi,\eta)|\leq\epsilon\}.$$

Then

$$\begin{aligned}\sup_\xi\int_{\mathbb{R}^2}\mathbf{1}_{E_{k,k_1,k_2;\epsilon}}(\xi,\eta)\,d\eta&\lesssim 2^{-\overline{k}/2}\epsilon\log(2+1/\epsilon)2^{4\min(k_1^+,k_2^+)},\\ \sup_\eta\int_{\mathbb{R}^2}\mathbf{1}_{E_{k,k_1,k_2;\epsilon}}(\xi,\eta)\,d\xi&\lesssim 2^{-\overline{k}/2}\epsilon\log(2+1/\epsilon)2^{4\min(k_1^+,k^+)}.\end{aligned} \tag{7.40}$$

(ii) Assume that $r_0\in[2^{-\mathcal{D}_1},2^{\mathcal{D}_1}]$, $\epsilon\leq 2^{\min(k,k_1,k_2,0)/2-\mathcal{D}_1}$, $\epsilon'\leq 1$ and let

$$E'_{k,k_1,k_2;\epsilon,\epsilon'}=\{(\xi,\eta)\in\mathcal{D}_{k,k_1,k_2},|\Phi(\xi,\eta)|\leq\epsilon,\big||\xi-\eta|-r_0\big|\leq\epsilon'\}.$$

Then we can write $E'_{k,k_1,k_2;\epsilon,\epsilon'} = E'_1 \cup E'_2$ *such that*

$$\sup_\xi \int_{\mathbb{R}^2} \mathbf{1}_{E'_1}(\xi,\eta)\,d\eta + \sup_\eta \int_{\mathbb{R}^2} \mathbf{1}_{E'_2}(\xi,\eta)\,d\xi \lesssim \epsilon \log(1/\epsilon) \cdot 2^{2\overline{k}}(\epsilon')^{1/2}. \tag{7.41}$$

(iii) Assume that $\epsilon \leq 2^{\min(k,k_1,k_2,0)/2-\mathcal{D}_1}$, $\kappa \leq 1$, $p, q \leq 0$, *and let*

$$E''_{k,k_1,k_2;\epsilon,\kappa} = \{(\xi,\eta) \in \mathcal{D}_{k,k_1,k_2}, |\Phi(\xi,\eta)| \leq \epsilon, |(\Omega_\eta\Phi)(\xi,\eta)| \leq \kappa\}.$$

Then

$$\begin{aligned}
&\sup_\xi \int_{\mathbb{R}^2} \mathbf{1}_{E''_{k,k_1,k_2;\epsilon,\kappa}}(\xi,\eta)\varphi_{\geq q}(\nabla_\eta\Phi(\xi,\eta))\,d\eta \lesssim 2^{8\min(|k_1|,|k_2|)}\epsilon\log(1/\epsilon)\cdot\kappa 2^{-q}2^{2\overline{k}},\\
&\sup_\eta \int_{\mathbb{R}^2} \mathbf{1}_{E''_{k,k_1,k_2;\epsilon,\kappa}}(\xi,\eta)\varphi_{\geq p}(\nabla_\xi\Phi(\xi,\eta))\,d\xi \lesssim 2^{8\min(|k_1|,|k|)}\epsilon\log(1/\epsilon)\cdot\kappa 2^{-p}2^{2\overline{k}}.
\end{aligned} \tag{7.42}$$

As a consequence, we can write $E''_{k,k_1,k_2;\epsilon,\kappa} = E''_1 \cup E''_2$ *such that*

$$\sup_\xi \int_{\mathbb{R}^2} \mathbf{1}_{E''_1}(\xi,\eta)\,d\eta + \sup_\eta \int_{\mathbb{R}^2} \mathbf{1}_{E''_2}(\xi,\eta)\,d\xi \lesssim \epsilon\log(1/\epsilon)\cdot\kappa 2^{12\overline{k}}. \tag{7.43}$$

Moreover, if $\kappa \leq 2^{-8\max(k,k_1,k_2)-\mathcal{D}_1}$ *then*

$$\begin{aligned}
&\sup_\xi \int_{\mathbb{R}^2} \mathbf{1}_{E''_{k,k_1,k_2;\epsilon,\kappa}}(\xi,\eta)\varphi_{\leq q}(\nabla_\eta\Phi(\xi,\eta))\,d\eta \lesssim \kappa 2^q 2^{8\overline{k}},\\
&\sup_\eta \int_{\mathbb{R}^2} \mathbf{1}_{E''_{k,k_1,k_2;\epsilon,\kappa}}(\xi,\eta)\varphi_{\leq p}(\nabla_\xi\Phi(\xi,\eta))\,d\xi \lesssim \kappa 2^p 2^{8\overline{k}}.
\end{aligned} \tag{7.44}$$

Proof. The condition (7.39) is natural due to (7.6), otherwise $|\Phi(\xi,\eta)| \approx 2^{\min(k,k_1,k_2)/2}$ in $\mathcal{D}_{k,k_1,k_2}$. Compare also with the condition $\epsilon \leq 2^{\min(k,k_1,k_2,0)/2-\mathcal{D}_1}$ in (ii) and (iii).

(i) By symmetry, it suffices to prove the inequality in the first line of (7.40). We may assume that $k_2 \leq k_1$, so, using (7.39),

$$k_1, \max(k,k_2) \in [\overline{k}-10, \overline{k}], \qquad k, k_2 \geq -\overline{k}-100. \tag{7.45}$$

Assume that $\xi = (s,0), \eta = (r\cos\theta, r\sin\theta)$, so

$$-\Phi(\xi,\eta) = -\sigma\lambda(s) + \nu\lambda(r) + \mu\lambda((s^2+r^2-2sr\cos\theta)^{1/2}) =: Z(r,\theta). \tag{7.46}$$

We may assume that $\epsilon \leq 2^{\min(k,k_2)}2^{\overline{k}/2-\mathcal{D}_1}$. Notice that

$$\Big|\frac{d}{d\theta}Z(r,\theta)\Big| = \Big|\lambda'((s^2+r^2-2sr\cos\theta)^{1/2})\frac{sr\sin\theta}{(s^2+r^2-2sr\cos\theta)^{1/2}}\Big|. \tag{7.47}$$

