LOCALIZATION (Math. Phys. seminar: 31/3/22)

Aim: Explain why path integrals can sometimes be reduced to local contributions from critical points of the action (classical traj's).

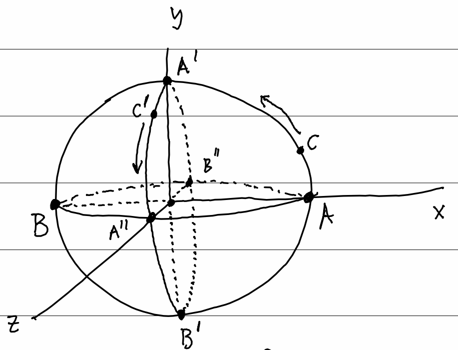
§1 Equivariant cohomology

Preliminaries $\mathbb{T}^m = \overbrace{S^1 \times \ldots \times S^1}^{m \text{ times}}$ the torus. (Usually denoted $\mathbb{T}$ below.)

Define $\mathbb{C}^\infty = \varinjlim_n \mathbb{C}^n$ and $S^\infty = \varinjlim_n S^{2n-1} \subseteq \mathbb{C}^\infty$.

We have a free action $S^1 \circlearrowright S^\infty$ w/ quotient $\mathbb{CP}^\infty = \varinjlim_n \mathbb{CP}^{n-1}$.
(scaling all the coordinates)

Claim: S^∞ is contractible. (for us it will be enough that $\pi_k(S^\infty) \equiv 0 \ \forall k$)



Contracting homotopy: in two steps:

① given $u = (x_1, x_2, x_3, \ldots) \in S^\infty$, (with $x_i \in \mathbb{R}$)
move to $u' = (0, x_1, x_2, x_3, \ldots)$ by

$$t \longmapsto \frac{(1-t)u + tu'}{\|(1-t)u + tu'\|} \quad (\text{since } u \neq \pm u')$$

② move u' to $e_0 = (1, 0, 0, 0, \ldots)$ by

$$t \longmapsto \frac{(1-t)e_0 + tu'}{\|(1-t)e_0 + tu'\|} \quad \left(\text{since } e_0 \text{ and } u' \text{ are independent}\right)$$

[Each individual S^k is being contracted to a point in S^{k+1}].

Claim: $ES^1 := S^\infty \to BS^1 := \mathbb{CP}^\infty$ is the universal S^1-bundle, i.e. $\forall$ simplicial complexes X, we have a bijection

$$\left\{ \begin{array}{c} \text{htpy classes of maps} \\ X \to BS^1 \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{iso classes of } S^1\text{-bundles} \\ \text{on } X \end{array} \right\}$$

$$\varphi \longmapsto \varphi^*(ES^1 \to BS^1)$$

Digression on classifying spaces & universal bundles

<u>Lemma:</u> Suppose $E \xrightarrow{p} B$ is a principal G-bundle w/ $\pi_k(E) = 0$ $\forall k$. Suppose (X, A) is a simplicial cx pair and $P \xrightarrow{q} X$ is a principal G-bundle. Then, any pullback diagram

$$\begin{tikzcd} P \arrow[d] & P_A \arrow[l, "\mathrm{incl}"'] \arrow[r, "\psi_A"] \arrow[d] \arrow[dl, phantom, "\llcorner"] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ X & A \arrow[l, "\mathrm{incl}"'] \arrow[r, "\phi_A"'] & B \end{tikzcd}$$

can be extended to a pullback diagram

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ X \arrow[r, "\phi"'] & B \end{tikzcd}$$

(i.e. $\psi|_{P_A} = \psi_A$, $\phi|_A = \phi_A$).

<u>Proof:</u> suffices to treat the case $X = \Delta^k$, $A = \partial\Delta^k$.

$P \to X$ can be trivialized (since Δ^k is contractible) and so, write $P = \Delta^k \times G$.

ψ_A is then just a G-equiv't map $\partial\Delta^k \times G \xrightarrow{\psi_A} E$, determined by its restriction to $\partial\Delta^k = \partial\Delta^k \times 1 \xrightarrow{\hat{\psi}_A} E$.

