MATHY PHYSICS FOR MATHEMATICIANS SEMINAR

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1. Mechanics to Supersymmetry Bootcamp

We start by establishing some common vocabulary, and gaining some intuition about the physicist's mindset. As you might have seen, the word "quantization" has many meanings, and the various meanings are related by loose reasoning. In this seminar, whenever someone presents a"theory", classical or quantum, we want to know with some precision what we're talking about. So our definition of a "quantum theory" will be one of the following: (1) states and operators, (2) path integral, (3) functorial field theory. As physicists do, we will learn how to use the one that's most convenient given a particular situation, and see that the transferring between perspectives sometimes gives/recovers interesting math.

(1) **Introduction to classical mechanics (Eric).** Introduce Lagrangian and Hamiltonian formulations of mechanics. Symmetries and Noether's theorem, Euler-Lagrange equations, symplectic/Poisson manifolds, the moment map, and symplectic reduction (see, for example, Chapters 3,4,8 of [Arn74], and any other parts that interest you or seem salient). Compute your favorite example to illustrate these concepts (e.g. planetary motions).
(2) **Introduction to quantization (Laura).** Basic setup of quantum mechanics in the Schrödinger and Hamiltonian pictures. Solve the harmonic oscillator completely, and explain the role of the creation/annihilation action. Solve two finite square wells and explain tunneling. For these examples you can look at Sections 2.3 and 2.6 in [Gri95]; the burden is to summarize the book (which is written for physics undergrads) using these examples in math words.
(3) **From particles to fields (Leo).** "Classical fields are maps". "Define" the path integral for QFT (see Wikipedia and Chapters 2 and 3 of [Ski18] for inspiration) and explain how this leads to the definition of functorial field theory (for example [Ati89]). Explain basic examples of $(1+1)$-dimensional theories. Consider $(1+1)$-dimensional theory of a scalar field on $\Sigma = S^1 \times \mathbb{R}_t$. Decouple into harmonic oscillator and solve. Consider the same theory with target a circle of radius R, and exhibit T-duality in this case. See Sections 11.1 and 11.2 of [HKK+03]. Optional: algebraic structure of a $(1+1)$-dimensional theory, Dijkgraaf-Witten theory (finite gauge theory), and Verlinde's formula.
(4) **The path integral (Shaoyun).** The goal is to relate the nonrigorous path integral to some of its rigorous manifestations in math. Explain the method of stationary phase, and the meaning of semiclassical limit. Explain the relation with things like Morse complex (Section 3 of [Wit82]) and Atiyah-Bott localization [AB84], highlighting the role of symmetries that make the path integral rigorous.
(5) **Electricity and magnetism (Linus).** Start from Maxwell's equations (see Wikipedia) and end up with U(1) Yang-Mills theory (in (3+1)-dimensions). What are its observables? Explain as many fancy perspectives of electric-magnetic duality as possible: Fouier(-Mukai) transform, mirror symmetry of Hitchin moduli space, (abelian) Hodge

theory. See Chapter 4 of [BZ21]. This talk will be swapped with talk (6), it will be okay.

We saw that the path integral picture has an advantage of making symmetries of the physical system manifest. Supersymmetry is just another type of symmetry, but it rotates bosonic fields and fermionic fields into each other. This feels physically unlikely/unmotivated, and it maybe is lol. But from a mathematical perspective we will see that many structures we already know/want to know can be understood via SUSY theories.

(6) **Witten's Morse theory.** Introduce supersymmetric quantum mechanics as the de Rham complex, the Witten index and Chern-Gauss-Bonnet. Dirac operators, Dirac spinors (at least in 4-dimensions), and a plausibility argument for the Atiyah-Singer index theorem. See [Wit82], [AG83], and Chapter 3 in [Ton22b].

(7) **Supersymmetry algebra.** Define the supersymmetry algebra (SUSY) (with emphasis on dimension 4). Explain how to construct representations of SUSY and how to build some of its representations, maybe with various other smaller dimensions and number of supersymmetries for intuition. All of this can be found in [Ton22a], but we have to translate it into math words. Relation with Hodge theory in the Riemannian and Kähler cases. Start building any SUSY theory in a dimension of your choice, you are allowed to fail, but the important thing is to point out what are the necessary ingredients.

(8) **Supersymemtric Maxwell theory.** How to build a supersymmetric U(1) gauge theory in dimension 4? Explain what twists mean, and single out what people call the A-twist and B-twist. Explain the relationship with geometric Langlands (or geometric class field theory in this case), in the sense of Kapustin-Witten/Elliott-Yoo.

(9) **2d supersymmetric sigma model.** How to build a 2d supersymmetric sigma model? Explain Witten's A/B-twists, and how it leads to mirror symmetry at the level of Gromov-Witten invariants and holomorphic periods. A good source is [Wit98], but if you know of a more mathematical source please let me know.

2. Geometric Langlands

Now that we have some basic picture of supersymmetric theories, we spend the rest of our time this semester (or maybe it will be the first thing we do next semester) looking at one specific supersymmetric theory: geometric Langlands. The main source here is um...Kapustin-Witten [KW07], and I found Elliot-Yoo's more algebraic interpretation also useful [EY18].

(10) **Geometric Langlands I: compactification and SYZ.** Start with a 4d supersymmetric gauge theory with gauge group G. Explain compactification, and argue that compactification along a Riemann surface produces a 2d supersymmetric sigma model with target the Hitchin moduli space. State the hyperkähler properties of the Hitchin moduli space, and SYZ mirror symmetry picture that leads to categorical Langlands.

(11) **Geometric Langlands II: branes and operators.** Introduce type BBB and type BAA branes, which are most relevant for number theory. Introduce Wilson and 't Hooft operators, and their eigenbranes. Try to be as sympathetic to number theorists as possible.

References

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