Assume that $|s-r| \geq 2^{\overline{k}-100}$, $s \in [2^{k-4}, 2^{k+4}]$, $r \in [2^{k_2-4}, 2^{k_2+4}]$. Then, for r, s fixed,

$$\big|\{\theta \in [0,2\pi] : |Z(r,\theta)| \leq \epsilon\}\big| \lesssim \sum_{b\in\{0,1\}} \frac{\epsilon}{\sqrt{2^{\overline{k}/2}2^{\min(k,k_2)}(\epsilon + Z(r,b\pi))}}. \tag{7.48}$$

Indeed, this follows from (7.47) since in this case $\big|\partial_\theta Z(r,\theta)\big| \approx 2^{\min(k,k_2)}2^{\overline{k}/2}|\sin\theta|$ for all $\theta \in [0,2\pi]$. Next, we observe that

$$\big|\{r \in [2^{k_2-4}, 2^{k_2+4}] : |s-r| \geq 2^{\overline{k}-100} \text{ and } |Z(r,b\pi)| \leq \kappa 2^{\min(k,k_2)}2^{\overline{k}/2}\}\big| \lesssim \kappa 2^{k_2}, \tag{7.49}$$

provided that $\overline{k}\geq 200$ and $b\in\{0,1\}$. Indeed, in proving (7.49) we may assume that $\kappa\leq 2^{-\mathcal{D}_1}$. Then we notice that the set in the left-hand side of (7.49) is nontrivial only if

$$\begin{aligned}&\text{either }\ \pm Z(r,b\pi)=\lambda(s)-\lambda(s\pm r)\pm\lambda(r)\ \text{ and }\ s\in[2^{\overline{k}-10},2^{\overline{k}+10}],\ r\in[2^{-\overline{k}-10},2^{-\overline{k}+10}],\\ &\text{or }\ \pm Z(r,b\pi)=\lambda(r)-\lambda(r\pm s)\pm\lambda(s)\ \text{ and }\ r\in[2^{\overline{k}-10},2^{\overline{k}+10}],\ s\in[2^{-\overline{k}-10},2^{-\overline{k}+10}].\end{aligned}$$

In all cases, the desired conclusion (7.49) follows easily, since $|\partial_r Z(r,b\pi)|$ is suitably bounded away from 0. Using also (7.48) it follows that

$$\big|\{\eta:\,|\eta|\in[2^{k_2-4},2^{k_2+4}],\ \big||\xi|-|\eta|\big|\geq 2^{\overline{k}-100}\text{ and }|\Phi(\xi,\eta)|\leq\epsilon\}\big|\lesssim\epsilon 2^{-\overline{k}/2}2^{4k_2^+}\tag{7.50}$$

provided that $|\xi|\in[2^{k-4},2^{k+4}]$, $\overline{k}\geq 200$, and (7.45) holds.

The case $\overline{k}\leq 200$ is easier. In this case we have $2^k,2^{k_1},2^{k_2}\approx 1$, due to (7.45). In view of Proposition 7.2 (iii), if $|Z(r,b\pi)|\leq\kappa\leq 2^{-2\mathcal{D}_1}$ and $\big|\partial_r Z(r,b\pi)\big|\leq 2^{-2\mathcal{D}_1}$ then s is close to γ_1, r is close to $\gamma_1/2$, $b=0$. As a consequence $\big|\partial_r^2 Z(r,b\pi)\big|\gtrsim 1$. It follows from Lemma 7.3 that

$$\big|\{r\in[2^{k_2-4},2^{k_2+4}]:\,|s-r|\geq 2^{\overline{k}-100}\ \text{ and }\ |Z(r,b\pi)|\leq\kappa\}\big|\lesssim\kappa^{1/2},$$

provided that $\overline{k}\leq 200$ and $\kappa\in\mathbb{R}$. Using (7.48) again it follows that

$$\big|\{\eta:\,|\eta|\in[2^{k_2-4},2^{k_2+4}],\ \big||\xi|-|\eta|\big|\geq 2^{\overline{k}-100}\text{ and }|\Phi(\xi,\eta)|\leq\epsilon\}\big|\lesssim\epsilon\log(2+1/\epsilon)\tag{7.51}$$

provided that $|\xi|\in[2^{k-4},2^{k+4}]$ and $\overline{k}\leq 200$.

Finally, we estimate the contribution of the set where $\big||\xi|-|\eta|\big|\leq 2^{\overline{k}-100}$. In this case we may assume that $k,k_1,k_2\geq\overline{k}-20$. We replace (7.48) by

$$\big|\{\theta\in[2^{-\mathcal{D}_1},2\pi-2^{-\mathcal{D}_1}]:\,|Z(r,\theta)|\leq\epsilon\}\big|\lesssim\frac{\epsilon}{\sqrt{2^{3\overline{k}/2}(\epsilon+Z(r,\pi))}},\tag{7.52}$$

which follows from (7.47) (since $\big|\partial_\theta Z(r,\theta)\big|\approx 2^{3\overline{k}/2}|\sin\theta|$ for all $\theta\in[2^{-\mathcal{D}_1},2\pi-2^{-\mathcal{D}_1}]$). The proof proceeds as before, by analyzing the vanishing of the function $r\to Z(r,\pi)$ (it is in fact slightly easier since $|Z(r,\pi)|\gtrsim 2^{3\overline{k}/2}$ if $\overline{k}\geq 200$). It follows that

$$\big|\{\eta:\,|\eta|\in[2^{k_2-4},2^{k_2+4}],\ \big||\xi|-|\eta|\big|\leq 2^{\overline{k}-100}\text{ and }|\Phi(\xi,\eta)|\leq\epsilon\}\big|\lesssim\epsilon\log(2+1/\epsilon)2^{\overline{k}/2}.$$

The desired bound in the first line of (7.40) follows using also (7.50)–(7.51).