$\hat{\psi}_A \in \pi_{k-1}(E) = 0 \Rightarrow \exists$ ext^n $\Delta^k \xrightarrow{\hat{\psi}} E$, extend this G-equiv'tly to get $P = \Delta^k \times G \xrightarrow{\psi} E$.

Project down to B to get $X = \Delta^k \xrightarrow{\phi} B$. ▨

<u>Cor:</u> Given any simplicial cx X, we have:

$$[X, B] \cong \{\text{principal } G\text{-bundles on } X\}/\text{iso} \qquad (\star)$$

$$[\varphi] \mapsto [\varphi^*(E \to B)]$$

homotopy classes of maps $X \to B$

→ We denote $E = EG$, $B = BG$. (these are <u>uniquely determined up to htpy</u> by the property $(\star)$).

G acts freely on the <u>right</u> on EG (convention).

Conclusion: $E\mathbb{T} = (S^\infty)^m \circlearrowleft \mathbb{T}$

$B\mathbb{T} = (\mathbb{CP}^\infty)^m$

think of $u_1, \dots, u_m$ as coordinates on the Lie algebra $\underline{t} \simeq \mathbb{R}^m$ of $\mathbb{T} = \mathbb{T}^m$.

always $\mathbb{C}$-coeffs

$H^*(B\mathbb{T}) = \mathbb{C}[u_1, \dots, u_m]$

$= \mathrm{Sym}\, \underline{t}^*_{\mathbb{C}}$

taking Spec recovers the complexified Lie algebra $\underline{t}_{\mathbb{C}}$.

- For a map $\mathbb{T} \xrightarrow{\varphi} \mathbb{T}'$, the induced map $H^*(B\mathbb{T}') \xrightarrow{\varphi^*} H^*(B\mathbb{T})$ is defined by the obvious map $\underline{t} \xrightarrow{d\varphi} \underline{t}'$ (or rather its complexified dual).
- The generator u_i lies in degree $+2$ and $u_i = c_1(L_i)$,

  $L_i =$ rank 1 $\mathbb{C}$-v.bundle over $B\mathbb{T}^m$ associated to the S^1-bundle given by the ith $S^\infty \to \mathbb{CP}^\infty$ $[L_i = \mathcal{O}_{\mathbb{CP}^\infty_i}(-1)]$

$\longrightarrow$ Below, we will always understand $G = \mathbb{T}$.

[Sometimes, we will restrict to $m = 1$.]

Borel construction: Given a G-space M, define the htpy quotient $M_G := (EG \times M)/G$

$$\begin{tikzcd} EG \arrow[d] & EG \times M \arrow[l] \arrow[r] \arrow[d] & M \arrow[d] \\ BG & M_G \arrow[l, "\pi"] \arrow[r, "\sigma"'] & M/G \end{tikzcd}$$

⚠ σ may not be a fibration in general.

Define the equiv't cohomology: $H^*_G(M) = H^*(M_G)$.

Via π^*: it is a graded module over $H^*_G = H^*(BG) = \mathbb{C}[u_1, \dots, u_m]$

When M is a compact manifold (or has a finite good cover), this module will be finitely gen $/ H^*_G$.

Note: $M \mapsto M_G$ is a functor $(G\text{-spaces}) \to (\text{spaces over } BG)$

Allows us to <u>extend</u> notions from ordinary cohomology to equivariant cohomology. <u>Roughly</u>:

- "ordinary data" on M $\longleftrightarrow$ given over one fibre of $\begin{array}{c} M_G \\ \downarrow \\ BG \end{array}$.
- "equiv't data" on M $\longleftrightarrow$ given over the whole M_G.

e.g. equiv't bundles $F \to M$ extend to bundles $F_G \to M_G$ (functorially) $\Rightarrow$ char. classes of equiv't bundle lie in equiv't cohomology.

There is a natural forgetful map $H^*_G(M) \longrightarrow H^*(M)$ induced by the inclusion of the fibre of M_G over a pt $\in BG$.

$\boxed{\text{Important example}}$: Suppose $V = \mathbb{C}$, a 1-dim'l $\mathbb{C}$-v. space w/ a linear representation of $\mathbb{T}$ on it.

