IN CLASS PROBLEMS

Spherical Geometry. Problem 1: Let (x, y) be a point in the plane.

(a) Let r be the distance of (x, y) to $(0, 0)$. What is a formula for r?

(b) Let θ be the angle to the x-axis of the line from $(0, 0)$ to (x, y). What is a formula for θ?

(c) Find a formula of x in terms of r and θ. Find a formula of y in terms of r and θ.

(d) A circle centered at $(0, 0)$ of radius r is defined by the equation $x^2 + y^2 = r^2$. Every point on the circle can be determined by a single parameter θ. Explain why this is the case.

Problem 2: Let (x, y, z) be a point in space.

(a) Let ρ be the distance of (x, y, z) to $(0, 0, 0)$. What is a formula for ρ?

(b) Let θ be the angle to the x-axis of the line from $(0, 0, 0)$ to (x, y, z). What is a formula for θ?

(c) Let ϕ be the angle to the z-axis of the line from $(0, 0, 0)$ to (x, y, z). What is the formula for ϕ.

(c) Find a formula of x, y and z in terms of ρ, ϕ and θ.

(d) A sphere centered at $(0, 0, 0)$ of radius ρ is defined by the equation $x^2 + y^2 + z^2 = \rho^2$. Every point on the sphere can be determined by two parameters θ and ϕ. Explain why this is the case.

Celestial Sphere. Problem 3: The celestial sphere rotates around a north pole. Give an informal argument for why the angle between the zenith (the point directly above you) and the north pole of the celestial sphere is 90° - your latitude.

Problem 4: Suppose that the sun moves along the ecliptic at a uniform rate and returns to its original location after 365.25 days. Assume that the ecliptic makes a 23.5° angle with the celestial equator and the celestial sphere makes a full revolution every day.

(a) If you are an observer on the North Pole, where will the sun be located 10 days after the spring equinox at noon (when the sun is directly on the horizon)? 100 days?

(b) Repeat the calculation in part (a) if you are an observer at a latitude of 40°.

Trigonometry. Problem 5: The law of cosines states that for any triangle ΔABC, with sides a, b and c,

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

We will prove this law:

(a) We can assume that the point $C = (0,0)$ and $B = (a,0)$, why is this the case?

(b) What is a formula for the point A, in terms of the angle at C and b?

(c) The distance between two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Find the length c using this formula and the answer in (b).

(d) Deduce the law of cosines from (c).

Problem 6: The law of sines states that for any triangle ΔABC,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

We will prove this law:

(a) Draw the triangle and its altitude, whose length is h.

(b) Find an expression for h in terms of a side and the sine of an angle.

(c) By symmetry, there should be another expression for h with a different side and angle. What is it?

(d) Use (b) and (c) to deduce the law of sines.

Problem 7: There are many laws in trigonometry that give formulas for sine and cosine of the sum of angles, difference of angles, etc. This problem gives a technique to deduce these laws without using any geometry. In mathematics, it is often useful to introduce the number $i := \sqrt{-1}$. The number i satisfies the same laws as any number and additionally $i^2 = -1$.

(a) Show that

$$(x + iy)(u + iv) = xu - yv + i(yu + xv).$$

The famous formula due to Euler states that:

$$e^{i\theta} = \sin\theta + i\cos\theta$$

(b) Use this formula and the laws of exponents to deduce a formula for $\cos(x+y)$ and $\sin(x+y)$.