Practice Midterm

- No calculators, books, or notes are permitted.
- Nothing should be on your desk but writing implements and a single one-sided reference page.
- If you have a question during the exam, you may leave the room and ask the proctor.
- You will receive points only for what is written on the numbered pages. Please use the back of pages as scratch paper.
- Please write neatly, show all your work, and justify all answers. Mysterious or illegible solutions will receive no credit.
- If you finish early, check your answers and wait until time is called.
- Please sign the Honor Pledge:

  I pledge my honor that I have not violated the Honor Code during this examination.

No exam without a signature will be graded.
1. (5 points) Define Lebesgue outer measure.

2. (5 points) Show that \((\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]\) is measurable and that \(m((\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]) = 1\).
3. Let \( \sim \) be a relation on \( \mathbb{R}^2 \) defined as \((x_1, x_2) \sim (y_1, y_2)\) if \(x_2 = y_2\).

(a) (2 points) Show that \( \sim \) is an equivalence relation on \( \mathbb{R}^2 \).

(b) (3 points) Find a bijective map from the set of equivalence classes of \( \sim \) to \((0, 1)\). (You must prove the map is bijective.)
4. (5 points) Prove that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every closed set $C \subset \mathbb{R}$, $f^{-1}(C)$ is closed.

5. (5 points) The symmetric difference of two sets $E$ and $F$ is defined as $E \Delta F = E \setminus F \cup F \setminus E$. Prove that a set $E$ is measurable if and only if for all $\epsilon > 0$, there exists an open set $U$ so that $m^*(U \Delta E) < \epsilon$. 
6. (5 points) Let \( f(x) = \frac{x}{6\pi} e^{3x^2-2x+1} \). Show that \( f(x) = 1 \) has a positive solution.

7. (5 points) Let \( E \) and \( F \) be closed sets, show that \( E + F = \{ x \in \mathbb{R} : x = a + b, a \in E, b \in F \} \) is measurable. **Hint:** First prove this for \( E \) and \( F \) compact.
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