(ii) We may assume that $\min(k,k_2)\geq -2\mathcal{D}_1$ and that $\epsilon'\leq 2^{-\mathcal{D}_1^2}$. Define

$$\begin{aligned}E_1'&:=\{(\xi,\eta)\in E'_{k,k_1,k_2;\epsilon,\epsilon'}:\,|\nabla_\eta\Phi(\xi,\eta)|\geq 2^{-20\mathcal{D}_1}\},\\ E_2'&:=\{(\xi,\eta)\in E'_{k,k_1,k_2;\epsilon,\epsilon'}:\,|\nabla_\xi\Phi(\xi,\eta)|\geq 2^{-20\mathcal{D}_1}\}.\end{aligned}\tag{7.53}$$

It is easy to see that $E'_{k,k_1,k_2;\epsilon,\epsilon'}=E_1'\cup E_2'$, using Proposition 7.2 (ii). By symmetry, it suffices to prove (7.41) for the first term in the left-hand side. Let $\xi=(s,0)$, $\eta=(r\cos\theta,r\sin\theta)$, and

$$\begin{aligned}E'_{1,\xi,1}&:=\{\eta:(\xi,\eta)\in E_1',\ |\sin\theta|\leq(\epsilon')^{1/2}2^{-2k_2}\},\\ E'_{1,\xi,2}&:=\{\eta:(\xi,\eta)\in E_1',\ |\sin\theta|\geq(\epsilon')^{1/2}2^{-2k_2}\}.\end{aligned}\tag{7.54}$$

It follows from Lemma 7.3 that $\big|E'_{1,\xi,1}\big|\lesssim\epsilon\cdot(\epsilon')^{1/2}$. Indeed, since $|\nabla_\eta\Phi(\xi,\eta)|\geq 2^{-20\mathcal{D}_1}$ and $|\sin\theta|\leq(\epsilon')^{1/2}2^{-2k_2}$, it follows from formula (7.46) that $|\partial_r[\Phi(\xi,\eta)]|\geq 2^{-21\mathcal{D}_1}$ in $E'_{1,\xi,1}$. The desired conclusion follows by applying Lemma 7.3 for every suitable angle θ.

To estimate $\left|E'_{1,\xi,2}\right|$ we use the formula (7.46). It follows from definitions that

$$E'_{1,\xi,2} \subseteq \{\eta : r \in [2^{k_2-4}, 2^{k_2+4}],\ \lambda(r) \in K_{s,r_0},\ |\sin\theta| \geq (\epsilon')^{1/2} 2^{-2k_2},\ |\Phi(\xi,\eta)| \leq \epsilon\},$$

where K_{s,r_0} is an interval of length $\lesssim \epsilon'$ and $k_2 \geq -2\mathcal{D}_1$. Therefore, using the formula (7.46) as before, $\left|E'_{1,\xi,2}\right| \lesssim 2^{2k_2}\epsilon(\epsilon')^{1/2}$, as desired.

(iii) For (7.42) it suffices to prove the inequality in the first line. We may also assume that (7.39) holds, and that $\kappa \leq 2^{q-2\max(k,k_1,k_2)-\mathcal{D}_1}$. Assume, as before, that $\xi = (s,0)$, $\eta = (r\cos\theta, r\sin\theta)$. Since

$$|(\Omega_\eta\Phi)(\xi,\eta)| = \frac{\lambda'(|\xi-\eta|)}{|\xi-\eta|}|(\xi\cdot\eta^\perp)|,$$

the condition $|(\Omega_\eta\Phi)(\xi,\eta)| \leq \kappa$ gives

$$|\sin\theta| \lesssim \kappa 2^{k_1-k-k_2-|k_1|/2}, \tag{7.55}$$

in the support of the integral. The formula (7.46) shows that

$$r^{-1}|\partial_\theta\Phi(\xi,\eta)| = \frac{\lambda'(|\xi-\eta|)}{|\xi-\eta|}s|\sin\theta| \lesssim \kappa 2^{-k_2}$$

in the support of the integral. Therefore $|\partial_r\Phi(\xi,\eta)| \geq 2^{q-4}$ in the support of the integral.

We assume now that θ is fixed satisfying (7.55). If $||k_2|-|k_1|| \geq 100$ then $|\partial_r\Phi(\xi,\eta)| \gtrsim 2^{|k_1|/2} + 2^{|k_2|/2}$ for all $(\xi,\eta) \in \mathcal{D}_{k,k_1,k_2}$, and the desired bound follows from (7.37), with $l=1$ and $n=1$. If $||k_2|-|k_1|| \leq 100$ then we use still use (7.37) to conclude that the integral is dominated by

$$C\epsilon 2^{-2q}2^{5|k_1|/2}\cdot\kappa 2^{k_1-k-|k_1|/2} \lesssim \epsilon\kappa 2^{-2q}2^{4|k_1|}.$$

This suffices to prove (7.42) if $2^q \geq 2^{-6\max(k,k_1,k_2)-\mathcal{D}_1}$. Finally, if

$$||k_2|-|k_1|| \leq 100, \quad 2^q \leq 2^{-6\max(k,k_1,k_2)-\mathcal{D}_1}, \quad \kappa \leq 2^{q-2\max(k,k_1,k_2)-\mathcal{D}_1},$$

then we would like to apply (7.38). For this it suffices to verify that for any θ fixed satisfying (7.55) the number of intervals (in the variable r) where $|\partial_r\Phi(\xi,\eta)| \leq 2^{q-4}$ is uniformly bounded. In view of Proposition 7.2 (iii) these intervals are present only when $k,k_1,k_2 \in [-10,10]$, $|s-\gamma_1| \ll 1$, $|r-\gamma_1/2| \ll 1$, and $\Phi(\xi,\eta) = \pm[\lambda(s)-\lambda(r)-\lambda((s^2+r^2-2sr\cos\theta)^{1/2})]$. In this case, however $|\partial_r^2\Phi(\xi,\eta)| \gtrsim 1$. As a consequence, for any s and θ there is at most one interval in r where $|\partial_r\Phi(\xi,\eta)| \leq 2^{q-4}$, and the desired bound follows from (7.38).

The decomposition (7.43) follows from (7.42) and Proposition 7.2 (iii), by setting $2^p = 2^q = 2^{-2\mathcal{D}_1}2^{-2\max(k,k_1,k_2)}$.

To prove the first inequality in (7.44), we may assume that $q \leq -5\max(k,k_1,k_2)-\mathcal{D}_1$ (due to (7.55)). In view of Proposition (7.2) (iii) we may assume that $k,k_1,k_2 \in [-10,10]$, $|s-\gamma_1| \ll 1$, $|r-\gamma_1/2| \ll 1$ and $\Phi(\xi,\eta) = \pm[\lambda(s)-\lambda(r)-\lambda((s^2+r^2-2sr\cos\theta)^{1/2})]$. As before, $|\partial_r^2\Phi(\xi,\eta)| \gtrsim 1$ in this case. As a consequence, for any s and θ fixed, the measure of the set of numbers r for which $|\partial_r\Phi(\xi,\eta)| \lesssim 2^q$ is bounded by $C2^q$, and the desired bound follows. □

We will also need a variant of Schur's lemma for suitably localized kernels.

