Final Solutions

1. (5 points) State the Weierstrass M-test.

   Solution: If $f_n\colon [a,b] \to \mathbb{R}$ are functions that satisfy $|f_n| \le M_n$, for $M_n \ge 0$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

2. (5 points) Suppose that $a_n \ge 0$ and $b_n \ge 0$ are sequences such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Show that

$$\sum_{n=1}^{\infty} a_n b_n$$

   converges.

   Solution: Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$. In particular the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded. Suppose that $a_n \le M$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} a_n b_n \le M \sum_{n=1}^{\infty} b_n < \infty.$$

   Since $a_n b_n \ge 0$, the comparison test applies and the sum converges.

3. (a) (5 points) Give an example of a sequence of integrable functions $f_n\colon [0,1] \to \mathbb{R}$ so that $f_n \to 0$ pointwise, but there exists $C \neq 0$ so that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = C$ (prove your answer).

   Solution: Define f_n as follows:

$$f_n(x) = \begin{cases} n & \text{if } x \in (0, 1/n) \\ 0 & \text{otherwise.} \end{cases}$$

   The limit of $f_n(x) = 0$ as $n \to \infty$ for any fixed $x \in [0,1]$ since, eventually, $x > 1/n$. While

$$\int_0^1 f_n(x)dx = 1$$

   for all $n \in \mathbb{N}$.

(b) (5 points) Give an example of a sequence of integrable functions $g_n\colon [0,\infty) \to \mathbb{R}$ so that there exists $M > 0$ with $|g_n(x)| \leq M$, $g_n \to 0$ pointwise, but $\lim_{n\to\infty} \int_0^\infty g_n(x)dx$ either does not exist or converges to a value that is not 0 (prove your answer).

Solution: Define g_n as follows:

$$g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in [0,n) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lim_{n\to\infty} g_n(x) \leq \lim_{n\to\infty} 1/n = 0$. While

$$\int_0^\infty g_n(x)dx = 1.$$

4. (10 points) Find the values of $x \in \mathbb{R}$ so that the series

$$\sum_{n=1}^{\infty} \left(\frac{x^n}{n+1} - \frac{1}{n} \right)$$

converges (prove your answer).

Solution: If $|x| > 1$, then

$$\lim_{n\to\infty} \left(\frac{x^n}{n+1} - \frac{1}{n} \right) = \infty$$

and the series diverges.

If $|x| < 1$, then for large n, eventually $|x|^n < \frac{1}{2}\frac{n+1}{n}$. So

$$\left| \frac{x^n}{n+1} - \frac{1}{n} \right| > \frac{1}{2n},$$

which diverges. So by the comparison test, the series diverges.

If $x = 1$, then $\frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)}$, which converges.

If $x = -1$, the series odd terms are all negative and bounded above by $\frac{-1}{n}$. So the series diverges. The even terms converges since they correspond to the $x = 1$ case. So the entire series diverges.

5. (a) (5 points) Find the power series and radius of convergence for $\int_0^x \log(1+t)dt$ around $c = 0$ (prove your answer).

Solution: Around $t = 0$,

$$\log(1+t) = \sum_{n=1}^{\infty} (-1)^n \frac{t^n}{n},$$

with radius of convergence 1. So we may integrate this series to get the power series for $\int_0^x \log(1+t)dt$ with the same radius of convergence. Thus

$$\int_0^x \log(1+t)dt = \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n(n+1)}.$$

(b) (5 points) Use your answer in the previous part to find a function whose derivative is $\log(1+x)$ (do not write your answer as a power series). **Hint:** $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

Solution: By the hint,

$$\int_0^x \log(1+t)dt = \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n} - \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

Both of these series look like $\log(1+t)$. Manipulate these series so they become $\log(1+t)$:

$$\begin{aligned}\sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n} - \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} &= x \sum_{n=1}^{\infty} \frac{x^n}{n} + \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} - x \\ &= x\log(1+x) + \log(1+x) - x.\end{aligned}$$

6. (10 points) Let $f\colon [0,1] \to \mathbb{R}$ satisfy for every $x, y \in [0,1]$,

$$|f(x) - f(y)| \leq C|x-y|^\alpha,$$

where $C > 0$ and $\alpha > 1$. Prove that f is constant.

Solution: The derivative of f is 0:

$$\lim_{y \to x} \frac{|f(x) - f(y)|}{|x-y|} \leq \lim_{y \to x} C|x-y|^{\alpha-1} = 0,$$

since $\alpha > 1$. So f is constant.

7. (10 points) Let $f\colon [a,b] \to \mathbb{R}$ be an integrable function. For a partition P of $[a,b]$ define the *left Riemann sum* as $A(f,P) = \sum_{i=0}^{n} f(t_i)(t_{i+1} - t_i)$, where $P = \{t_0, \ldots, t_n\}$. Let P_k be a sequence of partitions defined as $P_k = \{a, a + \left(\frac{b-a}{k}\right), a + 2\left(\frac{b-a}{k}\right), \ldots, b\}$. Prove that

$$\lim_{k \to \infty} A(f, P_k) = \int_a^b f(x)dx.$$

Solution: The left Riemann sum $A(f, P_k)$ satisfies

$$L(f, P_k) \leq A(f, L_k) \leq U(f, P_k),$$

where $L(f,P)$ and $U(f,P)$ are the lower and upper Riemann sums respectively. Since f is Riemann integrable,

$$\lim_{k \to \infty} L(f, P_k) = \lim_{k \to \infty} U(f, P_k) = \int_a^b f.$$

By the squeeze theorem,

$$\lim_{k \to \infty} A(f, P_k) = \int_a^b f.$$

8. (10 points) Let a_n and b_n be sequences. Prove that if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

$$\lim_{n\to\infty} a_n b_n = AB.$$

Solution: Since $(a_n)_{n\in\mathbb{N}}$ converges, it is bounded. Let $M > 0$ satisfy $|a_n| \leq M$ for $n \in \mathbb{N}$. Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ so that for $n \geq N$, $|a_n - a| \leq \epsilon/(2B)$ and $|b_n - b| \leq \epsilon/(2M)$. By the triangle inequality,

$$\begin{aligned} |a_n b_n - AB| &\leq |a_n b_n - a_n B| + |a_n B - AB| \\ &\leq |M||b_n - b| + |B||a_n - a| \\ &\leq \epsilon, \end{aligned}$$

for $n \geq N$.

9. (10 points) Use induction to show that for $n \in \mathbb{N}$,

$$x^n - 1 = (x-1)\sum_{k=0}^{n-1} x^k.$$

Solution: If $n = 1$, then both sides are $x - 1$. Suppose the statement is true for n. For $n + 1$, by the induction hypothesis,

$$\begin{aligned} x^{n+1} - 1 &= x^{n+1} - x^n + x^n - 1 \\ & x^n(x-1) + (x-1)\sum_{k=0}^{n-1} x^k \\ &= (x-1)\sum_{k=0}^{n} x^k. \end{aligned}$$