3

The Ptolemaic universe

Hipparchus

Hipparchus, who lived in the second century BC,[1] built an observatory and performed most of his work on the island of Rhodes and was perhaps the greatest astronomer of antiquity. He used observations to produce geometrical models with real quantitative predictive power. His theory of the motion of the Sun was extremely accurate and he produced a model for the Moon that worked well at new and full moons, thus enabling him to produce a theory of eclipses which, in the case of lunar eclipses, was very successful.

All of Hipparchus' works are lost except for his relatively unimportant *Commentary on the Phaenomena of Eudoxus and Aratus*,[2] though Ptolemy quotes his work often, sometimes verbatim. We also know of Hipparchus' work on the Sun through an introduction to astronomy written in AD first century by Geminus of Rhodes. Part of the reason for the lack of extant work by Hipparchus may well be the fact that Ptolemy's subsequent writings superseded those of his predecessor so totally, just as the existence of Euclid's *Elements* rendered obsolete all previous works on geometry.

Hipparchus attempted to use the eccentric circles and epicycles of Apollonius to develop models for the motion of the heavenly bodies that, in contrast to Babylonian theories, would enable future positions to be calculated for all times. For the Sun and the Moon he found that he could use just one such device, but for the planets he needed to combine the two. He was a great

[1] Ptolemy refers to observations made by Hipparchus between 161 and 126 BC.
[2] In about 275 BC, Aratus of Soli wrote a very popular poem (the *Phaenomena*, inspired by a more technical, but now lost, work of Eudoxus) describing the risings and settings of stars and weather signs in both heavenly and natural phenomena. It was later translated into Latin and remained widely read for over 1000 years. Kidd (1997) contains a translation and commentary.

observational astronomer who improved the design of the instruments used for observing the skies and used these instruments to compile a catalogue of about 850 stars.

As we have seen, quantitative calculations that arose from astronomical problems often involved the solution of triangles, and it was for this reason that the subject of trigonometry developed. In fact, trigonometry did not become a branch of mathematics separate from astronomy until the fifteenth century. Hipparchus, who is considered to be the founder of trigonometry, constructed a table of chords (equivalent to a table of sines) though we do not know how he did this.[3] He subdivided the circle into 360°, an idea introduced into Greek astronomy from Babylonia through the work of his contemporary, Hypsicles of Alexandria.

One of Hipparchus' greatest achievements was his discovery of the precession of the equinoxes. He discovered that the points at which the ecliptic crosses the celestial equator move slowly with respect to the stars. He was able to do this because Babylonian astronomical data became available in the Greek world and he examined systematically old observations and compared them with his new ones in order to discover changes that were too slow to be detected by astronomers using only data gathered during their own lifetimes. The arrival of large quantities of data covering observations made over many centuries helped transform Greek astronomy. It was now possible to take the geometrical models that had been developed and use them to produce procedures for accurate quantitative prediction. The work of Hipparchus, based as it was on a merging of Greek and Babylonian approaches, marks the transition between qualitative and quantitative mathematical astronomy.[4]

Hipparchus' contributions to astronomy were enormous. In the words of the French astronomer J.-B. J. Delambre:

> When we consider all that Hipparchus invented or perfected, and reflect upon the number of his works and the mass of calculations which they imply, we must regard him as one of the most astonishing men of antiquity, and as the greatest of all in the sciences which are not purely speculative, and which require a combination of geometrical knowledge with a knowledge of phenomena, to be observed only by diligent attention and refined instruments.[5]

[3] Toomer (1974) contains a plausible reconstruction. Ptolemy also constructed a table of chords and his method is preserved. This will be described later.

[4] Babylonian arithmetical methods continued to be used in the Greek world right up to Ptolemy's time. Indeed, Hipparchus used them to compute both solar and lunar longitudes while developing his geometrical schemes (see Jones (1991a)).

[5] Delambre *L'histoire de l'astronomie ancienne*, I (1817). Translation from Berry (1961).

This is probably a bit over the top. Delambre tends to credit Hipparchus with things that most modern scholars attribute to his successor, Ptolemy.

The distances of the Sun and Moon

In *On Sizes and Distances*, a work described by Pappus of Alexandria in his commentary on Ptolemy's *Almagest*, Hipparchus calculated the distances of the Sun and Moon from the Earth, measured in Earth radii. His method was based around the concept of diurnal solar parallax.

Just like the stellar parallax in a heliocentric universe (see Figure 2.6, p. 40), the diurnal parallax is the change in the apparent position of an object (in this case the Sun) as a result of the position of the observer. With stellar parallax it is the position of the Earth in its orbit that causes the difference, whereas for diurnal parallax it is the position of the observer on the Earth that is the cause. Thus, the longitude of the object O in Figure 3.1 (which can be thought of as a view from the celestial north pole) differs depending upon whether it is viewed from P_1 or P_2, two points at different positions on the Earth. Two observers are not required to determine this difference, however, since the object O takes part in the daily rotation of the heavens and its longitude will thus be affected by the time of day at which measurements are taken. Thus, if one makes observations of the sun 6 h apart, for example, the longitudes will not differ by the amount due to the motion of the Sun alone (which can, in principle, be calculated), but there will be a small error (the angle marked in the diagram) corresponding to the diurnal parallax. The solar parallax is a direct measurement of the distance of the Sun from the Earth in terms of the size of the Earth – the greater the distance of the Sun the smaller the parallax – but unfortunately Hipparchus was unable to measure it due to its very small magnitude.

While it is true that Hipparchus could not measure any solar parallax, he knew this did not mean it did not exist. Accordingly, he assumed a solar parallax of $7'$, on the basis that if it were bigger he would be able to measure it. From Figure 3.1,

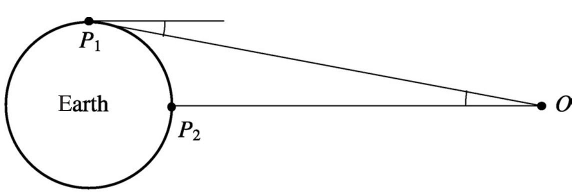

Fig. 3.1. Diurnal parallax.

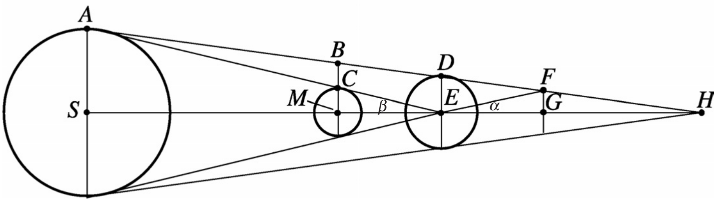

Fig. 3.2. Hipparchus' method for determining the relative distances of the Sun and Moon.

we see that if the marked angle is $7'$, the distance to the Sun is $1/\tan 7' \approx 490$ Earth radii, and this is the value Hipparchus used. He then obtained, using data from lunar eclipses, a lunar distance of $67\frac{1}{3}$ Earth radii (which is just over 10 per cent too big) and hence, a ratio of the Sun's distance to the Moon's distance of about $7\frac{1}{4}$ (which is worse than Aristarchus' estimate). He also obtained the lunar and solar radii as $\frac{1}{3}$ and $2\frac{1}{3}$ Earth radii, respectively, the former being quite close to the modern value of 0.27, although the latter is about 50 times too small.

The method Hipparchus used is described by Ptolemy and is illustrated in Figure 3.2.[6] The centres of the Sun, Moon and Earth are S, M and E, respectively, and the point G is on SH such that $|EG| = |EM|$. The lines SA, MC and ED are all perpendicular to SH and intersect the circles representing the Sun, Moon and Earth in A, C and D, respectively. The lines ADH and ACE are then good approximations to the lines tangent to the Sun and Earth, and Sun and Moon. The angle β (the apparent radius of the Moon) was determined through observation, and then Hipparchus determined the length of the Earth's shadow from the length of lunar eclipses. He took $\alpha/\beta = 2\frac{1}{2}$, where previously Aristarchus had used 2.

It is convenient to work in units of Earth radii so that $|ED| = 1$. First, we note that in reality α and β are very small, so that to a good approximation $\alpha/\beta = \tan\alpha/\tan\beta = |FG|/|MC|$. Now, MB, ED and FG are parallel so that $|MB| + |FG| = 2|ED| = 2$, and, hence,

$$|BC| = |MB| - |MC| = 2 - |FG| - |MC| = 2 - \left(\frac{\alpha}{\beta} + 1\right)|MC|. \tag{3.1}$$

Finally, we use similar triangles to obtain the result

$$\frac{1}{|BC|} = \frac{|EA|}{|CA|} = \frac{|ES|}{|MS|} = \frac{|ES|}{|ES| - |EM|}$$

[6] Ptolemy *Almagest*, Book V, 11. Further details can be found in Swerdlow (1969).

from which, using Eqn (3.1) and $|MC| = |EM| \tan\beta \approx |EM| \sin\beta$, we obtain

$$|ES| = \frac{|EM|}{\left(\frac{\alpha}{\beta} + 1\right) |EM| \sin\beta - 1}.$$

This expression is rearranged easily so as to give $|EM|$ in terms of $|ES|$. Hipparchus substituted into this expression the vales $|ES| = 490$, $\beta = 0; 16, 37^\circ$ and $\alpha/\beta = 2\frac{1}{2}$, from which he obtained the result $|EM| = 67\frac{1}{3}$ Earth radii.

