Practice Final Solutions

1. (5 points) For each $n \in \mathbb{N}$, let $s_n \in (0, \frac{1}{n})$. Let $E = \{s_n : n \in \mathbb{N}\}$.

   (a) Find $\inf E$, with proof.

   Solution: $\inf E = 0$: 0 is a lower bound. Suppose that $a > 0$, then by the Archimedean property, there exists $n \in \mathbb{N}$ so that $n > 1/a$. So $s_n < 1/n < a$. Therefore 0 is the greatest lower bound.

   (b) Does there exists $s \in E$ such that $s = \inf E$?

   Solution: No, since $s_n > 0$ for all $n \in \mathbb{N}$.

2. (5 points) Define $f\colon [0,1] \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ x, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Is f Riemann integrable on $[0,1]$? Prove your answer.

Solution: The function f is not Riemann integrable. Let P be a partition of $[0,1]$, then

$$U(f,P) = \sum_{i=0}^{n} M_i(t_{i+1} - t_i) = 1,$$

since $M_i = 1$. While

$$L(f,P) = \sum_{i=0}^{n} m_i(t_{i+1} - t_i) = \sum_{i=0}^{n} t_i(t_{i+1} - t_i).$$

Without loss of generality, $1/2 \in P$, then

$$\begin{aligned} L(f,P) &= \sum_{t_i < 1/2} t_i(t_{i+1} - t_i) + \sum_{t_i \geq 1/2} t_i(t_{i+1} - t_i) \\ &\leq \frac{1}{2}\frac{1}{2} + \frac{1}{2} = 3/4. \end{aligned}$$

Therefore $\sup L(f,P) \neq \inf U(f,P)$ and f is not integrable.

3. (5 points) Let f be a differentiable function on $\mathbb{R}$, and assume $|f'(x)| < \frac{1}{x^2+1}$ everywhere. Prove that f is bounded.

Solution: By the fundamental theorem of calculus,

$$f(x) = C + \int_0^x f'(t)dt,$$

where C is a constant. So

$$|f(x)| \leq C + \frac{|x|}{1+x^2} \leq C + 1$$

and f is bounded.

4. (5 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{n} x^n.$$

Include the endpoints. Prove your answer.

Solution: By the ratio test,

$$\lim_{n\to\infty} \left| \frac{x^{n+1}\log^2(n+1)n}{(n+1)\log^2 n x^n} \right| = \lim_{n\to\infty} |x| \frac{\log^2(n+1)n}{\log^2(n)(n+1)} = |x|.$$

So the series converges for $|x| < 1$. If $x = 1$, then $\log^2 n/n > 1/n$ and the series diverges. If $x = -1$, then by the alternating series test, the series converges.

5. (5 points) Let $p > 0$. Prove that $x^3 + px + q = 0$ has exactly one solution.

Solution: Firstly, $\lim_{x\to\infty} x^3 + px + q = \infty$ and $\lim_{x\to-\infty} x^3 + px + q = -\infty$. By the intermediate value theorem, there exists a solution to $x^3 + px + q = 0$. Secondly, the derivative is $x^2 + p > 0$. So the function is increasing and therefore injective. This means the solution is unique.

6. (5 points) Let $f_1\colon [0,1] \to \mathbb{R}$ be Riemann integrable. Define inductively

$$f_n(x) = \int_0^x f_{n-1}(t)dt,$$

for $x \in [0,1]$ and $n > 1$.

(a) Prove that f_n is Riemann integrable.

Solution: Since f_1 is integrable, f_2 is continuous and hence integrable. By induction, f_n is integrable, so f_{n+1} is continuous and hence integrable.

(b) Prove that f_n converges to 0 uniformly as $n \to \infty$.

Solution: Since f_1 is integrable, it is bounded and $|f_1(x)| \leq M$. Claim by induction that $f_n(x) \leq M\frac{x^{n-1}}{(n-1)!}$: The base case has been shown.

$$\begin{aligned} |f_n(x)| &\leq \int_0^x |f_{n-1}(t)|dt \leq \frac{M}{(n-2)!}\int_0^x t^{n-2} \\ &= \frac{M}{(n-1)!}x^{n-1}. \end{aligned}$$

Therefore $\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x)| = 0$ and f_n converges to 0 uniformly.

7. (5 points) Let the function $f\colon [0,\infty) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \sin(e^{\frac{1}{x}}) \cdot e^{-\frac{1}{x}}, & \text{if } x \in (0,\infty), \\ 0, & \text{if } x = 0. \end{cases}$$

Let the function $g : [0,\infty) \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} f'(x), & \text{if } x \in (0,\infty), \\ c, & \text{if } x = 0, \end{cases}$$

where $c \in \mathbb{R}$.

(a) Show that f is continuous at $x = 0$.

Solution: The function $\frac{\sin(e^{1/x})}{e^{1/x}}$ satisfies $0 < \left|\frac{\sin(e^{1/x})}{e^{1/x}}\right| < \frac{1}{e^{1/x}}$. So

$$0 \leq \left|\lim_{x\to 0} \frac{\sin(e^{1/x})}{e^{1/x}}\right| = \lim_{x\to 0} \frac{1}{e^{1/x}} = 0.$$

This gives that f is continuous at 0.

(b) For what values of c is the function g continuous at $x = 0$?

Solution: Compute $f'(x)$:

$$\begin{aligned} f'(x) &= -\frac{\cos(e^{1/x})}{x^2} + \frac{\sin(e^{1/x})e^{-1/x}}{x^2} \\ &= \frac{1}{x^2}\left(\frac{1}{6}(e^{1/x})^2 + \text{ higher order powers of } e^{1/x}\right), \end{aligned}$$

where this was derived by expanding sin and cos as power series in $e^{1/x}$. As $x \to 0$, $\left(\frac{e^{1/x}}{x}\right)^2$ does not have a limit. So there is no limit and g is never continuous at 0.

8. (5 points) Let A be a bounded set. Prove that $M = \sup A$ if and only if M is an upper bound for A and for each $\epsilon > 0$, there exists $a \in A$ such that $M - \epsilon < a$.

Solution: First, suppose that $M = \sup A$. Then M is the least upper bound for A. Fix $\epsilon > 0$, then $M - \epsilon$ is not an upper bound for A. This means there exists $a \in A$ so that $M - \epsilon < a$.

Second, suppose that M is an upper bound for A and for each ϵ there exists $a \in A$ with $M - \epsilon < a$. Let $M' = \sup A$, then $M' \leq M$. Let $\epsilon = M - M'$. If $\epsilon > 0$, there exists $a \in A$ so that $M' = M - \epsilon < a$. This is not possible and so $\epsilon = 0$. So $M = \sup A$.

9. (5 points) Let $f\colon [0,1] \to \mathbb{R}$ be Riemann integrable. Show that there exists

$c \in [0,1]$ so that

$$\int_0^c f(x)dx = \int_c^1 f(x)dx.$$

Solution: The function $g(x) = \int_0^x f(t)dt - \int_x^1 f(t)dt$ is continuous. $g(0) = -\int_0^1 f(t)dt$ and $g(1) = \int_0^1 f(t)dt$. By the intermediate value theorem, there exists $c \in [0,1]$ so that $g(c) = 0$. So

$$0 = \int_0^c f(x)dx - \int_c^1 f(x)dx.$$

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