It is therefore a character : $\exists\ (k_1, \dots, k_m) \in \mathbb{Z}^m$ s.t. $(\alpha_1, \dots, \alpha_m) \in \mathbb{T}$ acts by $v \in V \longmapsto \alpha_1^{k_1} \cdots \alpha_m^{k_m} . v \in V$.

View $V \to *$ as a $\mathbb{T}$-equiv't v. bundle over a pt $*$.

$\rightsquigarrow$ extend this to a vector bundle $V_\rho = (E\mathbb{T} \times V)/\mathbb{T}$ over $B\mathbb{T} = E\mathbb{T}/\mathbb{T}$.

$(k_1, \dots, k_m) = (0, \dots 0, \underset{\uparrow}{1}, 0 \dots, 0)$ recovers L_i.

i^{th} spot

$\Rightarrow$ In general, $c_1(V_\rho) = \sum_{i=1}^{m} k_i u_i \in H^2_{\mathbb{T}}$

More generally, if $\rho = \rho_1 \oplus \cdots \oplus \rho_\ell$, a sum of 1-dim'l $\mathbb{C}$-rep's then the Euler class $e(V_\rho) = c_1(V_{\rho_1}) \cdots c_1(V_{\rho_\ell}) \in H^{2\ell}_{\mathbb{T}}$.

Finite dimensional approximations

Define: $ES^1(n) = S^{2n-1}$ and $E\mathbb{T}^m(n) = (ES^1(n))^m$

$BS^1(n) = \mathbb{CP}^{n-1}$ $\qquad$ $B\mathbb{T}^m(n) = (BS^1(n))^m$

These are finite dimensional approximations for EG, BG

i.e.: Given $q \geq 1$, $\exists\, n = n(q) \geq 1$ s.t.

$$H^i(EG) \xrightarrow{\cong} H^i(EG(k)) \qquad \forall\, k \geq n,\ i \leq q$$
$$H^i(BG) \xrightarrow{\cong} H^i(BG(k)) \qquad \forall\, k \geq n,\ i \leq q.$$

- Similar statement also for $M_G(n) = (EG(n) \times M)/G \to BG(n)$. and for homology in place of cohomology.

Consequence: $H^*_G(M) = H^*(M_G) = \varprojlim_n H^*(M_G(n))$

is a natural isomorphism of $H^*_G = \varprojlim_n H^*(BG(n))$-algebras

Pushforwards in equiv't cohomology

Suppose $M \xrightarrow{f} M'$ is a G-equivariant map of compact oriented mflds. Write $\dim M = \dim M' + r$. $(r \in \mathbb{Z})$

Non-equiv'tly: Pushforward in homology: $H_*(M) \xrightarrow{f_*} H_*(M')$

Poincaré duality now gives: $H^*(M) \xrightarrow{f_*} H^{*-r}(M')$. (also written $\int_f$)

Equiv'tly: For fixed $i \geq 1$ and $i \ll k \leq l$, the following diagram commutes:

$$\begin{array}{ccc} H^i(M_G(k)) & \xrightarrow{f_*} & H^{i-r}(M'_G(k)) \\ \uparrow \cong & (\star\star) & \uparrow \cong \\ H^i(M_G(l)) & \xrightarrow{f_*} & H^{i-r}(M'_G(l)) \end{array}$$

⚠ requires a bit of work to check

Take $\varprojlim_k$ now to define: $H^*_G(M) \xrightarrow{f_*} H^{*-r}_G(M')$. This is an H^*_G-module map.

$\boxed{\text{Important remarks}}$:

- When $f : M \twoheadrightarrow M'$ is a submersion (of fibre dimension r) f_* integrates along the fibres of f (which explains the grading shift).
- when $f : M \hookrightarrow M'$ is the inclusion of a submfld (of codim r) with normal bundle ν_M, $f^* f_* (\cdot) = (\cdot) \cup \underbrace{e(\nu_M)}$ (which also explains the grading shift). Euler class of the normal bundle
- Every f can be factorized as a composite of the above two types: $M \xrightarrow{\text{graph}(f)} M \times M' \xrightarrow{\text{proj}^n} M'$
- All this extends verbatim to equiv't cohomology (using the finite dimensional approximations).