The motion of the Sun and precession

Perhaps Hipparchus' greatest contributions concerned the motion of the Sun. In *On the Length of the Year* he claimed that the length of the tropical year (the time between identical equinoxes or solstices) was constant, and he measured it at $365 + \frac{1}{4} - \frac{1}{300}$ days (365 days 5 h 55 min 12 s) which exceeds the modern value by about $6\frac{1}{2}$ minutes but, nevertheless, represents a significant improvement over the previous value of $365\frac{1}{4}$ days. This probably was done by taking the Babylonian value for the synodic month (i.e. 29; 31, 50, 8, 20 days) and using the approximate equality given by the Metonic cycle (i.e. 19 years = 235 months) and then checking the result with observations.[7]

Hipparchus was the first to attempt to calculate the parameters needed for the eccentric circle theory of Apollonius to agree with observations of the Sun's position, and his model of the solar motion, and the basic principles by which the parameters for the model were deduced, remained standard until the seventeenth century. The success of Hipparchus' solar model was due to being mathematically simple and yet very accurate. Provided the parameters in the model are calculated accurately, the errors in the predicted solar longitudes will not be detectable from naked-eye observations. Hipparchus' method for determining the parameters is illustrated in Figure 3.3, in which E is the Earth and the Sun S rotates around an eccentric circle centre O with an angular speed of $\omega = 1$ revolution per year $\approx 59' \, 8''$ per day. The points P_1, P_2, P_3, and P_4 represent the position of the Sun at the vernal equinox, the summer solstice, the autumnal equinox and the winter solstice, respectively. In order to be able to use the model to compute the position of the Sun, we need to calculate the longitude λ of the apogee A of the orbit of the Sun (the apogee is the point on the orbit furthest from the Earth), which we choose to measure from the vernal equinox, and the ratio of the eccentricity $|EO|$ to the radius $|OS|$ of the orbit of the Sun.

[7] See Swerdlow (1979).

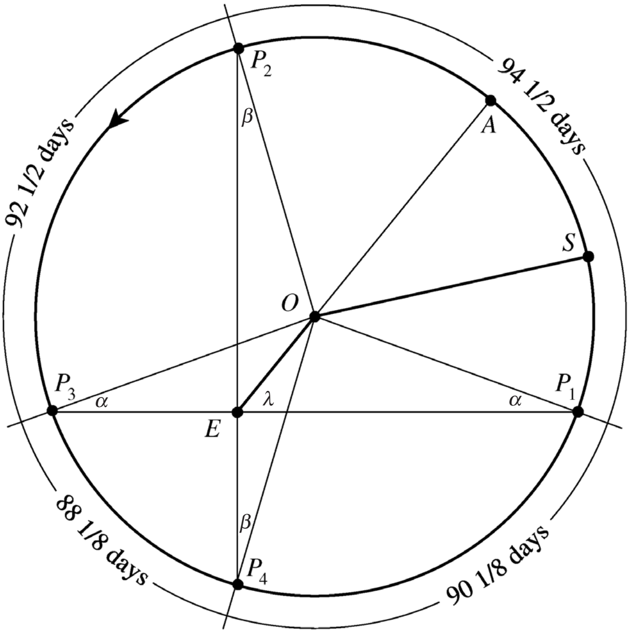

Fig. 3.3. Hipparchus' solar theory.

The technique Hipparchus used to do this can be described using modern trigonometry as follows. First, consider the triangles EOP_1 and EOP_2. The sine rule gives

$$\frac{|OP_1|}{\sin\lambda} = \frac{|OE|}{\sin\alpha} \qquad \text{and} \qquad \frac{|OP_2|}{\cos\lambda} = \frac{|OE|}{\sin\beta}.$$

From these equations it follows, since $|OP_1| = |OP_2| = |OS|$, that

$$\tan\lambda = \frac{\sin\alpha}{\sin\beta} \qquad \text{and} \qquad \frac{|EO|}{|OS|} = \frac{\sin\alpha}{\sin\lambda}. \tag{3.2}$$

To obtain λ and $|EO|/|OS|$ from these equations, Hipparchus needed to know the time between the vernal equinox and the summer solstice and that between the summer solstice and the autumnal equinox, and from his observations he was able to improve the known values for the lengths of the seasons. He measured the time taken for the Sun to travel from P_1 to P_2 as $94\frac{1}{2}$ days and from P_2 to P_3 as $92\frac{1}{2}$ days (the times from P_3 to P_4 and from P_4 to P_1 being $88\frac{1}{8}$ and $90\frac{1}{8}$ days, respectively). Hence, since $\angle P_1OP_2 = \alpha + \beta + 90^\circ$ and $\angle P_2OP_3 = \alpha - \beta + 90^\circ$,

$$\alpha + \beta + 90^\circ = \frac{189\omega}{2} \qquad \text{and} \qquad 2\alpha + 180^\circ = 187\omega,$$

simultaneous equations that can be solved for α and β. Finally Eqn (3.2) can be used to obtain the needed parameters. Hipparchus obtained the values

$$\lambda = 65^\circ\, 30' \qquad \text{and} \qquad \frac{|EO|}{|OS|} = \frac{1}{24}$$

which, given the rudimentary trigonometry at his disposal, are fairly impressive ($65^\circ\, 25'\, 39''$ and $1/24.17$ are more accurate solutions to the equations). Modern computations show that in Hipparchus' time the true value of the longitude of the Sun's apogee was nearer to 66°.[8]

Hipparchus made another hugely significant discovery concerning the Sun's motion. By comparing his own observations of the longitudes of certain stars with those made by Timocharis some 150 years previously, he discovered the phenomenon now known as the 'precession of the equinoxes'.[9] He noticed that the sidereal year (the time for the Sun to return to a particular fixed star) was slightly greater than the tropical year, and attributed this to a slow rotation of the stars, from west to east, about the poles of the ecliptic. A consequence of this is that the positions of the equinoxes on the celestial sphere gradually shift with time. According to Ptolemy,[10] Hipparchus gave the value of this motion as 1° per century (and Ptolemy used this value), although the actual value is about 1° every 72 years. This numerical inaccuracy was to have a significant influence over subsequent theories since, when precession was measured more accurately about 1000 years later, the differing values led many astronomers to believe that the rate of precession was a variable quantity.

The motion of the Moon

The motion of the Moon is much more complex than that of the Sun and it cannot be described by a simple eccentric circle mechanism. The reasons for this were known to Hipparchus and are stated by Ptolemy:[11] they are that first the Moon moves with varying speed in such a way that over the course of time it achieves its maximal speed for every value of its longitude λ, and, second, that

8 More details of how Hipparchus arrived at his solar model can be found in Jones (1991b), and Maeyama (1998) has analysed the effect of observational errors on the underlying parameters and hence on the predictions of the model. According to Jacobsen (1999) the maximum error in Hipparchus' theoretical values for the Sun's longitude was about $22'$.

9 A theory developed in the 1920s that credits the Babylonians with the discovery of precession was shown to be false by Neugebauer (1950).

10 Ptolemy *Almagest*, Book VII, 2. Ptolemy stated that Hipparchus wrote a work (now lost) entitled *On the Displacement of the Solstitial and Equinoctial Points*. Surprisingly perhaps, Hipparchus' discovery was mentioned by only a few Greek writers (see Dreyer (1953), p. 203); it took on a much greater significance in later centuries.

11 Ptolemy *Almagest*, Book IV, 2.

the Moon does not move on the ecliptic, but instead has latitudes which vary between $\pm 5^\circ$ of the ecliptic in such a way that the Moon achieves its maximal latitude for every value of λ. It is a simple matter to see that a model like that used by Hipparchus for the Sun will lead always to the maximal speed occurring for the same value of the longitude (at the perigee – the point on the orbit closest to the Earth) and so cannot account for the first of these observations. The second observation suggests that the orbit of the Moon is inclined at an angle of about 5° to the ecliptic. However, if the line joining the intersections of the Moon's orbit and the ecliptic is fixed, the maximum latitude always will occur for the same value of the longitude which is observed not to be the case. In order for an eclipse to occur, the Moon must be near one of its nodes and thus observations of eclipses can be used to give fairly accurate information on the nodes of the Moon's orbit. Lunar periods had been computed accurately by the Babylonians, and Hipparchus, by comparing his own observations with earlier ones, confirmed the Babylonian lunar periods as:[12]

$$\begin{aligned}
&1 \text{ synodic month} = 29;31,50,8,20 \text{ days},\\
&251 \text{ synodic months} = 269 \text{ anomalistic months},\\
&5458 \text{ synodic months} = 5923 \text{ draconitic months}.
\end{aligned}$$

Hipparchus' lunar model is illustrated in Figure 3.4. It consists of an epicycle carrying the Moon, M, the centre C of which rotates around a deferent circle, the deferent–epicycle system being inclined at an angle of 5° to the ecliptic and intersecting that circle in the two nodes A and B. The nodal line AOB was made to rotate in order to account for the fact that the longitude of the position of maximum latitude of the Moon changes gradually. The motion on the epicycle ensures that the Moon's speed is variable and by making the period of revolution of M around the epicycle different from the period of C around the deferent, we ensure that the longitude of the position of maximal speed (which occurs when M is at its closest to O) varies over time.

Hipparchus made the simplifying assumption that the motions in latitude caused by the 5° angle of the orbit could be treated separately from the motion in longitude due to the epicyclic system. This introduces only very minor errors. The rotation of the nodal line thus becomes irrelevant for the longitude theory, which is reduced to a simple two-dimensional deferent–epicycle system. In order to use the theory, various parameters have to be computed. Thus, for the longitude calculations, we require the rates of rotation of C around the deferent and M around the epicycle as well as the ratio of the radii of the epicycle and

[12] See Swerdlow and Neugebauer (1984), I, p. 198.

