Classical Mechanics Review

Mohan Swaminathan

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Everything here is based on the book [Arn78] by V. I. Arnol'd.

1 Newtonian mechanics

Studies the time evolution of systems of point particles in 3D space under a potential. This section is based on the content of [Arn78, Chapter 1].

1.1 Basic principles

- (Space and time) Space is 3D and time is 1D.

  The universe is a 4-dimensional affine space $\mathbb{A}^4$ and there is an affine linear submersion $t : \mathbb{A}^4 \to \mathbb{A}^1$ called **time**.

- (Inertial frames and Galilean relativity) There is a class of coordinate systems called **inertial frames** such that the laws of physics are the same in all of these at all times. The observer of any inertial frame differs from the observer of any other by a uniform straight line motion.

  There is a distinguished collection of coordinate systems $\mathbb{A}^4 \simeq \mathbb{R}^1_t \times \mathbb{R}^3_x$ (identifying $\mathbb{A}^1$ with $\mathbb{R}^1_t$ via the coordinate projection) such that any two are related by an element of the group generated by

  $$(t, x) \mapsto (t + t_0, x) \tag{1}$$
  $$(t, x) \mapsto (t, x + x_0 + tv_0) \tag{2}$$
  $$(t, x) \mapsto (t, Gx) \tag{3}$$

  where $t_0 \in \mathbb{R}^1$, $x_0, v_0 \in \mathbb{R}^3$ and $G \in SO(3)$. This symmetry group, called the **Galilean group**, is a 10D Lie group.

- (Newton's determinacy) The initial state of a system (i.e. the collection of initial positions and initial velocities) completely determines the states of the system for all time.

  If the system has N particles and, in an inertial frame, their **trajectories** are given by $x(t) = (x_1(t), \ldots, x_N(t))$, then there exists a (smooth)

function $F : \mathbb{R}^{3N}_x \times \mathbb{R}^{3N}_v \times \mathbb{R}^1_t \to \mathbb{R}^{3N}_x$ such that Newton's law

$$\ddot{x}(t) \equiv F(x(t), \dot{x}(t), t) \tag{4}$$

is obeyed. If this system is **closed** (e.g. represents the whole universe), then Galilean invariance shows that we must have, for all $t \in \mathbb{R}^1$, $x, v \in \mathbb{R}^{3N}$ and $A \in SO(3)$,

$$F(x, v, t) = F(x, v) \tag{5}$$

$$F(x, v) = F(\{x_i - x_j\}_{j<k}, \{v_i - v_j\}_{j<k}) \tag{6}$$

$$F(Ax, Av) = AF(x, v) \tag{7}$$

1.2 Some examples

Example 1 (Falling from a great height) Use $x > 0$ to denote the height of the object of mass m above the earth's surface. Then, using Newton's law of gravitation (experimental), we have

$$m\ddot{x} = -\frac{GMm}{(R+x)^2} = -mg\frac{R^2}{(R+x)^2} \tag{8}$$

where we note the (experimental) values

$$G \approx 6.67 \times 10^{-11}\,\mathrm{Nm}^2/\mathrm{kg}^2 \tag{9}$$

$$M \approx 6 \times 10^{24}\,\mathrm{kg} \tag{10}$$

$$R \approx 6.4 \times 10^{6}\,\mathrm{m} \tag{11}$$

$$g = \frac{GM}{R^2} \approx 9.8\,\mathrm{m/s}^2. \tag{12}$$

We would like to write the right side in terms of a potential function $U(x)$, i.e., $\ddot{x} = -U'(x)$. For $x \ll R$, we can take the potential to be $U(x) = gx$. For general x, we can take the potential to be $U(x) = -\frac{gR^2}{R+x}$. Note that the total mechanical energy (kinetic + potential)

$$E = m(\tfrac{1}{2}\dot{x}^2 + U(x)) \tag{13}$$

remains conserved throughout. If the object thrown with initial velocity v escapes to infinity (and has velocty v' there), then