§2 Localization theorem

$\mathbb{T}$-equiv't cohomology is a $\mathbb{C}[u_1, \dots, u_m]$-module, (finitely generated for compact $\mathbb{T}$-mflds M)

Q What is its structure? Rank?

Algebraic preliminaries

Let H be a f.g. module / $A := \mathbb{C}[u_1, \dots, u_m] = \mathrm{Sym}\, \underline{t}^*_{\mathbb{C}}$.
We define its support to be a subset of $\mathbb{C}^m = \underline{t}_{\mathbb{C}}$:

$$\mathrm{supp}\, H = \bigcap \{ V_f : f \in A \text{ s.t. } fH = 0 \}$$

where $V_f = \{ (u_1, \dots, u_m) \in \mathbb{C}^m \mid f(u_1, \dots, u_m) = 0 \}$.

$$\mathrm{supp}\, H \subset V_f \iff 0 = H_f =: H \otimes_A A[f^{-1}] \quad \text{"localization"}$$

$H_{gen} := H \underset{\mathbb{C}[u_1,\dots,u_m]}{\otimes} \underbrace{\mathbb{C}(u_1,\dots,u_m)}$ is a finite dimensional v. space over $K = \mathbb{C}(u_1,\dots,u_m) = \mathrm{Frac}(A)$.

$\leftarrow$ the field of rational functions in $u_1,\dots,u_m$

$\mathrm{rank}_A H := \dim_K H_{gen}$.

<u>Most basic version of localization theorem:</u>

Suppose $\mathbb{T} \circlearrowright M$ mfld and $F = M^{\mathbb{T}} \subseteq M$ is the fixed point locus.

Then, $\mathrm{rank}_{H^*_{\mathbb{T}}} H^*_{\mathbb{T}}(M) = \dim_{\mathbb{C}} H^*(F)$.

Local structure of a $\mathbb{T}$-manifold M

By averaging over $\mathbb{T}$, we endow M with an invariant Riemannian metric. Let $p \in M$ and let $K = \mathrm{Stab}_{\mathbb{T}}(p) \subseteq \mathbb{T}$.

The map $\begin{aligned} \mathbb{T}/K &\longrightarrow M \\ [g] &\longmapsto g\cdot p \end{aligned}$ is a diffeomorphism onto the orbit S of p.

Consider $N_{S/M} \cong (TS)^{\perp} \subseteq TM|_S$, the normal bundle, it has a linear G-action lifting the one on S.

$[N_{S/M} = N^K_{S/M} \oplus (N^K_{S/M})^{\perp}$ decomposition of K-equiv't bundles.$]$

Let $N_{S/M}(\varepsilon)$ be the ε-nbd of the zero section. For $0 < \varepsilon \ll 1$,

$\exp : N_{S/M}(\varepsilon) \xrightarrow{\cong} U \supseteq S \ni p$, a $\mathbb{T}$-inv't tubular nbd.

$\rightarrow$ $U^K \underset{\triangle}{=} U \cap \{q \in M \mid \mathrm{Stab}_{\mathbb{T}}(q) = K\} = \text{image of } (\exp|_{N^K_{S/M}(\varepsilon)})$ &

the normal bundle of U^K is given by pulling back $(N^K_{S/M})^{\perp}$ under $U^K \to S$.

$\triangle$ Uses that any point in U has stabilizer contained in K $\rightarrow$ $\exists$ $\mathbb{T}$-equiv't map $U \longrightarrow \mathbb{T}/K$.

(simply project $N_{S/M}(\varepsilon) \longrightarrow S$)

Key observations for localization theorem

- If M is a $\mathbb{T}$-manifold and $\exists$ equiv't map $M \to \mathbb{T}/K$, then $\operatorname{supp} H^*_{\mathbb{T}}(M) \subseteq \underline{k}_{\mathbb{C}} \subseteq \underline{t}_{\mathbb{C}}$.

  Proof: K_0 = identity component of K.

  Since we're using $\mathbb{C}$-coefficients $BK_0 \to BK$ induces iso on H^*.