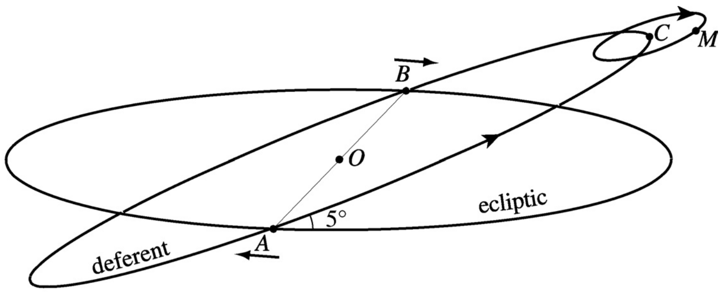

Fig. 3.4. Hipparchus' lunar theory.

deferent, whereas for the latitude calculations we require simply the rate of rotation of the nodal line. This latter parameter can be determined from direct observations of eclipses. It turns out that in order to reproduce the observed behaviour of the Moon, the nodal line must rotate approximately once every 18 years (the so-called 'Saros period').

The rates of rotation of the deferent and the epicycle also are easily obtained. The former is simply the rate required for C to complete 1 revolution in 1 sidereal month which turns out to be about $13^\circ\,10'\,35''$ per day, and the latter is chosen to ensure that the epicycle rotates once in each anomalistic month, which implies a rate of $13^\circ\,3'\,54''$ per day. The final parameter, the ratio of the radii of the epicycle and deferent, presents a far harder problem. Hipparchus developed a geometrical method that enabled this ratio to be obtained from three observations of the Moon[13] and he used it on two different sets of three observations, obtaining different answers. There has been some debate as to what value Hipparchus actually used[14] although, when the lunar model was taken over later by Ptolemy, he computed a ratio of $5\frac{1}{4} : 60$.

Hipparchus' parameters were based on Babylonian observations of lunar eclipses. As a result, the model worked well at full moons but, as Ptolemy demonstrated 300 years later with observations of the Moon at other points in its orbit, it did not work well away from the syzygies. With his solar and lunar theories, Hipparchus created the first coherent theory of eclipses. Durations could be determined about as accurately as they could be measured, but the time at which the eclipse would occur was predicted less well. This situation did not improve substantially until the work of Tycho Brahe at the end of the sixteenth century.

13 See, for example, Pedersen (1974), p. 172.
14 See Neugebauer (1959), Toomer (1967), Pedersen (1974).

The motion of the planets

Hipparchus did not achieve a satisfactory theory for the motion of the five planets. Partly this is due to the fact that he, unlike many of his predecessors, had examined sufficiently many observations over a period of many years to realize that the motions were exceedingly complex, with, for example, retrograde arcs, the lengths of which vary according to the position of the planet in its orbit. As Ptolemy later put it:

> . . . Hence it was, I think, that Hipparchus, being a great lover of truth, for all the above reasons, and especially because he did not yet have in his possession such a groundwork of resources in the form of accurate observations from earlier times as he himself has provided to us, although he investigated the theories of the sun and moon, and, to the best of his ability, demonstrated with every means at his command that they are represented by uniform circular motions, did not even make a beginning in establishing theories for the five planets, not at least in his writings which have come down to us. All that he did was to make a compilation of the planetary observations arranged in a more useful way, and to show by means of these that the phenomena were not in agreement with the hypotheses of the astronomers of that time.[15]

Ptolemy and the *Almagest*

Ptolemy (Latinized as Claudius Ptolemaeus) was one of the great scholars of antiquity, and mathematical astronomy was dominated by his ideas for nearly 1500 years following his death. Little is known of his life, but he taught in Alexandria and quoted the results of his observations made between AD 127 and 141. He was responsible for a number of great works, each of which place him among the most important ancient authors. The earliest of these is his masterpiece of mathematical astronomy, the *Almagest*,[16] and others include the *Tetrabiblos* (on astrology) and the *Geography* (on mathematical geography). These works exercised a colossal influence over mankind for the next 2000 years. Ptolemy's very wide range of interests is indicated by his works on other subjects, e.g. music, optics, and logic. His *Harmony*, in which he described musical consonances and their relationship to an underlying universal harmony, was later an inspiration to Kepler.[17]

The name *Almagest* – which is the name by which his astronomical treatise is usually known – is a corruption by medieval Latin translators of the Arabic

[15] PTOLEMY *Almagest*, Book IX, 2. Translation from Toomer (1984).
[16] All quotations from the *Almagest* are taken from the translation by Toomer (1984).
[17] See Martens (2000), Chapter 6.

word for 'the greatest', and was given to the book long after it was written. The original Greek title translates as *Mathematical Synthesis* or *Mathematical Collection* and the work, which survives in the original Greek, is sometimes referred to as the *Syntaxis*. Most of our knowledge of Greek astronomy is derived from this work, which was originally written around AD150, translated into Latin in the twelfth century, and first printed in the sixteenth century. The huge success of the *Almagest* resulted in the loss of most of the work of Ptolemy's predecessors, notably that of Hipparchus.

Ptolemy was, as far as we know, the first person to show how to convert observational data into numerical values for the parameters so as to make existing planetary models fit the observations. Unlike mathematical astronomers before him, who used unspecified observational data, Ptolemy used specific dated observations – indeed, many of the observations Ptolemy used, which span over 800 years, are only preserved through his work. The rigour with which Ptolemy developed his theories in the *Almagest* set a very high standard for future astronomical works.

The model of the Solar System as set out in the *Almagest* will be described in detail below. The various geometrical models he used are quite complicated and so, before describing these, we will begin with a brief description of the overall structure. In Ptolemy's universe, the spherical Earth is situated at the centre of the heaven, which is itself spherical. The Sun, Moon and planets all orbit the Earth and as to their order, Ptolemy wrote:

> . . . we see that almost all the foremost astronomers agree that all the spheres are closer to the earth than that of the fixed stars, and farther from the earth than that of the moon, and that those of the three [outer planets] are farther from the earth than those of the other [two] and the sun, Saturn's being greatest, Jupiter's the next in order towards the earth, and Mars' below that. But concerning the spheres of Venus and Mercury, we see that they are placed below the sun's by the most ancient astronomers, but by some of their successors these too are placed above [the sun's], for the reason that the sun has never been obscured by them [Venus and Mercury] either. To us, however, such a criterion seems to have an element of uncertainty, since it is possible that some planets might indeed be below the sun, but nevertheless not always be in one of the planes through the sun and our viewpoint, but in another [plane], and hence might not be seen passing in front of it, just as in the case of the moon, when it passes below [the sun] at conjunction, no obscuration results in most cases.[18]

Thus, Ptolemy decided to side with those astronomers who put Mercury and Venus nearer than the Sun as this naturally separates those planets which can be seen at any longitude with respect to the Sun and those which always remain

[18] Ptolemy *Almagest*, Book IX, 1.

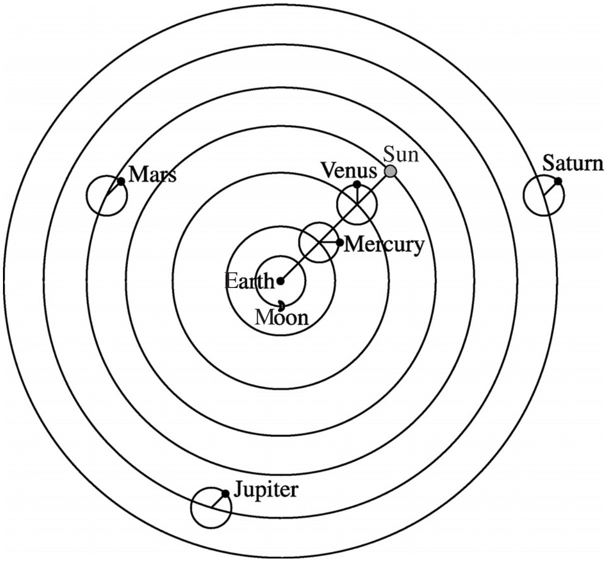

Fig. 3.5. A simplified view of Ptolemy's world system.

close to the Sun. Ptolemy's authority was such that the order of the planets that he proposed (Moon, Mercury, Venus, Sun, Mars, Jupiter, Saturn) was accepted by virtually all subsequent astronomers until the sixteenth century.

In the *Almagest*, the motion of each celestial body is considered in turn and, if all the models are put together, we get the world system sketched in Figure 3.5. The diagram illustrates the epicyclic nature of the planetary models, but not the many other elements used by Ptolemy to reflect more accurately the observational evidence. An important thing to note is the fact that, for Mercury and Venus, the line joining the Earth to the centre of their epicycles is the same as that joining the Earth to the Sun, whereas for the outer planets, which may appear anywhere with respect to the Sun, the radius connecting the planet to the centre of its epicycle is parallel to the Earth–Sun line. Thus, it is evident that the Sun does not simply orbit the Earth like the other planets, but has a much more significant role. The motion of the Sun also plays a role in the lunar theory.

As the Sun has an important function in the theories of all the heavenly bodies, it is logical to begin with a solar theory, and this is what Ptolemy does. He goes on then to consider the Moon and, finally, the planets. For each celestial body, Ptolemy describes the type of phenomena that must be accounted for, goes on to propose a geometrical model suitable for the purpose, shows how to use observations to derive the numerical values of the various geometrical parameters and, finally, produces tables to enable others to determine the position of the body on a given date. Ptolemy obtained data easily for the Sun, Moon, and all the planets from Venus to Saturn, but his over-reliance on the poor available data for Mercury – much the most difficult of the then-known planets to observe – led him to introduce a complicated geometrical device in order accurately to reproduce erroneous data.

The *Almagest* is a complete exposition of Greek mathematical astronomy divided into thirteen books. As Ptolemy himself says:

> We shall try to note down everything which we think we have discovered up to the present time; we shall do this as concisely as possible and in a manner which can be followed by those who have already made some progress in the field. For the sake of completeness in our treatment we shall set out everything useful for the theory of the heavens in the proper order, but to avoid undue length we shall merely recount what has been adequately established by the ancients. However, those topics which have not been dealt with [by our predecessors] at all, or not as usefully as they might have been, will be discussed at length, to the best of our ability.[19]

The first two books deal with the assumptions upon which the work is based and with mathematical methods. Books III and IV deal with the motion of the Sun and Moon, respectively. Book V contains a more advanced lunar theory (and a discussion, among other things, of the construction of an astrolabe) and in Book VI, Ptolemy presents his theory for eclipses. Books VII and VIII are taken up largely with his star catalogue[20] and, finally, Books IX–XIII deal with planetary motions. The organization is strictly logical, each book dependent only on those preceding it.