$$\frac{1}{2}v^2 - gR = \frac{1}{2}v'^2 \geq 0 \tag{14}$$

which allows us to calculate the **escape velocity**

$$v_0 = \sqrt{2gR} \approx 11.2\,\mathrm{km}/s. \tag{15}$$

Example 2 (mass on a spring) If a body of mass m is attached to a spring and the spring is extended by a small amount x, then (experimentally)

$$m\ddot{x} = -kx \tag{16}$$

for some constant $k > 0$ characteristic of the spring (with units N/m). As above, we introduce the potential function $U(x) = \frac{1}{2}kx^2$ and find that the total mechanical energy

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 \tag{17}$$

remains conserved. Introducing $\omega^2 = k/m$, we can solve the equation explicitly to get $x = A\cos(\omega t + \varphi)$ for some constants $A > 0$ (called the **amplitude**), $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ (called the **phase**) determined by the initial conditions. This motion is sometimes called a **simple harmonic motion**.

1.3 Conservative mechanical systems

More generally, if for a mechanical system of N particles of masses $m_1, \dots, m_N$, Newton's law can be written as

$$m_i\ddot{x}_i = -\nabla_{x_i} U \tag{18}$$

for some function $U = U(x) = U(x_1, \dots, x_N)$ called the **potential**, then we say that the system is **conservative**. We will mostly concentrate on conservative systems. Introducing the **total mechanical energy**

$$E(x, \dot{x}) = \sum_{i=1}^{N} \tfrac{1}{2}m_i\dot{x}_i^2 + U(x_1, \dots, x_N) \tag{19}$$

we find that for a solution $x = x(t)$ of Newton's equation we have

$$\tfrac{d}{dt}E(x(t), \dot{x}(t)) = \textstyle\sum_{i=1}^{N}(m_i\ddot{x}_i + \nabla_{x_i}U) \equiv 0 \tag{20}$$

and thus, energy is conserved.

2 Solving the equations of motion

Solving Newton's equation, also called the equations of motion, explicitly is very hard except in some special but important cases. We study some of these cases here. This section is based on [Arn78, Chapter 2].

2.1 Systems with one degree of freedom

If Newton's equation takes the form $\ddot{x} = f(x)$ for some function $f : \mathbb{R} \to \mathbb{R}$ called the **force**, then we say that the system has **one degree of freedom**. As usual, we can introduce the **kinetic energy**

$$T = \tfrac{1}{2}\dot{x}^2 \tag{21}$$

and the **potential energy**

$$U(x) = -\int_{x_0}^{x} f(\xi)\, d\xi. \tag{22}$$

Based on Newton's principle of determinacy, if we consider the **phase space** $\mathbb{R}^2_{x,y}$ of all possible intial conditions $(x(0), y(0) = \dot{x}(0))$ of the system, then we have a flow φ^t on this space determined by the vector field $(x, y) \mapsto (y, f(x))$. This is called the **phase flow**. For a solution $x = x(t)$ of Newton's equation, we have

$$(x(t), \dot{x}(t)) = \varphi^t(x(0), \dot{x}(0)). \tag{23}$$

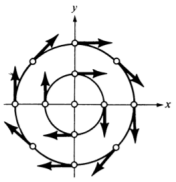
By conservation of energy, the flow lines are confined to the energy level sets