  Thus, consider $M \to \mathbb{T}/K \to pt$ and apply $H^*_{\mathbb{T}}$ to get

$$\begin{array}{ccccc} H^*_{\mathbb{T}} & \to & H^*_K & \longrightarrow & H^*_{\mathbb{T}}(M). \\ & & \parallel & & \\ & & H^*_{K_0} & & \end{array}$$

  $\Rightarrow$ the action of $\operatorname{Sym} \underline{t}^*_{\mathbb{C}}$ factors through $\operatorname{Sym} \underline{k}^*_{\mathbb{C}}$. ▨

- If H', H, H'' are f.g. modules over $\mathbb{C}[u_1, \dots, u_m]$ and $H' \to H \to H''$ is an exact sequence, then

  $\operatorname{supp} H \subseteq \operatorname{supp} H' \cup \operatorname{supp} H''$.

- The above two imply that if M is cpt, $F = M^{\mathbb{T}}$, then

  $\operatorname{supp} H^*_{\mathbb{T}}(M - F) \subseteq \bigcup_K \underline{k}_{\mathbb{C}}$ w/ the union over the (finitely many) subgroups $K \subsetneq \mathbb{T}$ occuring as stabilizers.

- The same also works for relative cohomology (of pairs), e.g. to $(M - U, \partial(M - U))$ for a tubular nbd $U \supseteq F$ as before.

$\boxed{\text{Theorem}}$ (Localization) The natural map $H^*_{\mathbb{T}}(M) \xrightarrow{i^*} H^*_{\mathbb{T}}(F) = H^*(F) \otimes H^*_{\mathbb{T}}$ has kernel & cokernel being torsion $H^*_{\mathbb{T}}$-modules. Thus, after localization (or passing to the fraction field of $H^*_{\mathbb{T}}$), i^* is an iso.

- Similar arguments $\Rightarrow$ $i_* : H^*_{\mathbb{T}}(F) \longrightarrow H^*_{\mathbb{T}}(M)$ iso mod torsion.

$\Rightarrow$ $i^* i_*$ is an iso mod torsion.

recall this is given by $-\cup e(\nu_F)$ by the "important remarks" earlier

Since $H^*_{\mathbb{T}}(F) \underset{H^*_{\mathbb{T}}}{\otimes} \mathrm{Frac}(H^*_{\mathbb{T}}) = H^*(F) \otimes \mathbb{C}(u_1, \dots, u_m)$,

it follows that
$$e(\nu_F) \in H^*(F) \otimes \mathbb{C}(u_1, \dots, u_m) = \bigoplus_{i \geq 0} H^i(F) \otimes \mathbb{C}(u_1, \dots, u_m)$$

must be invertible in this ring, i.e.,

its $H^0(F) \otimes \mathbb{C}(u_1, \dots, u_m) = \prod_j \left(\underbrace{H^0(F_j)}_{F = \coprod_{j=1}^{k} F_j \text{ connected components}} \otimes \mathbb{C}(u_1, \dots, u_m)\right)$ - component

must be of the form:
$$(1 \otimes \alpha_{F_1}, \dots, 1 \otimes \alpha_{F_k}) \quad \text{for} \quad \underset{1 \leq i \leq k}{\alpha_{F_i} \neq 0 \in \mathbb{C}[u_1, \dots, u_m]}$$

Can we see this independently?

Yes: just amounts to computing $e(\nu_{F_j}|_{x_j})$ for some $x_j \in F_j$.
$$\nu_{F_j}|_{x_j} = (T_{x_j}M) / \underbrace{(T_{x_j}M)^{\mathbb{T}}}_{T_{x_j}F_j} = \bigoplus \rho$$
ρ ← one-dimensional non-trivial $\mathbb{T}$-representations.

Now, use the "important example" from before to determine $e(\nu_{F_i}|_{x_i})$.