By late antiquity, the *Almagest* had become the standard textbook on astronomy, and remained as such for more than 1000 years. Perhaps the only scientific work to achieve greater dominance in history is Euclid's *Elements*. Ptolemy does not mention any physical interpretation of his system: his aim

19 PTOLEMY *Almagest*, Book I, 1.

20 In all, Ptolemy tabulated 1022 stars in 48 constellations, giving the longitude, latitude and magnitude of each. He was very interested in the question of whether the stars move in relation to each other and so he gave an extensive list of stars that lay in straight lines so that future observations easily could reveal any relative motion that exists. There is evidence to suggest that Ptolemy took the star catalogue of Hipparchus and added the correction in longitude given by Hipparchus' value for the precession. For the $2\frac{2}{3}$ intervening centuries this would amount to $2^\circ\ 40'$ which gives a good explanation of why Ptolemy's longitude values are consistently about 1° too small, since, had he used the correct value of the precession, he would have had to add $3^\circ\ 40'$. However, there is no direct evidence that Hipparchus ever produced a systematic catalogue of about 1000 stars as Ptolemy did. More details on this question can be found in Evans (1987) and Shevchenko (1990) and an excellent historical survey of the debate on this question, which has gone on for over 100 years, can be found in Evans (1998); (see also note 25 on p. 70).

solely is to represent the heavenly phenomena by purely kinematic hypotheses. He returned to this question in a later work, the *Planetary Hypotheses*. However, Ptolemy did use physical arguments to justify his choice of a fixed Earth at the centre of the Universe. He realized that the diurnal rotation of the heavens could equally well be accounted for by a rotating Earth, as espoused by Heraclides, but this was ruled out on the grounds that it would contradict Aristotelian physics.

Mathematics in the *Almagest*

It is unclear who first introduced the Babylonian system of numeration into Greek work, but certainly Hipparchus was familiar with the sexagesimal positional system. Sexagesimal numeration is used throughout the *Almagest*, with minor modifications to the basic Babylonian system, though Ptolemy used the traditional Greek form for fractions where precision was unnecessary. There is very little mathematical formalism in the *Almagest*; instead, Ptolemy gives detailed accounts of the procedures used in actual calculations. The formulas given below are therefore a modern shorthand way of representing what Ptolemy wrote out in words. The main mathematical interest in the *Almagest* comes from Ptolemy's use of trigonometry, a subject developed by the Greeks specifically for the solution of problems arising in astronomy.

The only trigonometric function used in the *Almagest* is the chord, and Ptolemy constructs a table of chords in Book I. It is almost certain that such tables existed long before Ptolemy (Theon of Alexandria tells us that both Hipparchus and Menelaus had written works on chords) but Ptolemy's table is the first surviving specimen and his account of its construction is the first treatise on trigonometry known to us. The Greek chord function is illustrated in Figure 3.6. Ptolemy's aim is to calculate the length of the chord AC for a given angle α, denoted in Eqn (3.3) by $\operatorname{ch}\alpha$ and, in order to facilitate the use of the sexagesimal system, he used a circle of radius 60. Since one-half of the chord divided by the radius is just the sine of half the angle α, this chord function is related to the modern sine function through

$$\operatorname{ch}\alpha = 120\sin\tfrac{1}{2}\alpha, \tag{3.3}$$

though it should be remembered that the chord is a length, whereas the sine is a ratio.

Ptolemy's first step was to use well-known properties of regular polygons to evaluate some special values of the chord function. In this way he obtained

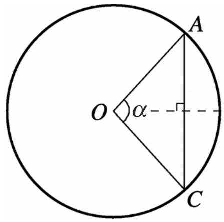

Fig. 3.6. The Greek chord.

$$\operatorname{ch} 36^\circ = 37;4,55, \qquad \operatorname{ch} 60^\circ = 1,0;0,$$
$$\operatorname{ch} 72^\circ = 1,10;32,3, \qquad \operatorname{ch} 90^\circ = 1,24;51,10,$$

from the properties of the decagon, hexagon, pentagon and square, respectively. Pythagoras' theorem and the fact that the angle inside a semicircle is a right angle[21] show that $\operatorname{ch}^2(180^\circ - \alpha) = 120^2 - \operatorname{ch}^2 \alpha$, and from this Ptolemy determined

$$\operatorname{ch} 120^\circ = 1,43;55,23, \qquad \operatorname{ch} 144^\circ = 1,54;7,37.$$

Next, Ptolemy proved what is now known as 'Ptolemy's theorem' but which, since it is so elementary, probably dates from an earlier period. Using arguments based on similar triangles Ptolemy showed that in a cyclic quadrilateral (a quadrilateral inscribed in a circle) the product of the diagonals is equal to the sum of the products of the opposite sides, or (see Figure 3.7(a))

$$|DB| \cdot |AC| = |AB| \cdot |DC| + |AD| \cdot |BC|.$$

Next, consider a cyclic quadrilateral with one side as diameter, as shown in Figure 3.7(b). An application of Ptolemy's theorem yields

$$\operatorname{ch}\beta \operatorname{ch}(180^\circ - \alpha) = 120 \operatorname{ch}(\beta - \alpha) + \operatorname{ch}(180^\circ - \beta) \operatorname{ch}\alpha.$$

Using Eqn (3.3), we see that this is equivalent to the modern trigonometric formula

$$\sin(x - y) = \sin x \cos y - \cos x \sin y,$$

where $x = \beta/2$, $y = \alpha/2$. In a similar way, Ptolemy derived the equivalent of the formulas

[21] EUCLID *Elements*, Book I, 47 and Book III, 31, respectively.

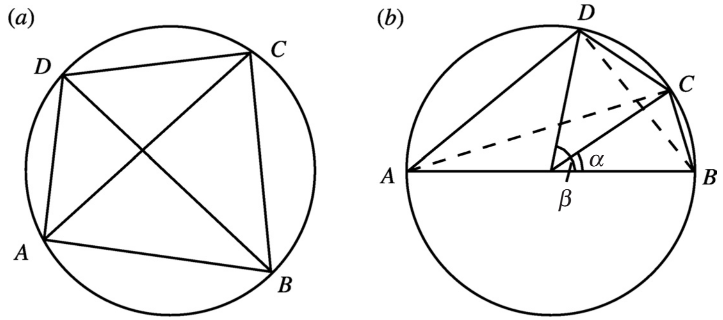

Fig. 3.7. Ptolemy's theorem.

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$2\sin^2 x = 1 - \cos 2x.$$

With these formulas and the chords already computed, it is possible to construct a table of chords in steps of 3° (or $3/2°$ or $3/4°$, etc.). However, Ptolemy's aim was to construct a table of chords in intervals of $1/2°$ and he achieved this with an ingenious argument: that for acute angles α and β with $\mathrm{ch}\,\alpha > \mathrm{ch}\,\beta$,

$$\frac{\mathrm{ch}\,\alpha}{\alpha} < \frac{\mathrm{ch}\,\beta}{\beta}, \tag{3.4}$$

a result known to Aristarchus and, since he had previously calculated

$$\mathrm{ch}\,\tfrac{3}{2}^\circ = 1;34,15, \qquad \mathrm{ch}\,\tfrac{3}{4}^\circ = 0;47,8,$$

Eqn (3.4) implies that

$$\tfrac{2}{3}\,\mathrm{ch}\,\tfrac{3}{2}^\circ < \mathrm{ch}\,1^\circ < \tfrac{4}{3}\,\mathrm{ch}\,\tfrac{3}{4}^\circ.$$

But to two sexagesimal places both $\frac{4}{3}\,\mathrm{ch}\,\frac{3}{4}^\circ$ and $\frac{2}{3}\,\mathrm{ch}\,\frac{3}{2}^\circ$ are equal to $1;2,50$. Hence, Ptolemy knew ch 1° and could compute

$$\mathrm{ch}\,\tfrac{1}{2}^\circ = 0;31,25.$$

Ptolemy was now able to construct a table of chords from $1/2°$ to 180° in steps of $1/2°$ (equivalent to a table of sines from $1/4°$ to 90° in steps of $1/4°$) accurate to two sexagesimal places.

With his table of chords, Ptolemy could use algorithms equivalent to the modern formulas (following the standard convention that the side opposite angle A is labelled a, etc.):

$$\sin A = \frac{a}{c}, \qquad \cos A \equiv \sin(90^\circ - A) = \frac{b}{c}, \qquad \tan A \equiv \frac{\sin A}{\cos A} = \frac{b}{a}.$$

Only right-angled triangles are solved in the *Almagest*, with oblique triangles being decomposed into right-angled ones, and in this way Ptolemy can solve any triangle (though somewhat clumsily by modern standards). Ptolemy's style throughout the *Almagest* suggests that this type of computation was fairly standard.

Of course, many of the astronomical calculations Ptolemy needed to perform concerned the angular distances between celestial bodies or, in other words, the positions of bodies on a spherical surface, for which spherical trigonometry is appropriate. Here, too, Ptolemy could use his table of chords. The geometry of the sphere, particularly with reference to astronomy, was one of the subjects taught within the Pythagorean quadrivium – and the subject was well advanced long before Ptolemy's time – but spherical trigonometry really came to prominence with the work of Menelaus of Alexandria in about AD100. The treatment of spherical trigonometry in the *Almagest* is based largely on the *Sphaerica* of Menelaus. In the first of three books, Menelaus introduced the concept of a spherical triangle – a figure formed by three arcs of great circles on a sphere, each arc being less than a semicircle – and proved some theorems about such triangles analogous to those Euclid had proved for plane triangles. The second book is concerned chiefly with astronomy and only indirectly with spherical geometry, while spherical trigonometry was the subject of the third book.