Lemma 7.5. *Assume that $n,p \leq -\mathcal{D}/10$, $k,k_1,k_2 \in \mathbb{Z}$, $l \leq \min(k,k_1,k_2,0)/2 - \mathcal{D}/10$, $\rho_1,\rho_2 \in \{\gamma_0,\gamma_1\}$. Then, with $\mathcal{D}_{k,k_1,k_2}$ as in* (2.3)*, and assuming that $\left\|\sup_{\omega\in\mathbb{S}^1}|\widehat{f}(r\omega)|\right\|_{L^2(rdr)} \leq 1$,*

$$\Big\|\int_{\mathbb{R}^2}\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\rho_1)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta\Big\|_{L^2_\xi} \lesssim 2^{(l+n)/2}\|g\|_{L^2}, \tag{7.56}$$

$$\Big\|\int_{\mathbb{R}^2}\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\rho_1)\varphi_p(|\eta|-\rho_2)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta\Big\|_{L^2_\xi}\\ \lesssim \min\{2^{l/2},2^{p/2}\}2^{(l+n)/2}\|g\|_{L^2}, \tag{7.57}$$

and

$$\Big\|\int_{\mathbb{R}^2}\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\Big\|_{L^2_\xi}\lesssim 2^{5|k_1|}2^{3l/4}(1+|l|)\|g\|_{L^2}. \tag{7.58}$$

Proof. In view of (7.6), we may assume that $\min(k,k_1,k_2)+\overline{k}\geq -100$, where $\overline{k}=\max(k,k_1,k_2)$.

We start with (7.56). We may assume that $\min(k,k_1,k_2)\geq -200$. By Schur's test, it suffices to show that

$$\begin{aligned}&\sup_\xi\int_{\mathbb{R}^2}\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\rho_1)|\widehat{f}(\xi-\eta)|\,d\eta\lesssim 2^{(l+n)/2},\\ &\sup_\eta\int_{\mathbb{R}^2}\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\rho_1)|\widehat{f}(\xi-\eta)|\,d\xi\lesssim 2^{(l+n)/2}.\end{aligned} \tag{7.59}$$

We focus on the first inequality. Fix $\xi\in\mathbb{R}^2$ and introduce polar coordinates, $\eta=\xi-r\omega$, $r\in(0,\infty)$, $\omega\in\mathbb{S}^1$. The left-hand side is dominated by

$$C\int_{\omega\in\mathbb{S}^1}\int_{2^{k_1-4}}^{2^{k_1+4}}\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-r\omega)\varphi_l(\Phi(\xi,\xi-r\omega))\varphi_n(r-\rho_1)|\widehat{f}(r\omega)|rdrd\omega,$$

for a constant C sufficiently large. Therefore it suffices to show that

$$\sup_{r,\xi}\int_{\omega\in\mathbb{S}^1}\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-r\omega)\varphi_l(\Phi(\xi,\xi-r\omega))\,d\omega\lesssim 2^{l/2}2^{|k_1|/2}, \tag{7.60}$$

which is easily verified as in Proposition 7.4, using the identity (7.46). Indeed for ξ and r fixed, and letting $\omega=(\cos\theta,\sin\theta)$, the absolute value of the $d/d\theta$ derivative of the function $\theta\to\Phi(\xi,\xi-r(\cos\theta,\sin\theta))$ is bounded from below by $c|\sin\theta|2^{k+k_1-k_2}2^{|k_2|/2}\gtrsim|\sin\theta|2^{-|k_1|/2}$. The bound (7.60) follows using also (7.38). The second inequality in (7.59) follows similarly.

We prove now (7.57). We may assume that $k,k_1,k_2\in[-80,80]$ and it suffices to show that

$$\begin{aligned}&\sup_\xi\int_{\mathbb{R}^2}\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\rho_1)\varphi_p(|\eta|-\rho_2)|\widehat{f}(\xi-\eta)|\,d\eta\lesssim 2^{n/2}\min(2^l,2^p),\\ &\sup_\eta\int_{\mathbb{R}^2}\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\rho_1)\varphi_p(|\eta|-\rho_2)|\widehat{f}(\xi-\eta)|\,d\xi\lesssim 2^{l+n/2}.\end{aligned}$$

We proceed as for (7.59) but replace (7.60) by

$$\begin{aligned}&\sup_{|\xi|\approx 1}\sup_r\int_{\omega\in\mathbb{S}^1}\varphi_l(\Phi(\xi,\xi-r\omega))\varphi_n(r-\rho_1)\varphi_p(|\xi-r\omega|-\rho_2)\,d\omega\lesssim\min\{2^l,2^p\},\\ &\sup_\eta\sup_r\int_{\omega\in\mathbb{S}^1}\varphi_l(\Phi(\eta+r\omega,\eta))\varphi_n(r-\rho_1)\varphi_p(|\eta|-\rho_2)\varphi_{\geq -90}(\eta+r\omega)\,d\omega\lesssim 2^l.\end{aligned} \tag{7.61}$$

The bounds (7.61) follow easily, using also the formula (7.46) to prove the 2^l bounds, once we notice that $|\sin\theta|\gtrsim 1$ in the support of the integrals. For this we only need to verify that the points ξ and η cannot be almost alligned; more precisely, we need to verify that if ξ and η are alligned then $|\Phi(\xi,\xi-\eta)|+\big||\xi-\eta|-\rho_2\big|+||\eta|-\rho_1|\gtrsim 1$. For this it suffices to notice that

$$\big|\pm\lambda(|\xi|)\pm\lambda(\rho_1)\pm\lambda(\rho_2)\big|\gtrsim 1\qquad\text{if}\qquad|\xi|\gtrsim 1\text{ and }\pm|\xi|\pm\rho_1\pm\rho_2=0.$$

Recalling that $\rho_1,\rho_2\in\{\gamma_0,\gamma_1\}$, it suffices to verify that $\lambda(2\gamma_0)-2\lambda(\gamma_0)\neq 0$, $\lambda(2\gamma_1)-2\lambda(\gamma_1)\neq 0$, $\lambda(\gamma_0+\gamma_1)-\lambda(\gamma_0)-\lambda(\gamma_1)\neq 0$, $\lambda(-\gamma_0+\gamma_1)+\lambda(\gamma_0)-\lambda(\gamma_1)\neq 0$. These claims follow from Lemma 7.1 (iv), since the numbers γ_0^2,γ_1^2, $\gamma_0\gamma_1$, and $\gamma_0(\gamma_1-\gamma_0)$ are not in the interval $[4/9,1/2]$.