We now get: for any $\phi \in H^*_{\mathbb{T}}(M)$, we have:
$$\phi = \sum_{1 \leq j \leq k} (i_{F_j})_* \left(\frac{(i_{F_j})^* \phi}{e(\nu_{F_j})} \right) \quad \text{in} \quad H^*_{\mathbb{T}}(M) \underset{\mathbb{C}[u_1, \dots, u_m]}{\otimes} \mathbb{C}(u_1, \dots, u_m)$$

Applying $(\pi_M)_* = \int_M$, we get
$$\int_M \phi = \sum_{1 \leq j \leq k} \int_{F_j} \frac{(i_{F_j})^* \phi}{e(\nu_{F_j})} \qquad \text{"integration formula"}$$

E.g. Suppose $\mathbb{T} \circlearrowright M$ w/ isolated fixed points, i.e., each $F_i = \{x_i\}$ for some $x_i \in M$. TM is naturally a $\mathbb{T}$-equiv't v. bundle and so,

$$\chi(M) = \int_M e(TM) = \sum_i \frac{e(TM)|_{x_i}}{e(T_{x_i}M)} = |F|.$$

Check: standard $\mathbb{T}^m$ action on $\mathbb{CP}^{m-1}$ $(\alpha_1, \dots, \alpha_m) \cdot [x_1 : \dots : x_m] = [\alpha_1 x_1 : \dots : \alpha_m x_m]$ has $m = \chi(\mathbb{CP}^{m-1})$ fixed points $[1:0:\dots:0], \dots, [0:0:\dots:0:1]$.

Consequence: $\chi(M) < 0 \Rightarrow \nexists$ $\mathbb{T}$-action on M w/ isolated fixed pts.

§3 Equivariant de Rham theory

We will now describe a deRham model for equivariant cohomology (without proof ☹).

Let $X_1, \dots, X_m$ be the basis of $\underline{t}$ dual to the basis $u_1, \dots, u_m$ of $\underline{t}^*$ which we have been using so far.

Let M be a $\mathbb{T}$-mfld. $\Omega^*(M) :=$ space of smooth diff'l forms on M.

$$\Omega^*(M)^{\mathbb{T}} := \text{space of } \mathbb{T}\text{-invariant diff'l forms on } M$$
$$= \{\alpha \in \Omega^*(M) \mid \mathcal{L}_{X_1}\alpha = \dots = \mathcal{L}_{X_m}\alpha = 0\}$$

Here, we are identifying X_i w/ the corresponding vector fields they generate on M and $\mathcal{L}$ is the Lie derivative. (derivative along the flow of X.)

Recall: Cartan's "magic formula": $\mathcal{L}_X \alpha = i_X d\alpha + d i_X \alpha$,

where $i_X \alpha = \alpha(X, -)$ is the interior product (or "contraction") w/ X. (reduces a k-form to a $(k-1)$-form)

We define the <u>equivariant de Rham complex</u> to be:

$$\Omega^*_{\mathbb{T}}(M) := \Omega^*(M)^{\mathbb{T}} \otimes_{\mathbb{C}} \operatorname{Sym} \mathfrak{t}^*_{\mathbb{C}} = \Omega^*(M)^{\mathbb{T}}_{\mathbb{C}}[u_1, \dots, u_m] \quad \text{equipped w/}$$

the differential:
$$D\alpha = d\alpha + i_{X_1}\alpha \cdot u_1 + \dots + i_{X_m}\alpha \cdot u_m \qquad \forall \alpha \in \Omega^*(M)^{\mathbb{T}}$$
$$Du_1 = \dots = Du_m = 0$$

s.t. D is a derivation w.r.t. the product on $\Omega^*_{\mathbb{T}}(M)$.

<u>Check:</u>
$$D^2\alpha = d^2\alpha + \underbrace{\sum_{k=1}^m i_{X_k} d\alpha \cdot u_k + \sum_{k=1}^m d(i_{X_k}\alpha)\cdot u_k}_{\sum_{k=1}^m (\mathcal{L}_{X_k}\alpha)\cdot u_k} + \sum_{1\le j,k\le m} \underbrace{i_{X_j} i_{X_k}}_{\text{antisymmetric in } j,k} \alpha \cdot \underbrace{u_j u_k}_{\text{symmetric in } j,k}$$

$= 0$. (<u>grading</u>: usual one on diff'l forms & $\deg(u_i) = +2 \ \forall i$)

