FRS 157: PROBLEM SET SOLUTIONS

Problem Set 1. Problem 1: Whittlesey, p.33: Exercises 1, 6

Solution: Exercise 1 has solutions in the book. Exercise 2: Suppose A, B and C are points on a sphere that do not lie on a great circle. A line intersects a sphere at most at two points so A, B and C are not on a line. Therefore there is a unique plane P that contains A, B and C. By Proposition 5.2, P either does not intersect the sphere, intersects the spehre in a single point or intersects the sphere in a circle. Since P contains A, B and C, it must intersect the sphere in a circle. Therefore A, B and C lie on a circle in the sphere. A circle is defined by 3 points and so it is unique. It is not a great circle by assumption.

Problem 2: Find the polar and latitude and longitude coordinates for the following points on the sphere of radius 1:

(a) $(1, 0, 0)$
(b) $(0, 1, 0)$
(c) $(0, 0, 1)$
(d) $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$
(e) $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

Solution:

(a) Polar: $(0, \pi/2)$, L and L: $(0, 0)$.
(b) Polar: $(\pi/2, \pi/2)$, L and L: $(\pi/2, 0)$.
(c) Polar: $(0, 0)$, L and L: $(0, \pi/2)$.
(d) Polar: $(\pi/4, \pi/2)$, L and L: $(\pi/4, 0)$.
(e) Polar: $(\pi/4, \cos^{-1}(1/\sqrt{3}))$, L and L: $(\pi/4, \sin^{-1}(1/\sqrt{3}))$.

Problem 3: If a great circle A on the sphere of radius 1 meets the equator at the point $(1, 0, 0)$ at an angle θ, find the other point on A that meets the equator. Find the two points on A that are farthest from the equator.

Solution: The intersection of two great circles gives antipodal points. So the other point on A that meets the equator is $(-1, 0, 0)$. The two points that are farthest from the equator will have no x-coordinate. So they are $(0, \cos\theta, \sin\theta)$ and $(0, -\cos\theta, -\sin\theta)$.

Problem Set 2. Problem 1: Suppose the Earth is a perfect sphere with radius R. Suppose that an observer is standing on a beach and looking towards the horizon.

(a) Draw a diagram that indicates how far a person of height h can see into the horizon.

(b) Use your diagram in part (a) to compute the distance an observer can see given h and R.

(c) If $R = 6371$km and $h = 2$m, use your answer in (b) to compute the distance an observer can see into the horizon. Compare this to if an observer is standing on a mountain so that $h = 2000$m.

Solution:

(b) If θ is the angle of how far a person can see, then $\cos\theta = \frac{R}{h+R}$. The distance that can be seen is then θR.

(c) Solve $\cos\theta = 6371/6371.002$. Then $\theta R \approx 5.05$ km.

Solve $\cos\theta = 6371/6373$. Then $\theta R \approx 159.62$ km.

Problem 2 (Diurnal Parallax): Let the radius of the Earth be R.

(a) Draw a diagram to indicate how to use parallax and the celestial sphere's rotation to measure the distance of a celestial object.

(b) Give a formula for the distance of the object to an observer based on two observations of the observer. The formula should only depend on R, the angle between the observations and the observed parallax angle.

(c) If the observer can only look with the naked eye and so can only detect differences of up to 1°, what is the farthest object for which the observer can tell its distance using this method?

Solution:

(b) The distance D to the celestial object can be expressed as

$$D = \frac{R \sin theta}{\sin\phi},$$

where R is the radius of the Earth, θ is the angle of rotations and ϕ is the observed difference.

(c) The largest θ can be chosen is $\pi/2$. Then the larges distance is when $\phi = 1°$ and so $D = \frac{R}{\sin 1°} \approx 365050$ km.

Problem Set 3. Problem 1: A simple model for the motion of a celestial object is the following: Suppose that the object is moving uniformly at a rate r_1 in a great circle that intersects the ecliptic at points A and B at an angle θ. Additionally, suppose that the points A and B move along the ecliptic at a uniform rate r_2.

(a) If the object starts at time 0 at the point $A = (0, 0)$ in longitude and latitude coordinates, what is the object's location at time t? **Hint:** First find where the object will be on a fixed great circle, then find where A and B will be after time t.

(b) If the sun moves along the ecliptic at a uniform rate r_3, find the times that the celestial object will coincide with the sun and the times when the object will be opposite the sun (these would correspond to solar and lunar eclipses respectively).

Solution:

(a) We will solve this problem in ecliptic coordinates. So the ecliptic is placed at the xy-plane of our coordinate system. Thus A has the following equation of motion:

$$A(t) = (\cos r_2 t, \sin r_2 t, 0).$$

To find the motion of the celestia, object we first find the principal vectors for the great circle. The first principle vector $v_1(t) = A(t)$. The second principle vector makes angle θ with the xy-plane and is orthogonal to v_1. This gives that

$$v_2(t) = (-\cos\theta \sin(r_2 t), \cos\theta \cos(r_2 t), \sin\theta).$$

The equation of motion for the celestial object is

$$\begin{aligned} c(t) &= v_1(t)\cos(r_1 t) + v_2(t)\sin(r_1 t) \\ &= (\cos(r_2 t)\cos(r_1 t) - \cos\theta \sin(r_2 t)\sin(r_1 t), \sin(r_2 t)\cos(r_1 t) + \cos\theta\cos(r_2 t)\sin(r_1 t), \sin\theta \sin(r_1 t)). \end{aligned}$$

In order to get the longitude and latitude coordinate you need to rotate this equation by the ecliptic angle.

(b) The equation of the sun in ecliptic coordinates is

$$s(t) = (\cos r_3 t, \sin r_3 t, 0).$$

We must solve $c(t) = \pm s(t)$. The z-coordinate is simplest and gives that

$$t = \frac{\pi}{r_1} k,$$

where k is any integer. Plugging this in to the x-coordinate,

$$\pm \cos(\frac{r_2}{r_1}\pi k) = \cos(\frac{r_3}{r_1}\pi k).$$

In the y-coordinate,

$$\pm \sin(\frac{r_2}{r_1}\pi k) = \sin(\frac{r_3}{r_1}\pi k).$$

Problem 2: Finish the computations done in class to prove that

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

and

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Hint: Use the property of exponentials, $e^{i(x+y)} = e^{ix}e^{iy}$.

Solution:

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$$

and

$$e^{ix}e^{iy} = \cos(x)\cos(y) - \sin(x)\sin(y) + i(\cos(x)\sin(y) + \sin(x)\cos(y)).$$

Equating the real and imaginary parts gives the equations.

Problem 3: Whittlesey, Section 21, Exercises 6 and 7.