Ptolemy's use of spherical trigonometry is based on two results (see Figure 3.8(a)). Suppose we have a spherical triangle ABE and another great circle intersecting the sides of this triangle (produced where necessary) at D, F, and C as in the figure, then

$$\sin CE \cdot \sin DF \cdot \sin BA = \sin AE \cdot \sin CF \cdot \sin DB,$$
$$\sin CA \cdot \sin DF \cdot \sin BE = \sin AE \cdot \sin CD \cdot \sin BF.$$

Here, $\sin CE$ means the sine of the angle subtended at the centre of the sphere by the arc CE, etc., and therefore is related directly to the Greek chord function. There are two other similar relations that can be derived, but Ptolemy did not mention this. The proof of this theorem rests upon the corresponding theorem for plane triangles, which is still referred to as Menelaus' theorem, although it probably predates Menelaus. The theorem (which is proved by Ptolemy) states

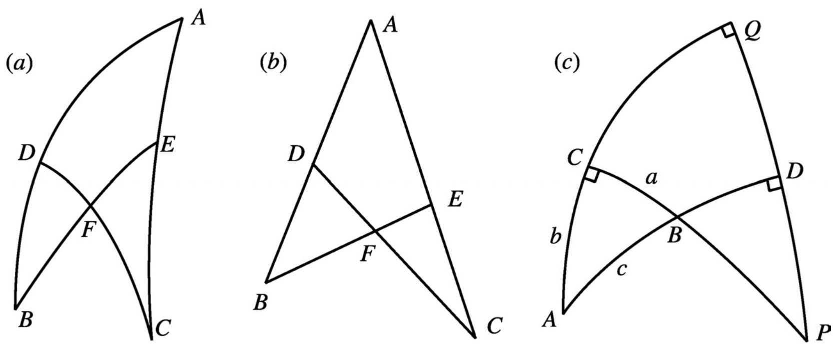

Fig. 3.8. Menelaus' theorem (a) for spherical triangles, (b) for plane triangles, and (c) Ptolemy's use for right-angled spherical triangles.

(see Figure 3.8(b)), that

$$|CE|\cdot|DF|\cdot|BA| = |AE|\cdot|CF|\cdot|DB|,$$
$$|CA|\cdot|DF|\cdot|BE| = |AE|\cdot|CD|\cdot|BF|.$$

As in the case of plane triangles, Ptolemy actually considered only right-angled spherical triangles, in which two of the great circles meet at 90°. By subdividing other triangles into two or more right-angled ones, he was able to solve them using the above theorem.[22]

For right-angled spherical triangles, Menelaus' theorem can be reduced to a relationship between three quantities. Thus, in Figure 3.8(c), ABC is a spherical triangle with right angle at C. The point A is the pole of the great circle PQ, and the arcs a, b and c are extended to meet this great circle in the points P, Q and D, respectively. From the second version of Menelaus' theorem for spherical triangles given above, using the facts that $QP = CP = AD = 90^\circ$, we obtain

$$\sin BC = \sin QD \cdot \sin AB.$$

Since A is the pole of QP, it follows that $\sin A = \sin QD$,[23] and so this equation can be written

$$\sin a = \sin A \sin c.$$

Similarly, Ptolemy used algorithms equivalent to the formulas

$$\tan a = \sin b \tan A, \qquad \cos c = \cos a \cos b, \qquad \tan b = \tan c \cos A.$$

[22] Details of Ptolemy's procedures can be found in Pedersen (1974) and Katz (1998) as well as in the *Almagest* itself.
[23] See, for example, Smart (1960).

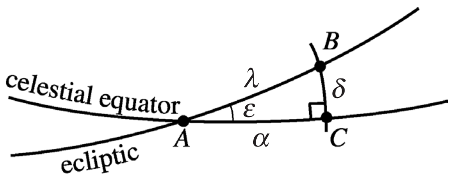

Fig. 3.9. The determination of the declination δ and right ascension α of the Sun; A is the vernal equinox and the Sun is at B.

As an example, consider Ptolemy's determination of the declination δ and right ascension α of the Sun for a given ecliptic longitude λ. In Figure 3.9, A is the vernal equinox, the Sun is at B, and the angle ε is the obliquity of the ecliptic, which Ptolemy took to be $23^\circ\ 51'\ 20''$.[24] The first and last of the four formulas listed above then show that

$$\sin\delta = \sin\varepsilon \sin\lambda, \qquad \tan\alpha = \tan\lambda \cos\varepsilon.$$

Solar theory

After the mathematical preliminaries, Ptolemy discusses the motion of the Sun, and here he uses exactly the same model as that developed by Hipparchus. Ptolemy made some observations of the dates of the equinoxes so as to see whether Hipparchus' value for the length of the tropical year was still correct after a 300-year period, and he concluded that it was and that it has the value ascribed to it by Hipparchus, i.e. 365 days 5 h 55 min 12 s – about $6\frac{1}{2}$ minutes too long.

Ptolemy also used his observations of the equinoxes to recompute the length of the seasons and again found agreement with Hipparchus' values of $94\frac{1}{2}$ days for the length of the spring season and $92\frac{1}{2}$ days for the length of summer. He thus obtained the same numerical parameters for the eccentric model of the Sun, i.e. $65^\circ\ 30'$ for the longitude of the solar apogee as measured from the vernal equinox, and $1/24$ for the eccentricity of the orbit. The fact that Ptolemy arrived at the same result for the solar apogee as Hipparchus had done

[24] In the middle of Book I of the *Almagest*, between his construction of a table of chords and his discussion of spherical trigonometry, Ptolemy discussed the angle that the ecliptic makes with the celestial equator, the obliquity ε. He described how it can be calculated and claimed to have found that $23\frac{5}{6}^\circ < \varepsilon < 23\frac{7}{8}^\circ$. Since the value he attributes to Eratosthenes, $22/83$ of a right angle ($23^\circ\ 51'\ 20''$), lies in this range, this is the value he adopted (see Goldstein (1983)). The actual value in Ptolemy's time was about $10'$ less than this (Wilson (1980), p. 63) and Ptolemy's failure to find a more accurate result, and hence perhaps discover the slow decrease in the obliquity over time, can be explained by the crudity of his measuring devices (see Britton (1969)).

300 years previously led him and other ancient astronomers to believe that this was an astronomical constant, whereas in fact, because of the precession of the equinoxes, in Ptolemy's time the true value was approximately 70°.[25]

Ptolemy used the geometrical solar theory to produce a table so that the position of the Sun at a given time can be calculated. From Figure 3.10 it is clear that, viewed from the Earth E, the Sun's longitude measured from its apogee A is α (the so-called true anomaly) and the mean anomaly $\overline{\alpha}$ is known, since the Sun S rotates uniformly around the centre of the eccentric circle, O, completing 1 revolution in 1 tropical year. It is also clear that $\alpha = \overline{\alpha} \pm \delta$ (the choice of sign depending on where the Sun is in its orbit) and so the position of S can be found once δ is known as a function of $\overline{\alpha}$. This is what Ptolemy tabulates. Ptolemy calls δ the *prosthaphaeresis*, which translates as the 'amount to be added and subtracted' but, following medieval usage, it is more commonly referred to as 'the equation of centre'.

In general, the term 'equation' is used to refer to any angle that must be added or subtracted from a mean motion in order to account for a particular geometrical feature. This is Ptolemy's style throughout – first the mean motions are described, and then the various small corrections, the equations, are calculated. In the case of the Sun, there was one such equation, but for the Moon and the planets there were more. Ptolemy's quantitative solar theory was used not only to determine the position of the Sun but was also an essential part of his theory for the other planets.

Because the Sun moves non-uniformly around the ecliptic, which is itself inclined to the celestial equator, the length of the solar day (the time between local noon on successive days) is not constant. Thus, when calculating the time difference in days between two events (such as a pair of eclipses) it is necessary to correct for this variation. The effect, however, is quite small and, until Ptolemy's accurate quantitative astronomy, the resulting discrepancies

[25] Some authors, notably Newton (1977), claim that Ptolemy must have fudged his data so as to reproduce Hipparchus' parameter values, but in fact Ptolemy's errors are not large when one takes into account that an error of 6 h in the length of the spring season can lead to an error of about 7° in the solar apogee (Peterson and Schmidt (1967), Maeyama (1998)). In fact, this demonstrates that the great accuracy achieved by Hipparchus was fortuitous. North (1994) goes so far as to say 'a modern tradition that Ptolemy was little more than a plagiarist of Hipparchus is hardly worth refuting' and Hamilton and Swerdlow's review of Newton's work in the *Journal for the History of Astronomy*, **12** (1981), 59–63 is deeply critical of the latter's approach. On the other hand, Britton (1992) has demonstrated convincingly that Ptolemy must have had access to rather more observations than are mentioned in the *Almagest* and that he sometimes used these selectively so as to reproduce agreement with predetermined values. Sheynin (1973b) suggests that Ptolemy might have selected those observations he believed to be the least susceptible to random or systematic error. Whatever the truth of the matter – and the debate continues (see Thurston (2002), Gingerich (2002)) – astronomers from the eighteenth century onwards certainly realized that Ptolemy's observations were not to be relied upon (see, for example, Wilson (1984), also note 20 on p. 63 of this text).