We now turn to (7.58). By Schur's lemma it suffices to show that

$$\begin{aligned}&\sup_\xi\int_{\mathbb{R}^2}\varphi_l(\Phi(\xi,\eta))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)|\widehat{f}(\xi-\eta)|\,d\eta\lesssim 2^{5|k_1|}2^{3l/4}(1+|l|),\\&\sup_\eta\int_{\mathbb{R}^2}\varphi_l(\Phi(\xi,\eta))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)|\widehat{f}(\xi-\eta)|\,d\xi\lesssim 2^{5|k_1|}2^{3l/4}(1+|l|).\end{aligned}\tag{7.62}$$

We show the first inequality. Introducing polar coordinates, as before, we estimate

$$\begin{aligned}&\int_{\mathbb{R}^2}\varphi_l(\Phi(\xi,\xi-r\omega))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-r\omega)|\widehat{f}(r\omega)|\,rdrd\omega\\&\lesssim\big\|\sup_\omega|\widehat{f}(r\omega)|\,\big\|_{L^2(rdr)}\Big\|\int_{\mathbb{S}^1}\varphi_l(\Phi(\xi,\xi-r\omega))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-r\omega)\,d\omega\Big\|_{L^2(rdr)}\\&\lesssim\|\varphi_{\leq l+2}(\Phi(\xi,\xi-\eta))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-\eta)\|_{L^2_\eta}\big\|\varphi_{\leq l+2}(\Phi(\xi,\xi-r\omega))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-r\omega)\big\|_{L^\infty_rL^2_\omega}\\&\lesssim 2^{5|k_1|}2^{3l/4}(1+|l|),\end{aligned}$$

using Proposition 7.4 (i) and (7.60). The second inequality in (7.62) follows similarly. □

7.4. **Iterated resonances.** In this subsection we prove a lemma concerning some properties of the cubic phases

$$\widetilde{\Phi}(\xi,\eta,\sigma)=\widetilde{\Phi}_{+\mu\beta\gamma}(\xi,\eta,\sigma)=\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\beta(\eta-\sigma)-\Lambda_\gamma(\sigma).\tag{7.63}$$

These properties are used only in the proof of Lemma 5.7 and Lemma 5.8.

Lemma 7.6. *(i) Assume that $\xi,\eta,\sigma\in\mathbb{R}^2$ satisfy*

$$\max(||\xi-\eta|-\gamma_0|,||\eta-\sigma|-\gamma_0|,||\sigma|-\gamma_0|)\leq 2^{-\mathcal{D}_1/2},\tag{7.64}$$

and

$$|\nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma)|\leq\kappa_1\leq 2^{-4\mathcal{D}_1}.\tag{7.65}$$

Then, for $\nu\in\{+,-\}$,

$$\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta)\gtrsim|\eta|.\tag{7.66}$$

Moreover,

$$\textit{if}\quad|\nabla_\xi\widetilde{\Phi}(\xi,\eta,\sigma)|\geq\kappa_2\geq 2^{\mathcal{D}_1}\kappa_1\quad\textit{then}\quad|\widetilde{\Phi}(\xi,\eta,\sigma)|\gtrsim\kappa_2^{3/2}.\tag{7.67}$$

(ii) Assume that $\xi,\eta,\sigma\in\mathbb{R}^2$ satisfy $|\xi-\eta|$, $|\eta-\sigma|$, $|\sigma|\in[2^{-10},2^{10}]$ and

$$\begin{aligned}&|\Phi_{+\mu\nu}(\xi,\eta)|=|\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta)|\leq 2^{-2\mathcal{D}_1},\\&|\Phi_{\nu\beta\gamma}(\eta,\sigma)|=|\Lambda_\nu(\eta)-\Lambda_\beta(\eta-\sigma)-\Lambda_\gamma(\sigma)|\leq 2^{-2\mathcal{D}_1}.\end{aligned}\tag{7.68}$$

If

$$|\nabla_{\eta,\sigma}\widetilde{\Phi}(\xi,\eta,\sigma)|\leq\kappa\leq 2^{-4\mathcal{D}_1}\tag{7.69}$$

then

$$\mu=-,\ \nu=\beta=\gamma=+,\qquad|\eta-2\sigma|+|\xi-\sigma|\lesssim\kappa,\qquad|\nabla_\xi\widetilde{\Phi}(\xi,\eta,\sigma)|\lesssim\kappa.\tag{7.70}$$

Proof. (i) If (7.64) and (7.65) hold then the vectors $\xi-\eta,\eta-\sigma,\sigma$ are almost aligned. Thus either $|\eta|\leq 2^{-\mathcal{D}_1/2+10}$ or $||\eta|-2\gamma_0|\leq 2^{-\mathcal{D}_1/2+10}$. We will assume that we are in the second case, $||\eta|-2\gamma_0|\leq 2^{-\mathcal{D}_1/2+10}$ (the other case is similar, in fact slightly easier because the inequality (7.66) is a direct consequence of (7.6)). Therefore either $||\xi|-3\gamma_0|\leq 2^{-\mathcal{D}_1/2+20}$, and in this case the desired conclusions are trivial, or $||\xi|-\gamma_0|\leq 2^{-\mathcal{D}_1/2+20}$. In this case (7.66) follows since $|\lambda(\gamma_0)\pm\lambda(\gamma_0)\pm\lambda(\gamma_0)|\gtrsim 1$; it remains to prove (7.67) in the case $\mu=-,\beta=\gamma=+$,

$$\begin{aligned}&\widetilde{\Phi}(\xi,\eta,\sigma)=\Lambda(\xi)+\Lambda(\xi-\eta)-\Lambda(\eta-\sigma)-\Lambda(\sigma),\\&||\eta|-2\gamma_0|\leq 2^{-\mathcal{D}_1/2+20},\qquad ||\xi|-\gamma_0|\leq 2^{-\mathcal{D}_1/2+20}.\end{aligned}\tag{7.71}$$