$$\tfrac{1}{2}y^2 + U(x) = c. \tag{24}$$

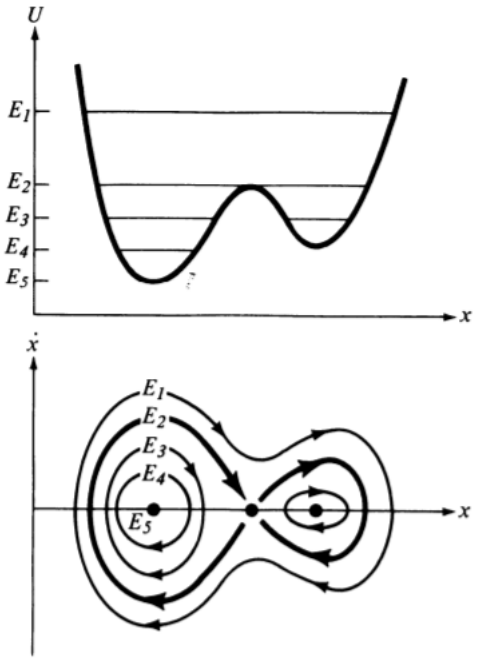
Equilibrium points are those points of the phase space where this vector field vanishes (in particular, the velocity y at such a phase point is zero).

By the implicit function theorem, each energy level set is smooth curve near any non-equilibrium point. This means that the qualitative study of trajectories is essentially reduced to studying energy level sets.

For example, the phase plane of $\ddot{x} = -x$ is shown below [Arn78, Figure 9].



For a more interesting example, consider the following [Arn78, Figure 10].



Using the conservation of energy, we can write

$$\dot{x} = \pm\sqrt{2(E_0 - U(x))} \tag{25}$$

where E_0 is the energy level of the trajectory. This allows us to find the dependence of x on t by integration.

For example, suppose that we have a trajectory $x(t)$ on an energy level E_0 and satisfies $x(t_0) = x_0$ and $x(t_1) = x_1$ on some time interval $[t_0, t_1]$ on which the motion is in the direction of increasing x. Then, separating variables and integrating gives the time duration of the path from x_0 to x_1:

$$t_1 - t_0 = \int_{x_0}^{x_1} \frac{dx}{\sqrt{2(E_0 - U(x))}}. \tag{26}$$

The concept of phase space and phase flow of course make sense even when there is more than one degree of freedom.

2.2 Motion in a central field

We look at a unit mass particle moving in the plane $\mathbb{R}^2$ under the influence of a **central field**, i.e.,

$$\ddot{x} = -\nabla_x U, \quad U = U(r). \tag{27}$$

where r, φ are polar coordinates on $\mathbb{R}^2$. Introduce the orthonormal frame

$$e_r = (\cos\varphi, \sin\varphi), \quad e_\varphi = (-\sin\varphi, \cos\varphi). \tag{28}$$

Then, for a path $x = x(t)$, written in polar coordinates as $(r(t), \varphi(t))$, we verify that

$$\dot{x} = \dot{r}e_r + r\dot{\varphi}e_\varphi \tag{29}$$

$$\ddot{x} = (\ddot{r} - r\dot{\varphi}^2)e_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})e_\varphi. \tag{30}$$

and for a trajectory, the equations of motion become

$$\ddot{r} - r\dot{\varphi}^2 = -U'(r) \tag{31}$$

$$2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \tag{32}$$

The second equation says that the **angular momentum** $M = r^2\dot{\varphi}$ is conserved. Said differently, **the particle sweeps out equal areas in equal times** – this is Kepler's 2nd law originally derived by careful observation. Using the conserved quantity M, we can rewrite the first equation of motion as

$$\ddot{r} = -V'(r) \tag{33}$$

by introducing the **effective potential energy** $V(r) = U(r) + \frac{M^2}{2r^2}$. This reduces us to the case of one degree of freedom!

Using this reduction, we can derive the elliptical orbits of planetary motion with the gravitational potential function $U(r) = -\frac{k}{r}$. See [Arn78, Section 8E] for the details.

3 Lagrangian mechanics

Describes the motion of a mechanical system using a configuration space (a manifold) and a Lagrangian function (defined on the tangent bundle of this manifold). The resulting equations of motion are invariant under the action of the diffeomorphism group of the configuration manifold. This section is based on [Arn78, Chapters 4 and 5].

