Eden Prywes | Mat 320 | Fall 2021

Practice Final

- You may use your notes and textbook but you may not consult other students or resources.
- Please write neatly, show all your work, and justify all answers. Mysterious or illegible solutions will receive no credit.
- You have 3 hours to complete the exam and 15 minutes afterwards to upload your solutions to Gradescope. Please mark your solutions on Gradescope.
- Please sign the Honor Pledge:

  I pledge my honor that I have not violated the Honor Code during this examination.

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No exam without a signature will be graded.

1. (5 points) Let $E \subset \mathbb{R}$ be a measurable set and suppose that $m(E) < \infty$. Show that for all $t \in [0,1]$, there exists disjoint subsets A and B of E that satisfy $m(A) = tm(E)$ and $m(B) = (1-t)m(E)$.

2. (5 points) Let $f\colon [0,1] \to [0,1]$ be strictly increasing and continuous and suppose that $f(0) = 0$ and $f(1) = 1$. Show that f is invertible and that f^{-1} is continuous.

3. (5 points) Let f and g be absolutely continuous functions from $[0,1]$ to $\mathbb{R}$. Show that fg is absolutely continuous.

4. (5 points) Give an example of a sequence of measurable functions $f_n\colon \mathbb{R} \to \mathbb{R}$ and a function $f\colon \mathbb{R} \to \mathbb{R}$ such that f_n converge to f in measure but not pointwise or in L^1.

5. (5 points) If $x \in \mathbb{Q} \setminus \{0\}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$, prove that $x + y$ and xy are in $\mathbb{R} \setminus \mathbb{Q}$.

6. (5 points) Prove the triangle inequality for the Lebesgue integral: If $E \subset \mathbb{R}$ is measurable and $f\colon E \to \mathbb{R}$ is integrable, then
$$\left| \int_E f \right| \leq \int_E |f|.$$

7. (5 points) Prove that every measurable function $f\colon \mathbb{R} \to \mathbb{R}$ is equal to the pointwise limit of a sequence of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ almost everywhere.

8. (a) (3 points) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and the derivative of f is continuous. Show that for every interval $[a,b] \subset \mathbb{R}$, the function f is of bounded variation on the interval $[a,b]$.

   (b) (2 points) Give an example of a function that satisfies the hypotheses of the first part of this question but is not of bounded variation on $\mathbb{R}$.