Eden Prywes | Mat 215 | Spring 2022

Name: ________________________________

Practice Final

- No calculators, books, or notes are permitted.
- Nothing should be on your desk but writing implements and a single one-sided reference page.
- If you have a question during the exam, you may leave the room and ask the proctor.
- You will receive points only for what is written on the numbered pages. Please use the back of pages as scratch paper.
- Please write neatly, show all your work, and justify all answers. Mysterious or illegible solutions will receive no credit.
- If you finish early, check your answers and wait until time is called.
- Please sign the Honor Pledge:

  I pledge my honor that I have not violated the Honor Code during this examination.

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  No exam without a signature will be graded.

1. (5 points) For each $n \in \mathbb{N}$, let $s_n \in (0, \frac{1}{n})$. Let $E = \{s_n : n \in \mathbb{N}\}$.

   (a) Find $\inf E$, with proof.

   (b) Does there exists $s \in E$ such that $s = \inf E$?

2. (5 points) Define $f\colon [0,1] \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ x, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   Is f Riemann integrable on $[0,1]$? Prove your answer.

3. (5 points) Let f be a differentiable function on $\mathbb{R}$, and assume $|f'(x)| < \frac{1}{x^2+1}$ everywhere. Prove that f is bounded.

4. (5 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{n} x^n.$$

   Include the endpoints. Prove your answer.

5. (5 points) Let $p > 0$. Prove that $x^3 + px + q = 0$ has exactly one solution.

6. (5 points) Let $f_1\colon [0,1] \to \mathbb{R}$ be Riemann integrable. Define inductively

$$f_n(x) = \int_0^x f_{n-1}(x)dx,$$

   for $x \in [0,1]$ and $n > 1$.

   (a) Prove that f_n is Riemann integrable.

   (b) Prove that f_n converges to 0 uniformly as $n \to \infty$.

7. (5 points) Let the function $f\colon [0,\infty) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \sin(e^{\frac{1}{x}}) \cdot e^{-\frac{1}{x}}, & \text{if } x \in (0,\infty), \\ 0, & \text{if } x = 0. \end{cases}$$

Let the function $g : [0,\infty) \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} f'(x), & \text{if } x \in (0,\infty), \\ c, & \text{if } x = 0, \end{cases}$$

where $c \in \mathbb{R}$.

(a) Show that f is continuous at $x = 0$.

(b) For what values of c is the function g continuous at $x = 0$?

8. (5 points) Let A be a bounded set. Prove that $M = \sup A$ if and only if M is an upper bound for A and for each $\epsilon > 0$, there exists $a \in A$ such that $M - \epsilon < a$.

9. (5 points) Let $f\colon [0,1] \to \mathbb{R}$ be Riemann integrable. Show that there exists $c \in [0,1]$ so that

$$\int_0^c f(x)dx = \int_c^1 f(x)dx.$$

This page left blank for more scratch work.