3.1 Euler–Lagrange equations

Let M be a smooth manifold and let $L : TM \times \mathbb{R}_t \to \mathbb{R}$ be a smooth function called the **Lagrangian**. It is customary to denote local coordinates on M by q and the induced local coordinates on TM as $(q, \dot{q})$. Write $L = L(q, \dot{q}, t)$.

This notation can be quite confusing when $\dot{q}$ can be mistaken for the time derivative of q in which case it doesn't appear as an independent variable. We will try to avoid this ambiguity though physics texts typically do not bother.

Given a path $\gamma : [t_0, t_1] \to M$, with $x_0 = \gamma(t_0)$ and $x_1 = \gamma(t_1)$, define the **action** of the path γ to be the quantity

$$\mathcal{L}[\gamma] = \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t), t)\, dt. \tag{34}$$

Regard $\mathcal{L}$ as a functional on the space $\mathcal{P}$ of all smooth paths $[t_0, t_1] \to M$ with fixed boundary conditions $t_i \mapsto x_i$ for $i = 0, 1$. We want to characterize critical points γ of this functional.

We have $T_\gamma \mathcal{P} = \Gamma_*([t_0, t_1], \gamma^* TM)$ where Γ_* indicates smooth sections vanishing on the boundary, i.e., $\{t_0, t_1\}$. Then, the linearization

$$\delta\mathcal{L}[\gamma] : T_\gamma \mathcal{P} \to \mathbb{R} \tag{35}$$

is determined by how it acts on sections η which are supported on (sufficiently small) closed subintervals $[a, b] \subset [t_0, t_1]$ and vanish at a and b. For simplicity, we do the computation after replacing $[t_0, t_1]$ by such a subinterval over which γ maps into a coordinate patch in M with local coordinate q. Let $\eta \in T_\gamma \mathcal{P}$. We compute in local coordinates

$$\delta\mathcal{L}[\gamma]\eta = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q}(\gamma, \dot{\gamma}, t) \cdot \eta + \frac{\partial L}{\partial \dot{q}}(\gamma, \dot{\gamma}, t) \cdot \dot{\eta} \right) dt \tag{36}$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q}(\gamma, \dot{\gamma}, t) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}}(\gamma, \dot{\gamma}, t) \right) \right) \eta\, dt \tag{37}$$

where we integrated by parts and used $\eta(t_i) = 0$ for $i = 0, 1$. Thus, γ is a critical point if and only if

$$\frac{\partial L}{\partial q} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \tag{38}$$

along γ in some (and therefore all) local coordinates q. This is called the **Euler–Lagrange equation** associated to the functional $\mathcal{L}$. This coordinate invariance is very useful in practice.

3.2 Hamilton's principle of least action

Motions of mechanical systems coincide with critical points of the action functional $\mathcal{L}$ where we take $L = T - U$. Here, T is the kinetic energy and U is the potential energy (which depends only on q and not $\dot{q}$).

Indeed, in Cartesian coordinates, writing $L = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{x}_i^2 - U(x_1, \ldots, x_N)$ Then, the Euler–Lagrange equations reduce to Newton's equations

$$m_i \ddot{x}_i = -\nabla_{x_i} U. \tag{39}$$

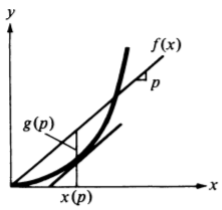
Probably better to call it something like "the principle of extremal action" but "least action" is the customary name. The key point of formulating the laws of motion in this way is that the Euler–Lagrange equations are independent of coordinates and thus, we can choose good coordinate systems (e.g. to take advantage of known conserved quantities). Some standard terminology in mechanics is recorded below for later reference.

- L: the Lagrangian,
- q_i: generalized coordinates,
- $\dot{q}_i$: generalized velocities,
- $p_i = \frac{\partial L}{\partial \dot{q}_i}$: generalized momenta,
- $F_i = \frac{\partial L}{\partial q_i}$: generalized forces,
- $\int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt$: action,
- $\frac{dp_i}{dt} = F_i$: Lagrange's equations.

