Final Solutions

1. (5 points) State the Weierstrass $M$-test.

Solution: If $f_n : [a, b] \to \mathbb{R}$ are functions that satisfy $|f_n| \leq M_n$, for $M_n \geq 0$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

2. (5 points) Suppose that $a_n \geq 0$ and $b_n \geq 0$ are sequences such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Show that

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Solution: Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n \to \infty} a_n = 0$. In particular the sequence $(a_n)_{n \in \mathbb{N}}$ is bounded. Suppose that $a_n \leq M$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} a_n b_n \leq M \sum_{n=1}^{\infty} b_n < \infty.$$

Since $a_n b_n \geq 0$, the comparison test applies and the sum converges.

3. (a) (5 points) Give an example of a sequence of integrable functions $f_n : [0, 1] \to \mathbb{R}$ so that $f_n \to 0$ pointwise, but there exists $C \neq 0$ so that $\lim_{n \to \infty} \int_0^1 f_n(x) dx = C$ (prove your answer).

Solution: Define $f_n$ as follows:

$$f_n(x) = \begin{cases} n & \text{if } x \in (0, 1/n) \\ 0 & \text{otherwise.} \end{cases}$$

The limit of $f_n(x) = 0$ as $n \to \infty$ for any fixed $x \in [0, 1]$ since, eventually, $x > 1/n$. While

$$\int_0^1 f_n(x) dx = 1$$

for all $n \in \mathbb{N}$. 
(b) (5 points) Give an example of a sequence of integrable functions \( g_n : [0, \infty) \to \mathbb{R} \) so that there exists \( M > 0 \) with \( |g_n(x)| \leq M \), \( g_n \to 0 \) pointwise, but \( \lim_{n \to \infty} \int_0^\infty g_n(x) \, dx \) either does not exist or converges to a value that is not 0 (prove your answer).

**Solution:** Define \( g_n \) as follows:

\[
g_n(x) = \begin{cases} 
\frac{1}{n} & \text{if } x \in [0, n) \\
0 & \text{otherwise.} 
\end{cases}
\]

Then \( \lim_{n \to \infty} g_n(x) \leq \lim_{n \to \infty} 1/n = 0 \). While

\[
\int_0^\infty g_n(x) \, dx = 1.
\]

4. (10 points) Find the values of \( x \in \mathbb{R} \) so that the series

\[
\sum_{n=1}^{\infty} \left( \frac{x^n}{n+1} - \frac{1}{n} \right)
\]

converges (prove your answer).

**Solution:** If \( |x| > 1 \), then

\[
\lim_{n \to \infty} \left( \frac{x^n}{n+1} - \frac{1}{n} \right) = \infty
\]

and the series diverges.

If \( |x| < 1 \), then for large \( n \), eventually \( |x|^n < \frac{1}{2} \cdot \frac{n+1}{n} \). So

\[
\left| \frac{x^n}{n+1} - \frac{1}{n} \right| > \frac{1}{2n},
\]

which diverges. So by the comparison test, the series diverges.

If \( x = 1 \), then \( \frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)} \), which converges.
If \( x = -1 \), the series odd terms are all negative and bounded above by \( \frac{1}{n} \). So the series diverges. The even terms converges since they correspond to the \( x = 1 \) case. So the entire series diverges.

5. (a) (5 points) Find the power series and radius of convergence for \( \int_0^x \log(1+t)dt \) around \( c = 0 \) (prove your answer).

Solution: Around \( t = 0 \),

\[
\log(1 + t) = \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n},
\]

with radius of convergence 1. So we may integrate this series to get the power series for \( \int_0^x \log(1+t)dt \) with the same radius of convergence. Thus

\[
\int_0^x \log(1+t)dt = \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{n(n+1)}.
\]

(b) (5 points) Use your answer in the previous part to find a function whose derivative is \( \log(1 + x) \) (do not write your answer as a power series). Hint:

\[
\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.
\]

Solution: By the hint,

\[
\int_0^x \log(1+t)dt = \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n} - \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n + 1}.
\]

Both of these series look like \( \log(1 + t) \). Manipulate these series so they become \( \log(1 + t) \):

\[
\sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n} - \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n + 1} = x \sum_{n=1}^{\infty} \frac{x^n}{n} + \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} - x
\]

\[
= x \log(1 + x) + \log(1 + x) - x.
\]
6. (10 points) Let $f: [0, 1] \to \mathbb{R}$ satisfy for every $x, y \in [0, 1]$,

$$|f(x) - f(y)| \leq C|x - y|^{\alpha},$$

where $C > 0$ and $\alpha > 1$. Prove that $f$ is constant.

**Solution:** The derivative of $f$ is 0:

$$\lim_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim_{y \to x} C|x - y|^{\alpha-1} = 0,$$

since $\alpha > 1$. So $f$ is constant.

7. (10 points) Let $f: [a, b] \to \mathbb{R}$ be an integrable function. For a partition $P$ of $[a, b]$ define the left Riemann sum as $A(f, P) = \sum_{i=0}^{n} f(t_i)(t_{i+1} - t_i)$, where $P = \{t_0, \ldots, t_n\}$. Let $P_k$ be a sequence of partitions defined as $P_k = \{a, a + (b-a)/k, a + 2(b-a)/k, \ldots, b\}$. Prove that

$$\lim_{k \to \infty} A(f, P_k) = \int_{a}^{b} f(x)dx.$$

**Solution:** The left Riemann sum $A(f, P_k)$ satisfies

$$L(f, P_k) \leq A(f, P_k) \leq U(f, P_k),$$

where $L(f, P)$ and $U(f, P)$ are the lower and upper Riemann sums respectively. Since $f$ is Riemann integrable,

$$\lim_{k \to \infty} L(f, P_k) = \lim_{k \to \infty} U(f, P_k) = \int_{a}^{b} f.$$

By the squeeze theorem,

$$\lim_{k \to \infty} A(f, P_k) = \int_{a}^{b} f.$$
8. (10 points) Let \( a_n \) and \( b_n \) be sequences. Prove that if \( \lim_{n \to \infty} a_n = A \) and \( \lim_{n \to \infty} b_n = B \), then

\[
\lim_{n \to \infty} a_n b_n = AB.
\]

**Solution:** Since \((a_n)_{n \in \mathbb{N}}\) converges, it is bounded. Let \( M > 0 \) satisfy \( |a_n| \leq M \) for \( n \in \mathbb{N} \). Fix \( \epsilon > 0 \) and choose \( N \in \mathbb{N} \) so that for \( n \geq N \), \( |a_n - a| \leq \epsilon/(2B) \) and \( |b_n - b| \leq \epsilon/(2M) \). By the triangle inequality,

\[
|a_n b_n - AB| \leq |a_n b_n - a_n B| + |a_n B - AB|
\leq |M||b_n - b| + |B||a_n - a|
\leq \epsilon,
\]

for \( n \geq N \).

9. (10 points) Use induction to show that for \( n \in \mathbb{N} \),

\[
x^n - 1 = (x - 1) \sum_{k=0}^{n-1} x^k.
\]

**Solution:** If \( n = 1 \), then both sides are \( x - 1 \). Suppose the statement is true for \( n \). For \( n + 1 \), by the induction hypothesis,

\[
x^{n+1} - 1 = x^{n+1} - x^n + x^n - 1
\]

\[
x^n(x - 1) + (x - 1) \sum_{k=0}^{n-1} x^k
\]

\[
= (x - 1) \sum_{k=0}^{n} x^k.
\]