Eden Prywes Mat 320 Fall 2021

Midterm Solutions

1. (5 points) State the definition for a uniformly continuous function $f\colon \mathbb{R} \to \mathbb{R}$.

   Solution: For all $\epsilon > 0$, there exists a $\delta > 0$ so that if $x, x' \in \mathbb{R}$ and $|x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$.

2. (5 points) Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $K \subset \mathbb{R}$ is compact, then $f(K)$ is compact.

   Solution: Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of $f(K)$. Since f is continuous, $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is a collection of open sets and $K \subset \bigcup_{\lambda\in\Lambda} f^{-1}(U_\lambda)$. Since K is compact, there exists a finite subcover $f^{-1}(U_{\lambda_1}), \ldots, f^{-1}(U_{\lambda_n})$. Therefore $f(K) \subset \bigcup_{i=1}^n U_{\lambda_i}$, which is a finite subcover of $\{U_\lambda\}_{\lambda\in\Lambda}$. Since the original open cover was arbitrary, $f(K)$ is compact.

3. Let E be a compact set and for all $n \in \mathbb{N}$, let $U_n = \{x \in \mathbb{R} : \inf_{y\in E} |x-y| < 1/n\}$.

   (a) (5 points) Prove that U_n is open.

   Solution: Let $x \in E$ and suppose that $|y-x| < 1/n$, then by the definition of U_n, $y \in U_n$. So $\bigcup_{x\in E}(x - 1/n, x + 1/n) \subset U_n$. Conversely, if $x \in U_n$, then $\inf_{x\in E} |y - x| < 1/n$ and there exists a point $x_0 \in E$ that satisfies $|x_0 - y| < 1/n$. So $U_n \subset \bigcup_{x\in E}(x - 1/n, x + 1/n)$.

   This gives that $U_n = \bigcup_{x\in E}(x - 1/n, x + 1/n)$. The union of open sets is open and so U_n is open.

   (b) (5 points) Show that $\lim_{n\to\infty} m(U_n) = m(E)$.

   Solution: $E \subset U_n$ for all $n \in \mathbb{N}$, so $E \subset \bigcap_{n\in\mathbb{N}} U_n$. If $x \in \bigcap_{n\in\mathbb{N}} U_n$, then the distance of x and E is 0 and since both E and $\{x\}$ are compact sets with zero distance they must intersect. So $x \in E$. This gives that $E = \bigcap_{n\in\mathbb{N}} U_n$.

By the definition of U_n, $U_{n+1} \subset U_n$ and since E is bounded $m(U_1) < \infty$. This allows us to use the monotone convergence of Lebesgue measure to conclude that

$$\lim_{n\to\infty} m(U_n) = m(\bigcap_{n\in\mathbb{N}} U_n) = m(E).$$

4. (5 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that is uniformly continuous on $\mathbb{Q}$. Show that f is continuous.

   Solution: Fix $\epsilon > 0$ and let δ be chosen so that if $x \in \mathbb{R}$ and $q \in \mathbb{Q}$ with $|x - q| < \delta$, then $|f(x) - f(q)| < \epsilon/2$.

   Suppose that $x, x' \in \mathbb{R}$ with $|x - x'| < \delta$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number q between x and x' (unless they are equal and $f(x) = f(x')$). Since

$$|x - q| < |x - x'| \quad \text{and} \quad |x' - q| < |x - x'|,$$

   we have that

$$|f(x) - f(x')| < |f(x) - f(q)| + |f(q) - f(x')| < \epsilon/2 + \epsilon/2 = \epsilon.$$

5. (5 points) Prove that if $E \subset \mathbb{R}$ is measurable and $m(E) < \infty$, then for all $\epsilon > 0$, there exists a finite union of open intervals $U = \cup_{i=1}^{n} I_n$ so that

$$m(E \setminus U) + m(U \setminus E) < \epsilon.$$

   Solution: Fix $\epsilon > 0$. Since E is measurable, there exists an open set U so that $m(U \setminus E) < \epsilon/2$. U is open and so $U = \bigcup_{k\in\mathbb{N}} I_k$, where I_k are disjoint open intervals. So

$$\infty > m(U) = \sum_{k\in\mathbb{N}} m(I_k).$$

Thus there exists K so that the tail sum, $\sum_{k\geq K+1} m(I_k) < \epsilon/2$. We claim that $O = \bigcup_{k=1}^{K} I_k$ is the finite collection of intervals that satisfies the problem.

$$m(O \setminus E) < m(U \setminus E) < \epsilon/2$$

since $O \subset U$.

$$m(E \setminus O) = m(E \cap (\bigcup_{k=K+1}^{\infty} I_k)) \leq \sum_{k\geq K+1} m(I_k) < \epsilon/2.$$

Therefore

$$m(O \setminus E) + m(E \setminus O) < \epsilon.$$