<u>Dictionary</u>

$(\Omega^*_{\mathbb{T}}(M), D)$	$\longleftarrow$ (cochain level) model for $\longrightarrow$	$H^*_{\mathbb{T}}(M)$
$\Omega^*_{\mathbb{T}}(M) \xrightarrow{u=0} \Omega^*(M)$	$\longleftrightarrow$	restriction $H^*_{\mathbb{T}}(M) \to H^*(M)$
$\mathbb{C}[u_1,\dots,u_m]$-mod str. on $\Omega^*_{\mathbb{T}}(M)$	$\longleftrightarrow$	$H^*_{\mathbb{T}}$-module str. on $H^*_{\mathbb{T}}(M)$
$\int_M : \Omega^*_{\mathbb{T}}(M) \to \mathbb{C}[u_1,\dots,u_m]$ for cpt oriented M	$\longleftrightarrow$	$(\pi_M : M \to pt)_* : H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}$.

⚠ A closed $\mathbb{T}$-inv't form α may not determine an equivariantly closed form. It does so only if $i_X\alpha = 0$.

→ However, it may be possible to <u>extend</u> α (by adding terms lying in the ideal gen'd by $u_1, \dots, u_m$) to an equiv'tly closed form.

§4 Moment maps

Now, suppose the $\mathbb{T}$-action preserves a symplectic form ω on M.

Recall: $\{1\text{-forms on } M\} \overset{\omega}{\longleftrightarrow} \{\text{vector fields on } M\}$

$\{\text{closed 1-forms on } M\} \overset{\omega}{\longleftrightarrow} \{\text{symplectic v-fields on } M\}$

$\{\text{exact 1-forms on } M\} \overset{\omega}{\longleftrightarrow} \{\text{Hamiltonian v.fields on } M\}$.

Given a fn $H: M \to \mathbb{R}$, the Hamiltonian v.f. X_H assoc. to H is given by $dH = \omega(X_H, \cdot)$

A moment map for the action $\mathbb{T} \circlearrowright M$ is a $\mathbb{T}$-inv't map:

$\mu: M \longrightarrow \underline{t}^*$ s.t. $\forall X \in \underline{t}$, $\langle \mu, X \rangle$ is a Hamiltonian fn for the X-flow on M.

(maybe easier to think of μ as: $\underline{t} \longrightarrow C^\infty(M, \mathbb{R})$)

Q Suppose $\mathbb{T} \circlearrowright M$ acts by Hamiltonian diffeos. Does $\exists \mu$?

A Yes! Just pick functions $\mu_1, \dots, \mu_m$ s.t. $d\mu_i = \omega(X_i, \cdot)$ and set $\mu = (\mu_1, \dots, \mu_m)$. Given $X = a_1 X_1 + \dots + a_m X_m$, it is clear that $\langle \mu, X \rangle = a_1 \mu_1 + \dots + a_m \mu_m$ is a Hamiltonian for X.

Why is μ invariant under $\mathbb{T}$?

Check: $d(\mathcal{L}_{X_i} \mu_j) = \mathcal{L}_{X_i} d\mu_j = \underbrace{\cancel{(\mathcal{L}_{X_i}\omega)}}_{\text{symplectic } \mathbb{T}\text{-action}}(X_j, \cdot) + \omega(\underbrace{\cancel{\mathcal{L}_{X_i} X_j}}_{\mathbb{T} \text{ is abelian} \Rightarrow [X_i, X_j] = 0}, \cdot)$

$\Rightarrow \mathcal{L}_{X_i} \mu_j \equiv$ loc. constt. But all orbits of X_i are closed

$\Rightarrow \mu_j$ can't always increase/decrease along the flow of $X_i \Rightarrow \mathcal{L}_{X_i} \mu_j \equiv 0$.

Now, observe that $\omega \in \Omega^*(M)^{\mathbb{T}}$.

Moreover the functions μ_i are also in $\Omega^*(M)^{\mathbb{T}}$.

