Practice Final Solutions

1. (5 points) Let $E \subset \mathbb{R}$ be a measurable set and suppose that $m(E) < \infty$. Show that for all $t \in [0,1]$, there exists disjoint subsets A and B of E that satisfy $m(A) = tm(E)$ and $m(B) = (1-t)m(E)$.

   Solution: The set $\{s \in \mathbb{R} : m((-\infty, s) \cap E) > tm(E)\}$ is nonempty since by downward monotone convergence of sets, $\lim_{n\to\infty} m((-\infty, -n) \cap E) = m(\emptyset)$. Note that this is only true since $m(E) < \infty$, otherwise the hypotheses of the downward monotone convergence theorem for sets are not satisfied. Since the set is nonempty, an infimum exists and let $a = \inf\{s \in \mathbb{R} : m((-\infty, s) \cap E) > tm(E)\}$. Again by downward monotone convergence of sets, $m((-\infty, a) \cap E) = tm(E)$ and by additivity of Lebesgue measure, $m([a, \infty) \cap E) = (1-t)m(E)$.

2. (5 points) Let $f : [0,1] \to [0,1]$ be increasing and continuous and suppose that $f(0) = 0$ and $f(1) = 1$. Show that f is invertible and that f^{-1} is continuous.

   Solution: To see that f is invertible it suffices to show that f is bijective. If $x < y$, then since f is increasing, $f(x) < f(y)$ and so f is injective. If $z \in [0,1]$, then by the intermediate value theorem, there exists $x \in [0,1]$ such that $f(x) = z$. So f is surjective.

   To show that f^{-1} is continuous it suffices to show that the image of a closed set by f is closed. Let $C \subset [0,1]$ be closed. Suppose that $y_n \in f(C)$ converge to $y \in [0,1]$. Since $y_n \in f(C)$, $y_n = f(x_n)$ for $x_n \in C$. The set C is closed and bounded so there exists a subsequence x_{n_k} that converges to $x \in C$. By the continuity of f, $f(x) = f(y)$ and therefore $y \in C$. This shows that C is closed and f is continuous.

3. (5 points) Let f and g be absolutely continuous functions from $[0, 1]$ to $\mathbb{R}$. Show that fg is absolutely continuous.

Solution: Fix $\epsilon > 0$. Since f and g are continuous on a compact interval, they are both bounded. Let $M = \sup_{x\in[0,1]} \max(f(x), g(x))$. Choose δ so that if

$$\sum_{k=1}^{n} |b_k - a_k| < \delta,$$

then

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon/(2M)$$

and

$$\sum_{k=1}^{n} |g(b_k) - g(a_k)| < \epsilon/(2M).$$

Such a δ exists by the absolute continuity of f and g. So

$$\begin{aligned}\sum_{k=1}^{n} |f(b_k)g(b_k) - f(a_k)g(a_k)| &\le \sum_{k=1}^{n} |f(b_k)g(b_k) - f(a_k)g(b_k)| + \sum_{k=1}^{n} |f(a_k)g(b_k) - f(a_k)g(a_k)| \\ &\le M\sum_{k=1}^{n} |f(b_k) - f(a_k)| + M\sum_{k=1}^{n} |g(b_k) - g(a_k)| \\ &< \epsilon.\end{aligned}$$

4. (5 points) Give an example of a sequence of measurable functions $f_n \colon \mathbb{R} \to \mathbb{R}$ and a function $f \colon \mathbb{R} \to \mathbb{R}$ such that f_n converge to f in measure but not pointwise or in L^1.

Solution: Let $f_{n,k}(x) = 2^n \chi_{[k2^{-n},(k+1)2^{-n}]}(x)$ for $n \in \mathbb{N}$ and $0 \le k \le 2^n - 1$. Then $f_{n,k}(x)$ converges to 0 in measure since

$$\lim_{k,n\to\infty} m(\{x : |f_{n,k}(x)| > \lambda\}) \le \lim_{n\to\infty} 2^{-n} = 0.$$

However, the integral of $f_{n,k}(x)$ is always 1 and so $f_{n,k}$ does not converge to 0 in L^1. Additionally, for all $x \in [0,1]$, there exists arbitrarily large indices such that $f_{n,k}(x) = 1$. So f does not converge to 0 pointwise.

5. (5 points) If $x \in \mathbb{Q} \setminus \{0\}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$, prove that $x + y$ and xy are in $\mathbb{R} \setminus \mathbb{Q}$.

   Solution: Suppose that $x + y = r \in \mathbb{Q}$, then $y = r - x \in \mathbb{Q}$, which is a contradiction. Similarly, suppose that $xy = p \in \mathbb{Q}$, then $y = p/x \in \mathbb{Q}$, which is a contradiction.

6. (5 points) Prove the triangle inequality for the Lebesgue integral: If $E \subset \mathbb{R}$ is measurable and $f\colon E \to \mathbb{R}$ is integrable, then

$$\left| \int_E f \right| \leq \int_E |f|.$$

   Solution: If f is positive, then the statement gives an equality. Otherwise, $f = f^+ - f^-$, where $f^+, f^- \geq 0$. So

$$\begin{aligned} \left| \int_E f \right| &\leq \left| \int_E f^+ - \int_E f^- \right| \\ &\leq | \int_E f^+ \Big| + \left| \int_E f^- \right| \quad = \int_E f^+ + \int_E f^- = \int_E f^+ + f^- = \int_E |f|. \end{aligned}$$

7. (5 points) Prove that every measurable function $f\colon \mathbb{R} \to \mathbb{R}$ is equal to the pointwise limit of a sequence of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ almost everywhere.

   Solution: Let $f_n = \chi_{[-n,n]}(\min(f^+, n) - \max(f^-, n))$. The functions f_n are integrable and converge pointwise to f. So it suffices to show that every integrable function g is the pointwise limit of a sequence of continuous functions. By a theorem in class, there exists a sequence of continuous functions g_n that converge to g in L^1. By another theorem in class there exists a subsequence g_{n_k} that converges to g pointwise almost everywhere.

8. (a) (3 points) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and the derivative of f is continuous. Show that for every interval $[a, b] \subset \mathbb{R}$, the function f is of bounded variation on the interval $[a, b]$.

   Solution: Fix $[a, b] \subset \mathbb{R}$. Since f' is continuous, the derivative f' is bounded on $[a, b]$ by a positive number M. By the mean value theorem,

$$|f(x) - f(y)| \leq M|x - y|$$

   for $x, y \in [a, b]$. Therefore f is Lipschitz on $[a, b]$ and has bounded variation.

   (b) (2 points) Give an example of a function that satisfies the hypotheses of the first part of this question but is not of bounded variation on $\mathbb{R}$.

   Solution: The function $f(x) = x$ does not have bounded variation on $\mathbb{R}$ since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.