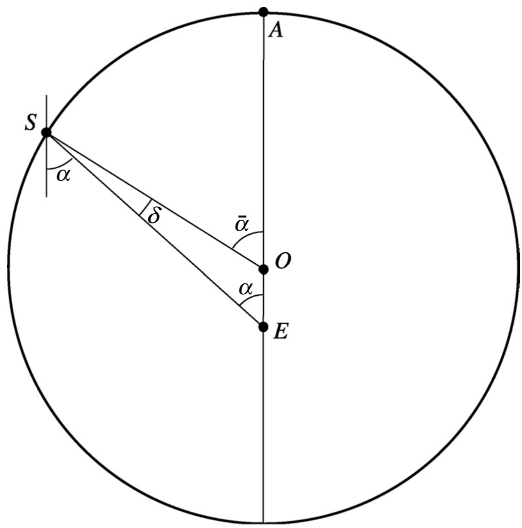

Fig. 3.10. The *prosthaphaeresis* angle δ.

(which never amount to more than about half an hour) were of little significance. Local time is determined by the position of the Sun with respect to meridians (great circles perpendicular to the celestial equator) and so the simplest way to appreciate the cause of the variation in the length of the day is to introduce the concept of the equatorial mean sun, a point that travels around the celestial equator at a uniform rate, once per tropical year.[26] This then leads to the idea of the mean solar day, a concept Ptolemy introduced.

Thus, in Figure 3.11, when the actual Sun is at S, the equatorial mean sun will be at $\overline{S}$ which, because S moves non-uniformly around the ecliptic and α is nonlinearly related to λ (see Figure 3.9), will at different times of the year be sometimes ahead of and sometimes behind A. The arc $A\overline{S}$ (the difference between the right ascension of the equatorial mean sun and that of the Sun itself) is known as the 'equation of time'. The discovery of the equation of time by Greek astronomers is just one example of the high level of sophistication they achieved, as it was deduced as a theoretical consequence of Hipparchus' solar theory, rather than from observation.

[26] The equatorial mean sun should not be confused with the mean sun, which is another important astronomical concept. The latter rotates uniformly around the ecliptic once each tropical year. The two clearly are related since they move around their respective circles at precisely the same rate. In fact, the right ascension of the equatorial mean sun is equal to the ecliptic longitude of the mean sun.

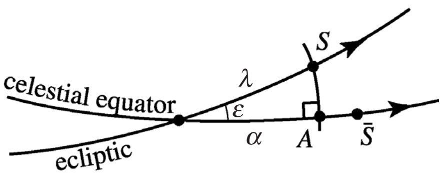

Fig. 3.11. The cause of the equation of time. The Sun is at S and $\overline{S}$ is the equatorial mean sun.

Lunar theory

By virtue of its proximity, small irregularities in lunar motion are discernible more easily than those of the planets, and Ptolemy was the first to discover that the Moon was subject to an anomaly that had not been detected by Hipparchus. As we have seen, Hipparchus treated the longitude and latitude theories for the Moon separately, and Ptolemy did the same but, whereas the latitude theory was not modified by Ptolemy, he found the longitude theory to be inadequate away from the syzygies.

> As far as concerns the [moon's] syzygies . . . we find that the hypotheses set out above for the first, simple anomaly is sufficient, even if we employ it just as it is, without any change. But for particular positions [of the moon] at other sun-moon configurations one will find that it is no longer adequate, since . . . we have discovered that there is a second lunar anomaly, related to its distance from the sun.[27]

Ptolemy showed that the discrepancy between the observed longitude of the Moon and that predicted by Hipparchus' theory depended on the position of the Moon relative to the Sun and was greatest at the quadratures, i.e. half moons. Thus, the Moon was subject to a second, independent irregularity in its motion which has become known as 'evection'.[28] In order to rectify the theory, Ptolemy introduced a mechanism by which the Moon's epicycle was brought closer to the Earth at the quadratures than at new and full moon. This device had the effect of incorporating the theory of the Sun into that for the Moon.

His solution is illustrated in Figure 3.12, and involves an epicycle on a deferent which is no longer centred on the earth E but is instead placed eccentrically. Moreover, the centre of the deferent O rotates around the Earth in a small circle with a constant angular speed with respect to the position of the mean sun, $\overline{S}$. This has the effect of moving the apogee of the deferent A from east to west

[27] Ptolemy *Almagest*, Book V, 1.
[28] The terminology is due to Ismael Boulliau in 1645.

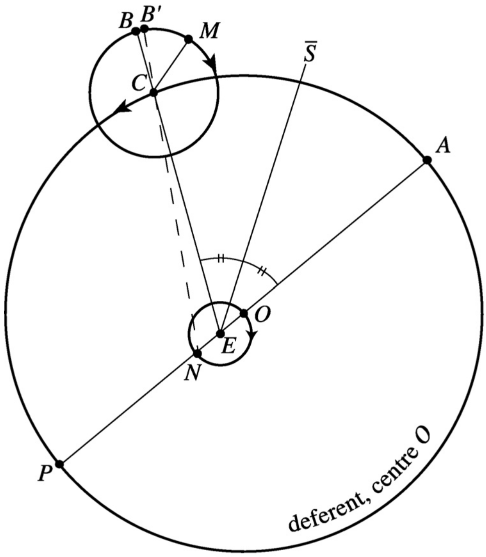

Fig. 3.12. Ptolemy's lunar theory.

relative to the mean sun. The centre of the Moon's epicycle C rotates around the deferent circle in such a way as to keep the angles $AE\overline{S}$ and $\overline{S}EC$ equal. Thus, relative to the mean sun, C rotates once around the deferent in an anti-clockwise direction (i.e. from west to east) in a synodic month, while O rotates around a circle, centred at E, at exactly the same rate in the opposite direction. Equivalently, if one considers the motion of the epicycle relative to the line EA, C rotates around the deferent twice each synodic month. At the syzygies, C and A coincide, whereas at the quadratures C and P coincide. It is clear that in the lunar model, the deferent rotates with uniform speed around the Earth, a point that is not its centre, and so Ptolemy was violating one of the fundamental principles on which Greek mathematical astronomy had been built, i.e. uniform circular motion. This radical shift, which Ptolemy did not even mention, was later a major source of criticism.

From observations[29] Ptolemy was able to determine the relative sizes of the circles in his lunar theory needed accurately to reproduce the phenomena. Letting $|EA|$ be 60 units, Ptolemy's skill with trigonometry enabled him to show that $|OE|$ should be 10; 19 and the radius of the epicycle was 5; 15.[30] In

[29] The claim that Ptolemy never made any lunar observations, in support of the general thesis of Newton (1977), is made by Goldstein (1982).
[30] Details can be found in Pedersen (1974).

the first version of his lunar theory, the Moon M was made to rotate around its epicycle at a uniform rate with respect to the line ECB (see Figure 3.12) once in each anomalistic month, exactly as in Hipparchus' theory. The angle BCM thus was a given linear function of time. In order to calculate the longitude of the Moon for a given time, Ptolemy had first to compute the longitude of the point C (the mean moon) from the mean lunar motion. Then he could calculate the angle $CE\bar{S}$, since the solar theory provided the longitude of the mean sun. Next, the distance $|CE|$ can be computed by solving the triangle EOC and, finally, $\angle CEM$ (the *prosthaphaeresis* angle for the Moon) can be found by solving the triangle CEM.[31]

The overall effect of Ptolemy's scheme was to leave the longitude at conjunction and opposition unchanged from that given by Hipparchus (since $\angle CEA = 0$ in these cases) and the theory accounted well for the observed position of the Moon at the quadratures. However, Ptolemy found that there were still noticeable errors at the octants – the points midway between the quadratures and the syzygies – and decided that a further modification was necessary. He did this by making the Moon rotate around its epicycle at a non-uniform rate with respect to ECB, but uniform with respect to NCB', where the point N is such that E is the midpoint of ON. For this new scheme, it is the angle $B'CM$ that is a given linear function of time, so in order to compute the *prosthaphaeresis* angle, Ptolemy first had to calculate $\angle BCB' = \angle NCE$. This he could do by solving the triangle NCE once the length $|EC|$ had been determined. This final peculiarity in Ptolemy's lunar theory, which has no effect on the position of the Moon at the syzygies or at quadrature and which was another source of criticism by later astronomers, is known as *prosneusis*. In actual fact, the effect of *prosneusis* was to make things better some of the time but worse at others. Ptolemy did not test his final theory against new observations away from the octant points.[32]

The lunar theory is not without its problems. One defect is that the ratio of the maximum to minimum lunar distances implied by the model is $64; 15/34; 7 \approx 1.9$ and so one should observe a doubling of the Moon's apparent diameter as it circles the Earth, whereas in actual fact a change of only 14 per cent is observed. Another objection that was raised by later astronomers was the fact that the epicyclic model seems to be incompatible with the phenomenon – mentioned

[31] The calculations involved in applying this procedure are very laborious and in order to make the application of his theory more straightforward Ptolemy produced (not just for the Moon but for all the celestial bodies) tables of values corresponding to the modern concept of functions of one or two variables. To make the computation of the tables less time-consuming he used various methods of interpolation (see Pedersen (1974), van Brummelen (1994)).

[32] The accuracy of Ptolemy's lunar theory is described in detail in Peterson (1969).

by Aristotle in *On the Heavens* – that the same side of the Moon is always visible to us. One plausible reason why neither of these 'defects' concerned Ptolemy is that he regarded his theory simply as a device for computing the latitude and longitude of the Moon rather than as a physical model of reality. However, this seems to be contradicted by the way Ptolemy later treated his models in the *Planetary Hypotheses* (see p. 81).

Two aspects of the lunar model are of particular interest. The coupling between Sun and Moon is a major shift from previous theories, and so is Ptolemy's abandonment of uniform circular motion. Prior to Ptolemy, uniform circular motion always had implied angular rotation around a circular path uniform with respect to the circle's centre. But in Ptolemy's final lunar scheme, the Moon revolves with uniform speed with respect to a different point. As we shall see, many later astronomers objected to this, the most notable being Copernicus.