In view of (7.65), the angle between any two of the vectors $\{\xi-\eta,\eta-\sigma,\sigma\}$ is either $O(\kappa_1)$ or $\pi+O(\kappa_1)$. Given $\sigma=ze$ for some $e\in\mathbb{S}^1$, we write $\eta=ye+\eta'$, $\xi=xe+\xi'$, with $e\cdot\eta'=e\cdot\xi'=0$ and $|\eta'|+|\xi'|\lesssim\kappa_1$. Notice that $|\widetilde{\Phi}(\xi,\eta,\sigma)-\widetilde{\Phi}(xe,ye,ze)|\lesssim\kappa_1^2$. Therefore, we may assume that

$$\begin{aligned}&|x-\gamma_0|+|y-2\gamma_0|+|z-\gamma_0|\leq 2^{-\mathcal{D}_1/2+30},\\&|\lambda'(y-z)-\lambda'(z)|\leq 2\kappa_1,\quad |\lambda'(y-x)-\lambda'(y-z)|\leq 2\kappa_1,\quad |\lambda'(x)-\lambda'(y-x)|\geq\kappa_2/2,\end{aligned}\tag{7.72}$$

and it remains to prove that

$$|\widetilde{\Phi}(xe,ye,ze)|=|\lambda(x)+\lambda(y-x)-\lambda(y-z)-\lambda(z)|\gtrsim\kappa_2^{3/2}.\tag{7.73}$$

Let $z'\neq z$ denote the unique solution to the equation $\lambda'(z')=\lambda'(z)$, and let $d:=|z-\gamma_0|$. Then $|z'-\gamma_0|\approx d$, in view of (7.10). Moreover $d\geq\sqrt{\kappa_1}$; otherwise $|y-z-\gamma_0|\lesssim\sqrt{\kappa_1}$, $|y-x-\gamma_0|\lesssim\sqrt{\kappa_1}$, so $|x-\gamma_0|\lesssim\sqrt{\kappa_1}$, in contradiction with the assumption $|\lambda'(x)-\lambda'(y-x)|\geq\kappa_2/2$. Moreover,

$$\text{there are }\sigma_1,\sigma_2\in\{z,z'\}\text{ such that }|y-z-\sigma_1|+|y-x-\sigma_2|\lesssim\kappa_1/d.\tag{7.74}$$

In fact, we may assume $d\geq 2^{-\mathcal{D}_1/4}\kappa_2^{1/2}$, since otherwise $|x-\gamma_0|+|y-x-\gamma_0|\lesssim d$, and hence $|\lambda'(x)-\lambda'(y-x)|\lesssim d^2$, which contradicts (7.65).

Now we must have $\sigma_1=z$; in fact, if $\sigma_1=z'$, then $x=z+z'-\sigma_2+O(\kappa_1/d)$, thus

$$|\lambda'(x)-\lambda'(\sigma_2)|\lesssim\kappa_1,$$

which again contradicts (7.72). Similarly $\sigma_2=z'$. Therefore

$$y=2z+O(\kappa_1/d),\quad x=2z-z'+O(\kappa_1/d),\quad y-x=z'+O(\kappa_1/d).\tag{7.75}$$

We expand the function λ at γ_0 in its Taylor series

$$\lambda(v)=\lambda(\gamma_0)+c_1(v-\gamma_0)+c_3(v-\gamma_0)^3+O(v-\gamma_0)^4,$$

where $c_1,c_3\neq 0$. Using (7.75) we have

$$\begin{aligned}\widetilde{\Phi}(xe,ye,ze)&=c_3[(x-\gamma_0)^3+(y-x-\gamma_0)^3-(z-\gamma_0)^3-(y-z-\gamma_0)^3]+O(d^4)\\&=c_3[(2z-z'-\gamma_0)^3+(z'-\gamma_0)^3-2(z-\gamma_0)^3]+O(d^4+\kappa_1 d).\end{aligned}$$

In view of (7.10), $z+z'-2\gamma_0=O(d^2)$. Therefore $\widetilde{\Phi}(xe,ye,ze)=24(z-\gamma_0)^3+O(d^4+\kappa_1 d)$ which shows that $|\widetilde{\Phi}(xe,ye,ze)|\gtrsim d^3$. The desired conclusion (7.73) follows.

(ii) The conditions $|\Phi_{\nu\beta\gamma}(\eta,\sigma)|\leq 2^{-2\mathcal{D}_1}$ and $|(\nabla_\sigma\Phi_{\nu\beta\gamma})(\eta,\sigma)|\leq\kappa$ show that η corresponds to a space-time resonance output. It follows from Lemma 7.2 (iii) that

$$|\eta-ye|+|\sigma-ye/2|\lesssim\kappa,\quad |y-\gamma_1|\lesssim 2^{-2\mathcal{D}_1},\quad \nu=\beta=\gamma,\tag{7.76}$$

for some $e\in\mathbb{S}^1$. Let $b\approx 0.207$ denote the unique nonnegative number $b\neq\gamma_1/2$ with the property that $\lambda'(b)=\lambda'(\gamma_1/2)$. The condition $|\nabla_\eta\widetilde{\Phi}(\xi,\eta,\sigma)|\leq\kappa$ shows that $\xi-\eta$ is close to one of the vectors $(\gamma_1/2)e, -(\gamma_1/2)e, be, -be$. However, $\lambda(b) \approx 0.465$, $\lambda(\gamma_1 + b) \approx 2.462$, $\lambda(\gamma_1 - b) \approx 1.722$, $\lambda(\gamma_1) \approx 2.060$. Therefore, the condition $|\Phi_{+\mu\nu}(\xi, \eta)| \le 2^{-2\mathcal{D}_1}$ prevents $\xi - \eta$ from being close to one of the vectors be or $-be$. Similarly $\xi - \eta$ cannot be close to the vector $(\gamma_1/2)e$, since $\lambda(\gamma_1/2) \approx 1.030$, $\lambda(3\gamma_1/2) \approx 3.416$. It follows that $|(\xi - \eta) + (\gamma_1/2)e| \lesssim 2^{-2\mathcal{D}_1}$, $||\xi| - \gamma_1/2| \lesssim 2^{-2\mathcal{D}_1}$, $\mu = -$, $\nu = +$. The condition $|\nabla_\eta \widetilde{\Phi}(\xi, \eta, \sigma)| \le \kappa$ then gives $|(\eta - \xi) - (\eta - \sigma)| \lesssim \kappa$, and remaining bounds in (7.70) follow using also (7.76). $\square$

References

[1] T. Alazard, N. Burq and C. Zuily. On the water waves equations with surface tension. *Duke Math. J.*, 158 (2011), no. 3, 413-499.

