Midterm Solutions

1. (5 points) State the definition of a countable set.

   Solution: A set A is countable if there exists a function $f\colon A \to \mathbb{N}$ that is bijective.

2. (5 points) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. If $\sum_{n=1}^{\infty} |a_n|$ converges, show that $\sum_{n=1}^{\infty} a_n$ converges and

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|.$$

   Solution: Let $n, m \in \mathbb{N}$ and let $n > m$.

$$\left|\sum_{i=m+1}^{n} a_i\right| \leq \sum_{i=m+1}^{n} |a_i|$$

   by the triangle inequality. Fix $\epsilon > 0$. By the Cauchy Criterion for sums, there exists $N \in \mathbb{N}$ so that if $n, m \geq N$, the right hand side of the inequality is less than ϵ. The Cauchy criterion for sums applied again gives that $\sum_{i=}^{\infty} a_i$ converges.

   Let s_N be the partial sums of the sequence a_n and t_N be the partial sums of the sequence $|a_n|$.

$$|s_N| = \left|\sum_{n=1}^{N} a_n\right| \leq \sum_{n=1}^{N} |a_n| = t_N,$$

   by the triangle inequality. By the order limit theorem, $\lim_{N\to\infty} |s_N| \leq \lim_{n\to\infty} t_N$.

3. (5 points) Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence. Recall that $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup\{a_n : n \geq N\}$. Show that there exists a subsequence of $(a_n)_{n\in\mathbb{N}}$ that converges to $\limsup_{n\to\infty} a_n$.

Solution: Since a_n is bounded, $\limsup_{n\to\infty} a_n$ exists. Let $b_k = \sup\{a_n : n \geq k\}$. By the definition of the supremum, there exists $a_{n_k} \in \{a_n : n \geq k\}$ such that

$$|a_{n_k} - b_k| = b_k - a_{n_k} < \frac{1}{k}.$$

Let $a = \limsup_{n\to\infty} a_n$. Fix $\epsilon > 0$. There exists $K \in \mathbb{N}$ so that if $k \geq K$, then $|b_k - a| < \epsilon/2$. There also exists $K' \in \mathbb{N}$ so that if $k \geq K'$, $|a_{n_k} - b_k| < 1/k < \epsilon/2$. So

$$|a_{n_k} - a| \leq |a_{n_k} - b_k| + |b_k - a| < \epsilon,$$

for $k \geq \max(K, K')$.

4. (5 points) Let A be a bounded and infinite set. Show that A has at least one limit point.

   Solution: Since A is infinite, there exists an infinite sequence of unique elements $a_n \in A$. Since A is bounded, $(a_n)_{n\in\mathbb{N}}$ is bounded. By the Bolzano-Weierstrass theorem, there exists a subsequence a_{n_k} that converges to a limit a. The a_{n_k} are all unique elements in A. So for all $\epsilon > 0$, $(a-\epsilon, a+\epsilon) \cap A$ contains infinitely many points in A. This means a is a limit point of A.

5. For $x \in \mathbb{R}$, define $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

   (a) (2 points) Show that for every $x \in \mathbb{R}$, the sum that defines e^x converges absolutely.

   Solution: Let $a_n = \frac{x^n}{n!}$.

   $$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|}{n+1} \to 0$$

   as $n \to \infty$. By the ratio test, the sum converges absolutely.

(b) (3 points) Suppose $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence that is not bounded. Show that for any $p \in \mathbb{N}$, $\lim_{n\to\infty} \frac{x_n^p}{e^{x_n}} = 0$.

Solution: For $p \in \mathbb{N}$ fixed and $x > 0$, $e^x \geq \frac{x^{p+1}}{(p+1)!}$ since the right hand side is a term in the sum that defines e^x and all the terms are positive. Since x_n is increasing and unbounded, eventually $x_n > 0$. So

$$\frac{x_n^p}{e^{x_n}} \leq \frac{(p+1)!x_n^p}{x_n^{p+1}} = \frac{(p+1)!}{x_n} \to 0$$

as $n \to \infty$. By comparison, the desired limit also converges to 0.

6. (5 points) Let $A \subset \mathbb{R}$, prove that $\text{int}(A) = \mathbb{R} \setminus (\overline{\mathbb{R} \setminus A})$.

Solution: $x \in \text{int}(A) \iff$ There exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset A \iff$ There exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus (\mathbb{R} \setminus A) \iff x \in \mathbb{R} \setminus (\overline{\mathbb{R} \setminus A})$.