Practice Midterm Solutions

1. (5 points) Define Lebesgue outer measure.

   Solutions: Let $E \subset \mathbb{R}$ be a set, the Lebesgue outer measure of E is defined as

$$m^*(E) = \inf\{\sum_{k=1}^{\infty} \ell(I_k) : E \subset \bigcup_{k=1}^{\infty} I_k \text{ and } I_k \text{ are open intervals}\}.$$

2. (5 points) Show that $(\mathbb{R}\setminus\mathbb{Q})\cap[0,1]$ is measurable and that $m((\mathbb{R}\setminus\mathbb{Q})\cap[0,1]) = 1$.

   Solution: The set $\mathbb{Q}$ is countable and therefore has outer measure 0. Sets with 0 outer measure are measurable and so $\mathbb{Q}$ is measurable. The complement of a measurable set is measurable and so $\mathbb{R} \setminus \mathbb{Q}$ is measurable. Closed sets are measurable and so $[0,1]$ is measurable. Finally, the intersection of measurable sets is measurable and so $(\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$ is measurable.

   To calculate $m((\mathbb{R}\setminus\mathbb{Q})\cap[0,1])$, use the additivity property of Lebesgue measure.

$$1 = m([0,1]) = m([0,1] \cap \mathbb{Q}) + m((\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]).$$

   $m([0,1] \cap \mathbb{Q}) = 0$ and so $m((\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]) = 1$.

3. Let $\sim$ be a relation on $\mathbb{R}^2$ defined as $(x_1, x_2) \sim (y_1, y_2)$ if $x_2 = y_2$.

   (a) (2 points) Show that $\sim$ is an equivalence relation on $\mathbb{R}^2$.

   Solution: We must show that $\sim$ satisfies reflexivity, symmetry and transitivity. Reflexive: $(x_1, x_2) \sim (x_1, x_2)$ since $x_2 = x_2$.

   Symmetric: If $(x_1, x_2) \sim (y_1, y_2)$, then $x_2 = y_2$ and so $y_2 = x_2$ and $(y_1, y_2) \sim (x_1, x_2)$.

   Transitive: If $(x_1, x_2) \sim (y_1, y_2)$ and $(y_1, y_2) \sim (z_1, z_2)$, then $x_2 = y_2 = z_2$. So $(x_1, x_2) \sim (z_1, z_2)$.

(b) (3 points) Find a bijective map from the set of equivalence classes of $\sim$ to $(0,1)$. (You must prove the map is bijective.)

Solution: Let $\mathbb{R}^2/\sim$ be the set of equivalence classes for $\sim$ and define the map $f\colon \mathbb{R}^2/\sim \to \mathbb{R}$ as $f([(x_1,x_2)]) = x_2$. We must show that the map does not depend on the specific representative of the equivalence class and that it is bijective.

The map does not depend on the specific representative: Suppose $[(x_1,x_2)] = [(y_1,y_2)]$, then $x_2 = y_2$ and $f([(x_1,x_2)]) = f([(y_1,y_2)])$.

Bijective: We can construct and inverse map $g\colon \mathbb{R} \to \mathbb{R}^2/\sim$ as $g(x) = [(0,x)]$. Then $f \circ g(x) = f([(0,x)]) = x$ and $g \circ f([(x_1,x_2)]) = g(x_2) = [(0,x_2)] = [(x_1,x_2)]$. So g is the inverse of f and f is bijective.

Finally, $\mathbb{R}$ is bijective to $(0,1)$ via the map $h(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x$. So $\mathbb{R}^2/\sim$ is bijective to $(0,1)$.

4. (5 points) Prove that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every closed set $C \subset \mathbb{R}$, $f^{-1}(C)$ is closed.

Solution: Suppose f is continuous. C is closed so $\mathbb{R}\setminus C$ is open and $f^{-1}(\mathbb{R}\setminus C)$ is open. If $x \in f^{-1}(\mathbb{R}\setminus C)$, then $f(x)$ is not in C and $x \in \mathbb{R}\setminus f^{-1}(C)$. Similarly, if $x \in \mathbb{R}\setminus f^{-1}(C)$, then $f(x)$ is not in C and $x \in f^{-1}(\mathbb{R}\setminus C)$. Thus $\mathbb{R}\setminus f^{-1}(C) = f^{-1}(\mathbb{R}\setminus C)$ and $\mathbb{R}\setminus f^{-1}(C)$ is open. This gives that $f^{-1}(C)$ is closed.

Conversely, the argument works the exact same way by proving that $f^{-1}(U)$ is open for every open set $U \subset \mathbb{R}$.

5. (5 points) The symmetric difference of two sets E and F is defined as $E\Delta F = E\setminus F \cup F\setminus E$. Prove that a set E is measurable if and only if for all $\epsilon > 0$, there exists an open set U so that $m^*(U\Delta E) < \epsilon$.

Solution: If E is measurable, then for all $\epsilon > 0$, there exists an open set U that contains E and satisfies $m^*(U \setminus E) < \epsilon$. So $m^*(U \Delta E) = m^*(U \setminus E) < \epsilon$.

Conversely, Fix $\epsilon > 0$ and let U be on open set that satisfies $m^*(E \Delta U) < \epsilon/4$. So $m^*(E \setminus U) < \epsilon/4$ and by the definition of outer measure, there exists an open set V that contains $E \setminus U$ and satisfies $m^*(V) < \epsilon/2$. $E \subset U \cup V$, the set $U \cup V$ is open and

$$m^*((U \cup V) \setminus E) \leq m^*(U \setminus E) + m^*(V) < \epsilon.$$

This gives that E is measurable.

6. (5 points) Let $f(x) = \frac{x}{6\pi} e^{3x^2 - 2x + 1}$. Show that $f(x) = 1$ has a positive solution.

   Solution: $f(0) = 0$ and $f(100) > 1$. By the intermediate value theorem there exists $c \in (0, 100)$ so that $f(c) = 1$.

7. (5 points) Let E and F be closed sets, show that $E + F = \{x \in \mathbb{R} : x = a + b, a \in E, b \in F\}$ is measurable. **Hint:** First prove this for E and F compact.

   Solution: First suppose E and F are compact, then $E + F$ is bounded. If $a_n + b_n \to c$ for $a_n \in E$ and $b_n \in F$, then by the compactness of E and F, a_n and b_n have subsequences that converge to $a \in E$ and $b \in F$ respectively. Hence $c = a + b \in E + F$. So $E + F$ is closed and therefore compact. This gives that $E + F$ is measurable.

   Now let E and F be arbitrary closed sets. Let $E_n = [-n, n] \cap E$ and let $F_n = [-n, n] \cap F$. E_n and F_n are compact and $E = \bigcup_n E_n$, $F = \bigcup_n F_n$. So $E + F = \bigcup_n E_n + F_n$. The countable union of measurable sets is measurable so $E + F$ is measurable.