Thus, $\omega^\# = \omega - (\mu_1 u_1 + \dots + \mu_m u_m) \in \Omega^*_{\mathbb{T}}(M)$.

$$
\begin{aligned}
D\omega^\# &= D(\omega - (\mu_1 u_1 + \dots + \mu_m u_m)) \\
&= d\omega + \sum_{k=1}^{m} (i_{X_k}\omega - d\mu_k) u_k \quad = 0.
\end{aligned}
$$

Running this in reverse also shows that:

$$
\left\{ \begin{array}{l} \text{choice of moment} \\ \text{map } \mu \text{ for } \mathbb{T}\text{-action on} \\ (M,\omega) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{choice of } \mathbb{T}\text{-equiv'tly} \\ \text{closed lift } \omega^\# \\ \text{for } \omega \in \Omega^2(M) \end{array} \right\}
$$

Duistermaat–Heckman formula

Suppose now that $m = 1$, i.e., a Hamiltonian S^1-action on (M^{2n}, ω) w/ moment map (Hamiltonian) $\mu : M \longrightarrow \mathbb{R}$.

Take $\omega^\# := \omega - \mu u$ as before.

Consider $e^{\omega^\#} = e^{\omega} e^{-\mu u} \in \hat{H}^{*,\text{even}}_{S^1}(M) := H^{*,\text{even}}_{S^1}(M) \otimes_{\mathbb{C}[u]} \mathbb{C}[[u]]$

Assume the S^1-action has isolated fixed pts $\{p_1, \dots, p_N\} \subseteq M$.

We apply the integration formula for localization:

$$
\int_M \frac{\omega^n}{n!} e^{-\mu u} = \sum_{k=1}^{N} \frac{e^{-\mu(p_k)u}}{e_{p_k} u^n}
$$

for some integers $0 \neq e_{p_k}$ $(1 \leq k \leq N)$.

$e_{p_k} u^n = e(T_{p_i} M)$

Plugging in $u = \frac{-i}{\hbar}$, we get the <u>Duistermaat-Heckman formula</u>:

LHS looks like a path integral

$$\boxed{\int_M e^{\frac{i}{\hbar}\mu} \cdot \frac{\omega^n}{n!} = \sum_{k=1}^{N} (i\hbar)^n \frac{e^{\frac{i}{\hbar}\mu(p_k)}}{e_{p_k}}}$$

RHS will be identified w/ the stationary phase approxima^n.

Fix $1 \leq k \leq N$. Let's write $p = p_k$ and determine $e_p \in \mathbb{Z}$.

Split $T_pM = \nu_1 \oplus \cdots \oplus \nu_n$ w/ coords $x_1, y_1, \ldots, x_n, y_n$ near p

w/ $\nu_i = \text{span}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right)$, $X = 2\pi \sum_j m_j \left(-y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}\right)$ and

$\omega|_p = \sum_j dx_j \wedge dy_j|_p$. $\quad (m_j \in \mathbb{Z}$, all nonzero$)$

By the "important example", $e(T_pM) = \prod_{j=1}^{n} m_j \cdot u^n$.

$\Rightarrow e_p = \prod_{j=1}^{n} m_j \in \mathbb{Z}$. Set $l = \#$ negative m_j's (for $1 \leq j \leq n$).

$\Rightarrow$ Upto second order near p:

$$\mu = \mu(p) - 2\pi \sum_{j=1}^{n} m_j \left(\frac{x_j^2 + y_j^2}{2}\right) + \ldots \Rightarrow \det \text{Hess}_p(\mu) = (2\pi)^{2n} \prod_{j=1}^{n} m_j^2$$

Now, the <u>stationary phase approximation</u> term contributed by the the critical point p to the oscillatory integral $\int_M e^{\frac{i}{\hbar}\mu} \frac{\omega^n}{n!}$ is:

$$\boxed{(2\pi\hbar)^n \left|\det(\text{Hess}_p(\mu))\right|^{-1/2} e^{i\frac{\pi}{4}\text{sign}(\text{Hess}_p(\mu))} e^{\frac{i}{\hbar}\mu(p)}}$$

Now, using: $\text{sign}(\text{Hess}_p(\mu)) = (2n-2l) - 2l = 2n - 4l$, we get: $\frac{(i\hbar)^n e^{\frac{i}{\hbar}\mu(p)}}{e_p}$.

Thus, the stationary phase approximation is <u>exact</u> here!

References

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