Having now described his theories for the motion of the Sun and Moon, Ptolemy was in a position to give a detailed theory of lunar and solar eclipses, which he did in Book VI of the *Almagest*. He began with a discussion of lunar parallax, which is significant when making observations of the Moon, and from his measurements he arrived at the conclusion that the radius of the deferent in his lunar model is about 49 Earth radii. Once he had a value for the distance to the Moon in terms of the size of the Earth, he was in a position to calculate the distance to the Sun by reversing the procedure used by Hipparchus for calculating the lunar distance from an assumed solar distance.[33] His final result was that the mean distance to the Sun is 1210 Earth radii (corresponding to a parallax of about $3'$), which is about 19 times too small.

Planetary theory

Books IX–XIII of the *Almagest* are devoted to the motion of the planets, with longitudes and latitudes being considered separately. For the longitude theories, there are two anomalies to model. The first is manifested by the varying speed of the planet as it travels round the ecliptic and is thus similar to the anomaly in the Sun's motion. This suggests an eccentric deferent as a suitable geometrical scheme to account for the irregularity. The second anomaly is the phenomenon of retrograde motion and this ultimately is linked to the motion of the Sun. The superior planets always reach the centre of their retrograde arcs when they are

[33] See p. 54. Ptolemy's method is described in detail in van Helden (1985). Unlike the procedure adopted by Hipparchus, Ptolemy's approach is extremely sensitive to errors in the measured parameters.

in opposition to the Sun, whereas this happens at conjunction for the inferior planets. As Apollonius had shown, retrograde motion can be modelled by an epicyclic theory, and so some combination of eccentric deferent and epicycle suggests itself.

However, it turns out that this is insufficient by itself accurately to predict the correct positions at which the retrograde motions begin and the correct angular widths of the retrograde arcs.[34] To solve the problem, Ptolemy introduced a further modification to Apollonius' scheme and separated the centre of the deferent circle from the centre of uniform rotation. Thus, he introduced a new point – the equant – about which the centre of the deferent rotated with uniform angular speed. Nowhere does Ptolemy state how he devised this construction, but it works astonishingly well – a Ptolemaic equant produces planetary longitudes differing from modern theory by less than $10'$ of arc, even for the comparatively large eccentricity of Mars. The discovery of the equant mechanism thus represents one of the major achievements of Greek mathematical astronomy.[35] Brilliant it may have been, but the incorporation of the equant introduced a major problem into the science of astronomy because it violated the principle of uniform circular motion. To Ptolemy, it clearly was more important to reproduce accurately the phenomena than to stick rigidly to accepted philosophical dogma. Others did not necessarily share Ptolemy's attitude and the status of the equant was one of the main concerns of future astronomers.

Although Ptolemy described his theories for the inferior planets first, it is perhaps more helpful to begin with his scheme for the superior planets – Mars, Jupiter and Saturn – since the Mercury theory is relatively complicated. The geometrical model that Ptolemy finally arrived at is shown in Figure 3.13. Each planet P is carried round a deferent circle, centre O, on an epicycle, centre C, which rotates in the same sense as the motion of C around the deferent. Just as in the theory of the Moon, the Earth E is not situated at O, and the distance between the Earth and the deferent centre $|EO|$ is known as the 'eccentricity' of the model. The motion of C around the deferent is uniform with respect to the new point Q – the equant – which is chosen so that O bisects EQ.[36] The mean longitude $\bar{\lambda}$ (measured from the vernal equinox V) increases at a uniform

[34] A nice explanation of why this is the case is given in Evans (1984).

[35] Did Ptolemy devise it? It is not possible to say with certainty and he certainly does not claim explicitly that it is his idea. On the other hand, Ptolemy is good at giving credit where credit is due and he does not mention anyone else as the discoverer of the equant idea. For a strident defence of the idea that this construction was not due to Ptolemy, see Rawlins (1987).

[36] It is unclear as to why Ptolemy chose to put the centre of the deferent exactly equidistant from the Earth and the equant. It seems most likely (see Wilson (1973), Pedersen (1974)) that the observations suggested that this was the best position for the Venus model and he extended it to the superior planets by analogy.

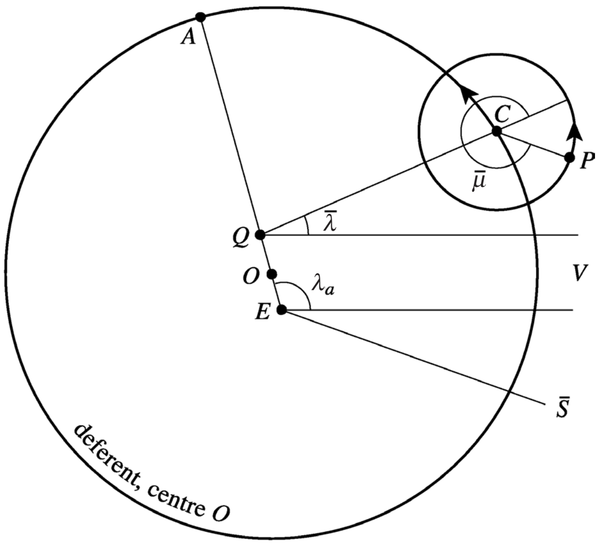

Fig. 3.13. Ptolemy's theory for the superior planets.

rate, with the epicycle rotating once around the deferent in the time it takes the planet to make one circuit around the ecliptic (its zodiacal period), i.e. 687 days for Mars, 11.86 years for Jupiter and 29.46 years for Saturn. The motion of the planet around its epicycle is uniform relative to the line QC, so that the angle $\overline{\mu}$ in Figure 3.13 (the mean epicyclic anomaly) increases uniformly with time t.

For each of the superior planets, observations show that the zodiacal and synodic periods, T_c and T_p respectively, satisfy (see Table 1.2, p. 9)

$$\frac{1}{T_\mathrm{c}} + \frac{1}{T_\mathrm{p}} = \frac{1}{T},$$

where T is the tropical year. In order to incorporate this feature into his model, Ptolemy made the planet rotate once around its epicycle each synodic period. We have then (ignoring irrelevant constants) $\overline{\lambda} = t/T_\mathrm{c}$ and $\overline{\mu} = t/T_\mathrm{p}$ and, hence, the longitude of the mean sun $\overline{S}$ is given by

$$t/T = \overline{\lambda} + \overline{\mu}.$$

Some elementary geometry reveals that this ensures that the line CP is always parallel to the line connecting the Earth E to the mean sun. The point A at which the line EOQ extended intersects the celestial sphere is the apogee of the deferent, and in Ptolemy's model it is assumed fixed with respect to the stars. In other words, its longitude λ_a increases slowly due to the effect of precession.

Finally, Ptolemy needed to determine the relative dimensions of his model, and he found that, with a deferent radius of 60, the required double eccentricities were given by $|EQ| = 12$ for Mars, $|EQ| = 5;30$ for Jupiter and $|EQ| = 6;50$ for Saturn. The values Ptolemy used for the radii of the epicycles were 39; 30 for Mars, 11; 30 for Jupiter, and 6; 32 for Saturn. By using a complicated series of trigonometrical calculations, Ptolemy could then compute the true longitude of the planet (the angle VEP in Figure 3.13) at any given time.

The above geometrical arrangement is inappropriate for the inferior planets (Mercury and Venus) since they follow the Sun as it travels around the heavens, never deviating from it by more than about 29 and 47°, respectively. The mean positions of these planets are the same as that of the Sun, and so in Ptolemy's models for the inferior planets the centre C of the epicycle rotates uniformly around an equant Q, such that QC is parallel to the line connecting the Earth E to the mean sun $\overline{S}$. Ptolemy's model for Venus is shown in Figure 3.14, and it can be seen that, apart from the different treatment of the coupling to the motion of the Sun, the scheme has the same essential features as his theories for the superior planets. The epicycle is carried around a deferent, the centre O of which is not the Earth, and the motion is uniform with respect to an equant Q which again is chosen so that O is the midpoint of EQ. It is interesting to note that in his model for Venus, Ptolemy in effect introduced an equant for the motion of the Sun, something he had not found to be necessary in the solar theory itself. He was, of course, unaware that the solar deferent and the deferent of Venus represent one and the same motion – the rotation of the Earth around the Sun. The apogee of the deferent A is assumed fixed with respect to the stars, exactly as before.

With a deferent radius of $R = 60$, the epicycle's radius was taken to be $r = 43;10$ and the double eccentricity as $|EQ| = 2;30$. The required size of the epicycle can, at least approximately, be determined from the maximum elongation (i.e. angular distance from the Sun) of the planet. Assuming zero eccentricity so that E and O coincide, the maximum elongation θ occurs when $\angle OPC$ is a right angle, i.e. when $r/R = \sin\theta$. Ptolemy's values give

$$\theta = \sin^{-1}\frac{43;10}{60} \approx 46^\circ,$$

in agreement with the value given by the Roman writer Pliny in AD first century. The effect of a non-zero value for the eccentricity on this simple argument is, however, quite complex.

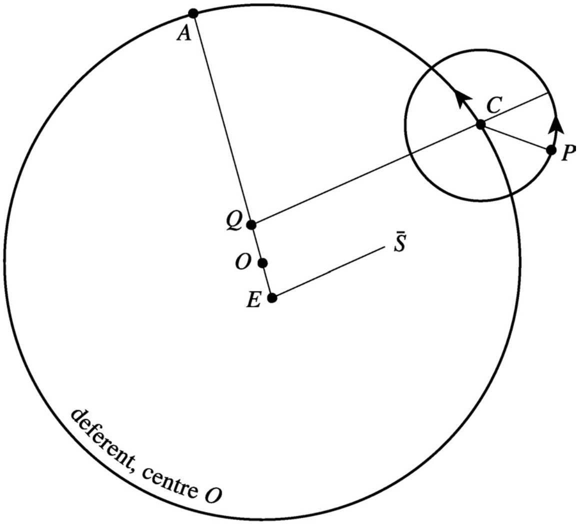

Fig. 3.14. Ptolemy's theory for Venus.