[2] T. Alazard, N. Burq and C. Zuily. On the Cauchy problem for gravity water waves. *Invent. Math.*, to appear. *arXiv:1212.0626.*

[3] T. Alazard and J.M. Delort. Global solutions and asymptotic behavior for two dimensional gravity water waves. Preprint. *arXiv:1305.4090.*

[4] T. Alazard and J.M. Delort. Sobolev estimates for two dimensional gravity water waves Preprint. *arXiv:1307.3836.*

[5] T. Alazard and G. Métivier. Paralinearization of the Dirichlet to Neumann operator, and regularity of three-dimensional water waves. *Comm. Partial Differential Equations*, 34 (2009), no. 10-12, 1632-1704.

[6] S. Alinhac. Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Comm. Partial Differential Equations*, 14 (1989), no. 2, 173-230.

[7] B. Alvarez-Samaniego and D. Lannes. Large time existence for 3D water-waves and asymptotics. *Invent. Math.* 171 (2008), no. 3, 485-541.

[8] D.M. Ambrose and N. Masmoudi. The zero surface tension limit of two-dimensional water waves. *Comm. Pure Appl. Math.*, 58 (2005), no. 10, 1287-1315.

[9] K. Beyer and M. Günther. On the Cauchy problem for a capillary drop. I. Irrotational motion. *Math. Methods Appl. Sci.* 21 (1998), no. 12, 1149-1183.

[10] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo and J. Gómez-Serrano. Finite time singularities for the free boundary incompressible Euler equations. *Ann. of Math.*, 178 (2013), 1061-1134.

[11] A-L Cauchy. Mémoire sur la th'eorie de la propagation des ondes à la surface d'un fluide pesant d'une profondeur indéfinie. *Mém. Présentés Divers Savans Acad. R. Sci. Inst. France (Prix Acad. R. Sci., concours de 1815 et de 1816)*, I:3-312 (1827).

[12] A. Cheng, D. Coutand and S. Shkoller. On the motion of Vortex Sheets with surface tension. *Comm. on Pure and Appl. Math.*, 61 (2008), no. 12, 1715-1752.

[13] H. Christianson, V. Hur, and G. Staffilani. Strichartz estimates for the water-wave problem with surface tension. *Comm. Partial Differential Equations*, 35 (2010), no. 12, 2195-2252.

[14] D. Christodoulou and H. Lindblad. On the motion of the free surface of a liquid. *Comm. Pure Appl. Math.*, 53 (2000), no. 12, 1536-1602.

[15] D. Christodoulou and S. Klainerman. The global nonlinear stability of the Minkowski space. *Princeton Mathematical Series*, 41. Princeton University Press, Princeton, NJ, 1993. x+514 pp.

[16] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao. Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$. *J. Amer. Math. Soc.*, 16 (2003), no. 3, 705-749.

[17] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao. Resonant decompositions and the I-method for the cubic nonlinear Schrödinger equation on $\mathbb{R}^2$. *Discrete Contin. Dyn. Syst.*, 21 (2008), no. 3, 665-686.

[18] A. Constantin. Nonlinear Water Waves with applications to Wave-Current interactions and Tsunamis. CBMS-NSF Regional Conference Series in Applied Mathematics, 81. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. xii+321 pp.

[19] A. Constantin and D. Lannes. The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations. *Arch. Ration. Mech. Anal.* 192 (2009), no. 1, 165-186.

[20] D. Coutand and S. Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *Journal of. Amer. Math. Soc.*, 20 (2007), no. 3, 829-930.

[21] D. Coutand and S. Shkoller. On the finite-time splash and splat singularities for the 3-D free-surface Euler equations. *Commun. Math. Phys.*, 325 (2014), 143-183.

[22] W. Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations*, 10 (1985), no. 8, 787-1003

[23] W. Craig and C. Sulem. Normal Form Transformations for Capillary-Gravity Water Waves. In *Hamiltonian Partial Differential Equations and Applications*, Vol 75 of the series Fields Institute Communications, 73-110.

[24] W. Craig, U. Schanz and C. Sulem. The modulational regime of three-dimensional water waves and the Davey-Stewartson system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14 (1997), no. 5, 615-667.

[25] W. Craig, C. Sulem and P.-L. Sulem. Nonlinear modulation of gravity waves: a rigorous approach. *Nonlinearity* 5 (1992), no. 2, 497-522.

[26] W. Craig and C. E. Wayne. Mathematical aspects of surface waves on water. *Uspekhi Mat. Nauk*, 62 (2007) 95-116.

[27] A.D.D. Craik. The origins of water wave theory. *Annual review of fluid mechanics*. Vol. 36, 1-28. Annu. Rev. Fluid Mech., 36, Annual Reviews, Palo Alto, CA, 2004.

[28] T. De Poyferré and Q.H. Nguyen. A paradifferential reduction for the gravity-capillary waves system at low regularity and applications. Preprint *arXiv:1508.00326*.

[29] J.M. Delort. Global existence and asymptotic behavior for the quasilinear Klein-Gordon equation with small data in dimension 1. *Ann. Sci. École Norm. Sup.* 34 (2001), no. 1, 1-61.

[30] Y. Deng. Multispeed Klein-Gordon systems in dimension three. Preprint *arXiv:1602.01820*.

[31] Y. Deng, A. D. Ionescu and B. Pausader. The Euler-Maxwell system for electrons: global solutions in 2D. Preprint (2015).

[32] Y. Deng, A. D. Ionescu, B. Pausader. and F. Pusateri. Global solutions for the $3D$ gravity-capillary water waves system, I: energy estimates. Preprint (2016).

[33] K. Friedrichs and D. Hyers. The existence of solitary waves. *Comm Pure Appl. Math.* 7 (1954), 517-550.

[34] R. Gerber. Sur les solutions exactes des équations du mouvement avec surface libre d'un liquide pesant. *J Math Pure Appl.* 34 (1955), 185-299.

[35] P. Germain and N. Masmoudi. Global existence for the Euler-Maxwell system. *Ann. Sci. Éc. Norm. Supér.* (4) 47 (2014), no. 3, 469-503.

[36] P. Germain, N. Masmoudi and J. Shatah. Global solutions for the gravity surface water waves equation in dimension 3. *Ann. of Math.*, 175 (2012), 691-754.

