Practice Final

- You may use your notes and textbook but you may not consult other students or resources.
- Please write neatly, show all your work, and justify all answers. Mysterious or illegible solutions will receive no credit.
- You have 3 hours to complete the exam and 15 minutes afterwards to upload your solutions to Gradescope. Please mark your solutions on Gradescope.
- Please sign the Honor Pledge:

  I pledge my honor that I have not violated the Honor Code during this examination.

No exam without a signature will be graded.
1. (5 points) Let $E \subset \mathbb{R}$ be a measurable set and suppose that $m(E) < \infty$. Show that for all $t \in [0, 1]$, there exists disjoint subsets $A$ and $B$ of $E$ that satisfy $m(A) = tm(E)$ and $m(B) = (1 - t)m(E)$.

2. (5 points) Let $f : [0, 1] \rightarrow [0, 1]$ be strictly increasing and continuous and suppose that $f(0) = 0$ and $f(1) = 1$. Show that $f$ is invertible and that $f^{-1}$ is continuous.

3. (5 points) Let $f$ and $g$ be absolutely continuous functions from $[0, 1]$ to $\mathbb{R}$. Show that $fg$ is absolutely continuous.

4. (5 points) Give an example of a sequence of measurable functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ and a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f_n$ converge to $f$ in measure but not pointwise or in $L^1$.

5. (5 points) If $x \in \mathbb{Q} \setminus \{0\}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$, prove that $x + y$ and $xy$ are in $\mathbb{R} \setminus \mathbb{Q}$.

6. (5 points) Prove the triangle inequality for the Lebesgue integral: If $E \subset \mathbb{R}$ is measurable and $f : E \rightarrow \mathbb{R}$ is integrable, then

$$\left| \int_E f \right| \leq \int_E |f|.$$

7. (5 points) Prove that every measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ is equal to the pointwise limit of a sequence of continuous functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ almost everywhere.

8. (a) (3 points) Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and the derivative of $f$ is continuous. Show that for every interval $[a, b] \subset \mathbb{R}$, the function $f$ is of bounded variation on the interval $[a, b]$.

(b) (2 points) Give an example of a function that satisfies the hypotheses of the first part of this question but is not of bounded variation on $\mathbb{R}$. 