When it came to Mercury, however, Ptolemy found that he could not use the same geometrical scheme. Of the planets known to Ptolemy, Mercury is by far the hardest to observe, and its orbit deviates significantly from a circle. The observations Ptolemy used for his theory of Mercury – some of which were very inaccurate – led him to believe that Mercury's orbit had two perigees. Thus, he came up with a geometrical device that modelled this phenomenon, and this is shown in Figure 3.15. The line EQO points to the apogee of Mercury's orbit, the longitude of which is fixed with respect to the stars, and the equant Q is the midpoint of EO. The centre of the epicycle C rotates uniformly around the equant, following the mean sun, in such a way that it lies always on a deferent circle, the centre D of which is rotating in the opposite sense around a small circle centred on O. The rates of rotation are chosen so that the angles DOA and AQC always are equal. This construction causes C to move in an oval orbit that has two points of closest approach to the Earth, P_1 and P_2. For a deferent radius of 60, Ptolemy calculated the radius of Mercury's epicycle as $22; 30$ and $|EQ|$ as $6; 0$. Based on the simple argument described above for Venus, an epicycle radius of $22; 30$ corresponds to a maximum elongation of 22°, which is again the value quoted by Pliny.[37] In his later work, the *Planetary Hypotheses*,

[37] Detailed discussions of the empirical basis for Ptolemy's theories of the inferior planets can be found in Wilson (1973), Swerdlow (1989), and the accuracy of the Mercury model is investigated in Nevalainen (1996).

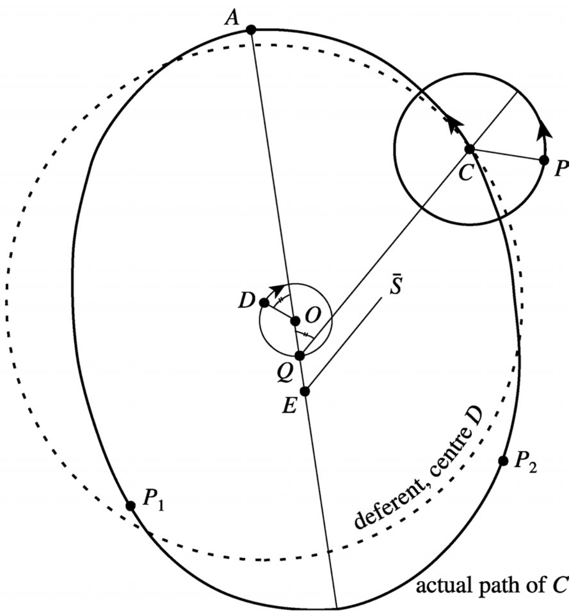

Fig. 3.15. Ptolemy's theory for Mercury.

Ptolemy justified the increased complexity of the motion of Mercury, as well as that of the Moon, by pointing out that these two celestial bodies are closest to the Earth and, hence, closest to the changeable air.

The planetary theories described above are all dedicated to the determination of longitudes; latitudes are treated separately in Book XIII of the *Almagest*. Unlike the simple latitude theory for the Moon, Ptolemy's theory for planetary latitudes is cumbersome and complicated. This is hardly surprising since the plane of the orbit of the Moon passes (very nearly) through the centre of the Earth, whereas the planes of the planetary orbits pass through the centre of the Sun, and Ptolemy was attempting to construct a geocentric latitude theory. For the superior planets, Ptolemy accomplished this by tilting the deferent with respect to the ecliptic and also inclining the plane of the epicycle with respect to the deferent. Now, for these planets we know that the motion of the planet around the epicycle actually is modelling the motion of the Earth around the Sun, so the epicycle should be in a plane parallel to the ecliptic. Ptolemy had no such knowledge, of course, and so had to determine two inclinations for each planet. The system Ptolemy used for the inferior planets was similar, except in this case the inclination of the deferent with respect to the ecliptic was an oscillatory function of time.[38]

The *Almagest* represents the culmination of 500 years of Greek mathematical astronomy, and the result is ingenious, mathematically sophisticated and tolerably accurate. It formed the foundation for subsequent developments and for the teaching of astronomy, and was the dominant influence in theoretical astronomy until the sixteenth century. As a scientific theory it is still *ad hoc* in that it explains only those phenomena built into the model, and new observations can be accommodated simply by tweaking the parameters.

The *Handy Tables* and the *Planetary Hypotheses*

After writing the *Almagest*, Ptolemy wrote two further astronomical works, the *Handy Tables* and the *Planetary Hypotheses*. The former contains the procedures that need to be followed to compute the positions of heavenly bodies from Ptolemy's theories, but with no discussion of the theoretical models on which they are based, while the latter contains his physical interpretations of the mathematical models developed in the *Almagest*. In the *Planetary Hypotheses*, parts of which have only become well known in the West fairly recently,[39] Ptolemy made two assumptions. The first was that the geometrical devices he had devised accurately to predict the observed phenomena actually exist in the heavens. The second was that he assumed there were no empty spaces between the mechanisms for each of the heavenly bodies. As some of the geometrical schemes Ptolemy had used in the *Almagest* contradicted Aristotelian natural philosophy – notably the equant construction – the first of his assumptions came under repeated attack during the Middle Ages. The second was not so controversial, but Ptolemy himself actually violated it by leaving an empty space between Venus and the Sun. Many later astronomers repeated Ptolemy's calculations to determine the dimensions of the Universe and managed, one way or another, to remove this void from the theory. Ptolemy's world view was as influential in the field of cosmology as the *Almagest* was in theoretical astronomy.

In the *Planetary Hypotheses*, Ptolemy deliberately simplified some of his geometrical constructions so as to make them easier to understand; some of the simplifications just take the form of approximating parameters by more convenient values, but there are also some changes of substance, notably in the latitude theories for the planets, which actually were improved in the process.

[38] Details of Ptolemy's latitude theories for the planets can be found in, for example, Pedersen (1974), Riddell (1978), Jacobsen (1999).
[39] See Goldstein (1967).

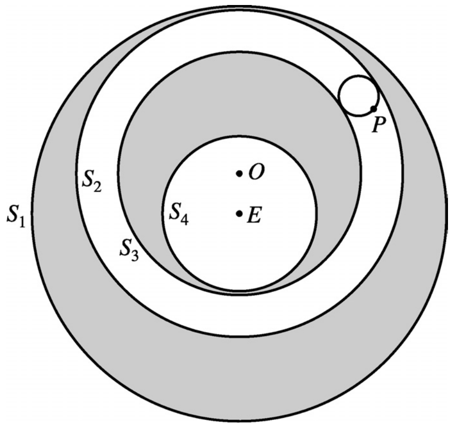

Fig. 3.16. Ptolemy's physical interpretation for a superior planet.

A schematic illustration of Ptolemy's physical interpretation of the motion of a superior planet is shown in Figure 3.16. The figure depicts four spherical surfaces: S_1 and S_4 centred on the Earth E (the centre of the Universe) and S_2 and S_3 centred on O (the centre of the deferent (see Figure 3.13)). The planet P is situated on a sphere (the epicycle) which is confined to the spherical annulus between S_2 and S_3. The shaded regions between S_1 and S_2 and between S_3 and S_4 are filled with the ether, which serves to transmit the required motion between the various spheres.

If we consider the diagram as representing the motion of Saturn, then Ptolemy now could fit his whole planetary scheme together without spheres intersecting simply by placing Jupiter inside S_4, and so on. On the assumption that the spheres fit as closely together as possible, we thus have a method for calculating the size of the Universe and this was done by Ptolemy (see Table 3.1).[40] In the *Almagest*, Ptolemy had already calculated the distances of the Sun and the Moon, and so he began with the Moon and worked outwards arriving at a maximum distance for Venus of 1079 Earth radii. But the minimum distance to the Sun was 1160 Earth radii, and so Ptolemy was forced to leave a gap between the spheres of Venus and the Sun.[41] Working outwards from the Sun, he arrived at a distance to the outer sphere of Saturn of 19 865 Earth radii (about 120 000 000 km in modern units, less than the true value for the radius of the Earth's orbit) and he rounded this to 20 000 Earth radii as the distance

[40] The figures are quoted from Goldstein and Swerdlow (1970).
[41] Ptolemy was aware that minor adjustments to some of the measured parameters could be used to close the unwanted gap, but he did not bother to make them (see van Helden (1985), p. 23).

Table 3.1. *Greatest and least distances, measured in Earth radii, according to Ptolemy.*

	d_{min}	d_{max}
Moon	33	64
Mercury	64	166
Venus	166	1 079
Sun	1 160	1 260
Mars	1 260	8 820
Jupiter	8 820	14 187
Saturn	14 187	19 865
Fixed stars	20 000	

to the fixed stars. From a modern perspective, Ptolemy's value for the size of the Solar System is hopelessly wrong but, seen in the context of his own time, he was actually the first person to suggest that the dimensions of the Universe were unimaginably large. The number of spheres necessary to construct the whole system was thirty-four, and Ptolemy thus claimed that he had produced a simpler system than any of his predecessors. Despite its many deficiencies, Ptolemy's cosmology reigned supreme – with modifications as to its detail – until the demise of geocentric astronomy in the seventeenth century.

No Greek astronomer after Ptolemy made any significant advance on Ptolemy's work. People were becoming more sceptical about the value of this type of endeavour, and working conditions for those involved in rational scientific enquiry gradually deteriorated. Commentaries on the *Almagest* were written in AD fourth century by Theon of Alexandria and by Pappus, but they added little. Progress in mathematical astronomy had to wait until the revival of scholarly activity in the Islamic civilization that grew up following the death of Mohammed in AD 632.