Midterm Solutions

1. (5 points) State the definition of a countable set.

**Solution:** A set $A$ is countable if there exists a function $f: A \rightarrow \mathbb{N}$ that is bijective.

2. (5 points) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence. If $\sum_{n=1}^{\infty} |a_n|$ converges, show that $\sum_{n=1}^{\infty} a_n$ converges and

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n| .$$

**Solution:** Let $n, m \in \mathbb{N}$ and let $n > m$.

$$\left| \sum_{i=m+1}^{n} a_i \right| \leq \sum_{i=m+1}^{n} |a_i|$$

by the triangle inequality. Fix $\epsilon > 0$. By the Cauchy Criterion for sums, there exists $N \in \mathbb{N}$ so that if $n, m \geq N$, the right hand side of the inequality is less than $\epsilon$. The Cauchy criterion for sums applied again gives that $\sum_{i=1}^{\infty} a_i$ converges.

Let $s_N$ be the partial sums of the sequence $a_n$ and $t_N$ be the partial sums of the sequence $|a_n|$.

$$|s_N| = \left| \sum_{n=1}^{N} a_n \right| \leq \sum_{n=1}^{N} |a_n| = t_N,$$

by the triangle inequality. By the order limit theorem, $\lim_{N \to \infty} |s_N| \leq \lim_{n \to \infty} t_N$.

3. (5 points) Let $(a_n)_{n \in \mathbb{N}}$ be a bounded sequence. Recall that $\limsup_{n \to \infty} a_n = \lim_{N \to \infty} \sup \{ a_n : n \geq N \}$. Show that there exists a subsequence of $(a_n)_{n \in \mathbb{N}}$ that converges to $\limsup_{n \to \infty} a_n$. 

Solution: Since \( a_n \) is bounded, \( \limsup_{n \to \infty} a_n \) exists. Let \( b_k = \sup\{a_n : n \geq k\} \). By the definition of the supremum, there exists \( a_{n_k} \in \{a_n : n \geq k\} \) such that

\[
|a_{n_k} - b_k| = b_k - a_{n_k} < \frac{1}{k}.
\]

Let \( a = \limsup_{n \to \infty} a_n \). Fix \( \epsilon > 0 \). There exists \( K \in \mathbb{N} \) so that if \( k \geq K \), then \( |b_k - a| < \epsilon/2 \). There also exists \( K' \in \mathbb{N} \) so that if \( k \geq K' \), \( |a_{n_k} - b_k| < 1/k < \epsilon/2 \). So

\[
|a_{n_k} - a| \leq |a_{n_k} - b_k| + |b_k - a| < \epsilon,
\]

for \( k \geq \max(K, K') \).

4. (5 points) Let \( A \) be a bounded and infinite set. Show that \( A \) has at least one limit point.

Solution: Since \( A \) is infinite, there exists an infinite sequence of unique elements \( a_n \in A \). Since \( A \) is bounded, \( (a_n)_{n \in \mathbb{N}} \) is bounded. By the Bolzano-Weierstrass theorem, there exists a subsequence \( a_{n_k} \) that converges to a limit \( a \). The \( a_{n_k} \) are all unique elements in \( A \). So for all \( \epsilon > 0 \), \( (a - \epsilon, a + \epsilon) \cap A \) contains infinitely many points in \( A \). This means \( a \) is a limit point of \( A \).

5. For \( x \in \mathbb{R} \), define \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

(a) (2 points) Show that for every \( x \in \mathbb{R} \), the sum that defines \( e^x \) converges absolutely.

Solution: Let \( a_n = \frac{x^n}{n!} \).

\[
\frac{|a_{n+1}|}{|a_n|} = \frac{|x|}{n + 1} \to 0
\]
as \( n \to \infty \). By the ratio test, the sum converges absolutely.
(b) (3 points) Suppose \((x_n)_{n \in \mathbb{N}}\) is an increasing sequence that is not bounded. Show that for any \(p \in \mathbb{N}\), \(\lim_{n \to \infty} \frac{x_p^n}{e^{x_n}} = 0\).

**Solution:** For \(p \in \mathbb{N}\) fixed and \(x > 0\), \(e^x \geq \frac{x^{p+1}}{(p+1)!}\) since the right hand side is a term in the sum that defines \(e^x\) and all the terms are positive. Since \(x_n\) is increasing and unbounded, eventually \(x_n > 0\). So

\[
\frac{x_p^n}{e^{x_n}} \leq \frac{(p + 1)! x_p^n}{x_n^{p+1}} = \frac{(p + 1)!}{x_n} \to 0
\]

as \(n \to \infty\). By comparison, the desired limit also converges to 0.

6. (5 points) Let \(A \subset \mathbb{R}\), prove that \(\mathrm{int}(A) = \mathbb{R} \setminus (\mathbb{R} \setminus A)\).

**Solution:** \(x \in \mathrm{int}(A) \iff\) There exists \(\epsilon > 0\) such that \((x - \epsilon, x + \epsilon) \subset A \iff\)

There exists \(\epsilon > 0\) such that \((x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus (\mathbb{R} \setminus A) \iff x \in \mathbb{R} \setminus (\mathbb{R} \setminus A)\).