A generalized coordinate q_i is called **cyclic** if the Lagrangian doesn't depend on it, i.e., the corresponding generalized force $F_i = \frac{\partial L}{\partial q_i}$ vanishes. If q_i is cyclic, then the corresponding generalized momentum p_i is conserved along motions of the system.

3.3 Legendre transform

Let $y = f(x)$ be a smooth convex function, i.e., $f''(x) > 0$. We construct new function $g = g(p)$ called the **Legendre transform** of f as follows. Define $F(p, x) = xp - f(x)$. Given p, find the point $x = x(p)$ where $f'(x) = p$ (equivalently, $F(p, x)$ has a maximum here) which is unique if it exists. Define $g(p) = F(p, x(p))$. See below [Arn78, Figure 43].



The key example for us is $f(x) = \frac{1}{2}mx^2$ and $g(p) = \frac{p^2}{2m}$. It's a fact that the Legendre transform applied to a convex function yields a convex function and applying it twice results in the original function. We also have **Young's inequality** almost by definition

$$px \leq f(x) + g(p) \tag{40}$$

for all x, p.

For many variables, there are analogous definitions. Explicitly, consider any smooth convex function $f : V \to \mathbb{R}$ on a vector space V and suppose that $f' : V \to V^*$ is bijective (and therefore a diffeomorphism by convexity). Define $F(p, x) = \langle p, x\rangle - f(x)$ for $p \in V^*$ and $x \in V$. Then, for any $p \in V^*$, we define $g(p) = F(p, x(p))$, where $x(p)$ is the unique point satisfying $f'(x) = p$.

The important point for us is that one-half of a positive definite quadratic form (i.e., corresponding to an inner product) on V is sent by the Legendre transform to one-half of the quadratic form corresponding to the dual inner product on V^*.

3.4 Hamilton's equations

By a Legendre transform, we convert Lagrange's equations $\dot{p} = \frac{\partial L}{\partial q}$ (with $p = \frac{\partial L}{\partial \dot{q}}$) to Hamilton's equations which are much more symmetric.

3.4.1 Derivation

Work in generalized coordinates $q \in \mathbb{R}^n$. Assume that the Lagrangian function $L = L(q, \dot{q}, t)$ is convex as a function of $\dot{q}$ (and that the equation $p = \frac{\partial L}{\partial \dot{q}}$ determines $\dot{q}$ as a function of p, q, t).

The Legendre transform of L with respect to $\dot{q}$ is the function $H = H(p, q, t)$ given by the formula

$$H(p, q, t) = p\dot{q} - L(q, \dot{q}, t) \tag{41}$$

where we express $\dot{q}$ as a function of p, q, t via the relation $p = \frac{\partial L}{\partial \dot{q}}$. Now, compute dH as the total differential of $p\dot{q} - L$ to get

$$dH = \dot{q}\, dp + p\, d\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial t}dt \tag{42}$$

$$= \dot{q}\, dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt \tag{43}$$

which gives

$$\frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{44}$$

Thus, if Lagrange's equations $\dot{p} = \frac{\partial L}{\partial q}$ are satisfied by a path $q(t)$ then the system

$$\dot{q} = \frac{\partial H}{\partial p} \tag{45}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{46}$$

called **Hamilton's system of equations** is satisfied by the corresponding path $(q(t), p(t))$. Conversely, given a solution $(a(t), b(t))$ of Hamilton's equations, we must verify the identity

$$b(t) = \frac{\partial L}{\partial \dot{q}}(a(t), \dot{a}(t), t) \tag{47}$$

to establish the equivalence of Lagrange's and Hamilton's equations.

References

[Arn78] V. I. Arnol'd, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics, vol. 60, Springer-Verlag, New York-Heidelberg, 1978, Translated from the Russian by K. Vogtmann and A. Weinstein. MR 0690288