[37] P. Germain, N. Masmoudi and J. Shatah. Global solutions for capillary waves equation in dimension 3. *Comm. Pure Appl. Math.*, 68 (2015), no. 4, 625-687.

[38] Y. Guo, A. D. Ionescu and B. Pausader. Global solutions of the Euler-Maxwell two-fluid system in 3D. To appear in *Ann. of Math.* arXiv:1303.1060.

[39] S. Gustafson, K. Nakanishi and T. Tsai. Scattering for the Gross-Pitaevsky equation in 3 dimensions. *Commun. Contemp. Math.* 11 (2009), no. 4, 657-707.

[40] J. Hunter, M. Ifrim and D. Tataru. Two dimensional water waves in holomorphic coordinates. Preprint *arXiv:1401.1252*.

[41] M. Ifrim and D. Tataru. Two dimensional water waves in holomorphic coordinates II: global solutions. Preprint *arXiv:1404.7583*.

[42] M. Ifrim and D. Tataru. The lifespan of small data solutions in two dimensional capillary water waves. Preprint *arXiv:1406.5471*.

[43] A. Ionescu and B. Pausader. The Euler-Poisson system in 2D: global stability of the constant equilibrium solution. *Int. Math. Res. Not.*, (2013), no. 4, 761-826.

[44] A. Ionescu and B. Pausader. Global solutions of quasilinear systems of Klein-Gordon equations in 3D. *J. Eur. Math. Soc.* 16 (2014), no. 11, 2355-2431.

[45] A. D. Ionescu and F. Pusateri. Nonlinear fractional Schrödinger equations in one dimension. *J. Funct. Anal.*, 266 (2014), 139-176.

[46] A. D. Ionescu and F. Pusateri. Global solutions for the gravity water waves system in 2D. *Invent. Math.* 199 (2015), no. 3, 653-804.

[47] A. D. Ionescu and F. Pusateri. Global analysis of a model for capillary water waves in 2D. Preprint *arXiv:1406.6042*.

[48] A. D. Ionescu and F. Pusateri. Global regularity for 2d water waves with surface tension. *arXiv:1408.4428*.

[49] F. John. Blow-up for quasilinear wave equations in three space dimensions. *Comm. on Pure Appl. Math.*, 34 (1981), no. 1, 29-51.

[50] T. Kano and T. Nishida. Sur les ondes de surface de l'eau avec une justification mathématique des équations des ondes en eau peu profonde. *J. Math. Kyoto Univ.* 19 (1979), no. 2, 335-370.

[51] S. Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38 (1985), no. 3, 321-332.

[52] S. Klainerman. The null condition and global existence for systems of wave equations. *Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*, 293-326. Lectures in Appl. Math., 23, Amer. Math. Soc., Providence, RI, 1986.

[53] D. Lannes. Well-posedness of the water waves equations. *J. Amer. Math. Soc.* 18 (2005), no. 3, 605-654.

[54] D. Lannes. The water waves problem. Mathematical analysis and asymptotics. Mathematical Surveys and Monographs, Vol. 188. American Mathematical Society, Providence, RI, 2013. xx+321 pp.

[55] T. Levi-Civita. Détermination rigoureuse des ondes permanentes d'ampleur finie. *Math. Ann.*, 93 (1925), no. 1, 264-314.

[56] H. Lindblad. Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. of Math.*, 162 (2005), no. 1, 109-194.

[57] V. I. Nalimov. The Cauchy-Poisson problem. *Dinamika Splosn. Sredy Vyp.* 18 Dinamika Zidkost. so Svobod. Granicami (1974), 10-210, 254.

[58] G. Schneider and C. E. Wayne. The rigorous approximation of long-wavelength capillary-gravity waves. *Arch. Ration. Mech. Anal.* 162 (2002), no. 3, 247-285.

[59] J. Shatah. Normal forms and quadratic nonlinear Klein-Gordon equations. *Comm. Pure and Appl. Math.*, 38 (1985), no. 5, 685-696.

[60] J. Shatah and C. Zeng. Geometry and a priori estimates for free boundary problems of the Euler equation. *Comm. Pure Appl. Math.*, 61 (2008), no. 5, 698-744.

[61] J. Shatah and C. Zeng. Local well-posedness for the fluid interface problem. *Arch. Ration. Mech. Anal.* 199 (2011), no. 2, 653-705.

[62] D. J. Struik. Détermination rigoureuse des ondes irrotationelles périodiques dans un canal à profondeur finie. *Math. Ann.* 95 (1926), no. 1, 595-634.

[63] T. C. Sideris. Formation of singularities in three-dimensional compressible fluids. *Comm. Math. Phys.* 101 (1985), no. 4, 475-485.

[64] G.G. Stokes. On the theory of oscillatory waves. *Trans. Cambridge Philos. Soc.*, 8 (1847), 441-455.

[65] C. Sulem and P.L. Sulem. The nonlinear Schrödinger equation. Self-focussing and wave collapse. *Applied Mathematical Sciences*, 139. Springer-Verlag, New York, 1999.

[66] G.I. Taylor. The instability of liquid surfaces when accelerated in a direction perpendicular to their planes I. *Proc. Roy. Soc. London* A 201 (1950) 192-196.

[67] N. Totz and S. Wu. A rigorous justification of the modulation approximation to the 2D full water wave problem. *Comm. Math. Phys.* 310 (2012), no. 3, 817-883.

[68] X. Wang. Global infinite energy solutions for the 2D gravity water waves system. Preprint *arXiv: 1502.00687*.

[69] X. Wang. On 3D water waves system above a flat bottom. Preprint *arXiv:1508.06223*.

[70] X. Wang. Global solution for the 3D gravity water waves system above a flat bottom. Preprint *arXiv:1508.06227*.

[71] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130 (1997), 39-72.

[72] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.*, 12 (1999), 445-495.

[73] S. Wu. Almost global wellposedness of the 2-D full water wave problem. *Invent. Math.*, 177 (2009), 45-135.

[74] S. Wu. Global wellposedness of the 3-D full water wave problem. *Invent. Math.*, 184 (2011), 125-220.

[75] H. Yosihara. Gravity waves on the free surface of an incompressible perfect fluid of finite depth. *Publ. Res. Inst. Math. Sci.* 18 (1982), 49-96.

[76] V. E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Zhurnal Prikladnoi Mekhaniki i Teckhnicheskoi Fiziki* 9 (1968), no.2, 86-94. *J. Appl. Mech. Tech. Phys.*, 9, 1990-1994.

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