Practice Midterm Solutions

1. (5 points) State the Bolzano Weierstrass theorem.

   Solution: If $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$, then there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ that converges.

2. (5 points) Let $E \subset \mathbb{R}$ be a set. Show that if U is an open set and $U \subset E$, then $U \subset \text{int}(E)$.

   Solution: Let $x \in U$. Since U is open, there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset U \subset E$. Therefore $x \in \text{int}(E)$. So $U \subset \text{int}(E)$.

3. Let $a_n = \frac{2^n}{n!}$

   (a) (2 points) Does the sequence $(a_n)_{n\in\mathbb{N}}$ converge (prove your answer)?

   Solution: First, prove by induction that $\frac{2^n}{n!} \leq \frac{2}{n}$, whenever $n \geq 1$. For $n = 1$, $2^n/n! = 2 = 2/n$. This proves the base case. Next, suppose $2^n/n! \leq 2/n$. Then,

$$\begin{aligned} \frac{2^{n+1}}{(n+1)!} &= \frac{2 * 2^n}{(n+1)n!} \\ &\leq \frac{2}{n+1}\frac{1}{n} \\ &\leq \frac{2}{n+1}. \end{aligned}$$

   Therefore $2^n/n! \leq 2/n$ for all $n \geq 1$. The limit converges to 0 by comparison with $2/n$.

   (b) (3 points) Does the series $\sum_{n=1}^{\infty} a_n$ converge (prove your answer)?

   Solution: The ratio can be computed,

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}n!}{2^n(n+1)!} = \frac{2}{n+1}.$$

This converges to 0 as n goes to ∞ and by the ratio test the series converges absolutely.

4. Let A and B be nonempty sets and suppose that for any $a \in A$ and $b \in B$, $a \leq b$.

   (a) (2 points) Show that $\sup A$ and $\inf B$ exist.

   Solution: $\sup A$ exists as long as A is bounded above. Since B is nonempty, there exists $b \in B$ and b is an upper bound for A. Thus $\sup A$ is finite. Similarly, $\inf B$ exists as long as B is bounded below. Since A is nonempty, there exists $a \in A$ and a is a lower bound for B. So $\inf B$ is finite.

   (b) (3 points) Prove that $\sup A \leq \inf B$.

   Solution: Suppose that $b \in B$, then b is an upper bound for A. Therefore the least upper bound $\sup A \leq b$. So $\sup A$ is a lower bound for B and $\sup A \leq \inf B$.

5. (a) (2 points) Give an example of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ so that $a_n \geq 0$, $\lim_{n\to\infty} a_n = 0$, $|b_{n+1} - b_n| \leq a_n$, but $(b_n)_{n\in\mathbb{N}}$ diverges.

   Solution: Define $a_n = 1/n$ and $b_n = \sum_{i=1}^{n-1} \frac{1}{i}$.

   (b) (3 points) Suppose that $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges. If

$$|b_{n+1} - b_n| \leq a_n,$$

   show that $(b_n)_{n\in\mathbb{N}}$ converges to a limit.

   Solution: Use the Cauchy criterion. Fix $\epsilon > 0$ Let $m > n \in \mathbb{N}$, such that $\sum_{i=n}^{m-1} a_i < \epsilon$. Then

$$\begin{aligned}|b_m - b_n| &= \left|\sum_{i=n}^{m-1} b_{i+1} - b_i\right| \\ &\leq \sum_{i=n}^{m-1} a_i < \epsilon.\end{aligned}$$

So by the Cauchy criterion for sequences, $(b_n)_{n\in\mathbb{N}}$ converges.