MAT 320: PROBLEM SET 9

DUE MONDAY NOVEMBER 22

Problem 1: Suppose the $I_1, \dots, I_n$ are closed and bounded intervals that cover a set $E \subset \mathbb{R}$. Prove that there exists a disjoint subcollection $I_{i_1}, \dots, I_{i_k}$ such that

$$E \subset \bigcup_{j=1}^{k} 3I_{j_k},$$

where $3I$ is the interval with the same midpoint as I, but three times the length. **Hint:** Use a similar method to choose the intervals as in the proof for the Vitali Covering theorem.

Problem 2: Prove that a function $f\colon (a,b) \to \mathbb{R}$ that is differentiable is continuous. Show that there exists a function $f\colon [0,1] \to \mathbb{R}$ that is monotone (and hence differentiable almost everywhere) but not continuous on $\mathbb{Q} \cap [0,1]$. **Hint:** Construct a function that has a discontinuity of size 2^{-n} at each rational point.

Problem 3: A function $f\colon \mathbb{R} \to \mathbb{R}$ is convex if for all $t \in (0,1)$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. Show that a convex function is continuous and if the convex function is monotone, then the derivative f', defined almost everywhere, is monotone (and hence differentiable almost everywhere).

Problem 4: Chapter 5.2 Problem 11.

Problem 5: Chapter 6.1 Problem 2.

Problem 6: Chapter 6.2 Problem 9.