Problem 1: Let \( \{a_n\}_{n \in \mathbb{N}} \) and \( \{b_n\}_{n \in \mathbb{N}} \) be positive sequences.

(i) Show that
\[
\limsup_{n \to \infty} a_n b_n \leq \limsup_{n \to \infty} a_n \limsup_{n \to \infty} b_n.
\]

(ii) Give an example where the inequality in (i) is a strict inequality.

(iii) Show that if \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} b_n = b \), then \( \lim_{n \to \infty} a_n b_n = ab \).

Problem 2: Let \( a_n = cr^n \), where \( c, r \in \mathbb{R} \setminus \{0\} \). Show that the series \( \sum_{n=1}^{\infty} a_n \) converges if and only if \( |r| < 1 \).

Problem 3: Tonelli’s theorem states: Let \( \{a_{n,m}\}_{n,m \in \mathbb{N} \times \mathbb{N}} \) be a sequence indexed by \( \mathbb{N} \times \mathbb{N} \). If \( a_{n,m} \geq 0 \), then
\[
\sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} a_{n,m} \right) = \sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} a_{n,m} \right).
\]
Show that the equality does not necessarily hold if the terms \( a_{n,m} \) are not assumed to be non-negative.

Problem 4: Let \( E \subset \mathbb{R} \). A function \( f: E \to \mathbb{R} \) is called Lipschitz if there exists a constant \( C \geq 0 \) so that for all \( x, y \in E \), the function satisfies \( |f(x) - f(y)| \leq C|x - y| \).

(i) Show all Lipschitz functions are continuous.

(ii) Show that \( f(x) = \sqrt{x} \) defined on \( (0, 1) \) is not Lipschitz.

(iii) Let \( f: [0, 1] \to \mathbb{R} \) be a function. Show that if there exists a constant \( C \geq 0 \) so that for all \( x, y \in [0, 1] \)
\[
|f(x) - f(y)| \leq C|x - y|^2,
\]
then \( f \) is constant. Hint: Show that \( f \) is differentiable at every point in \( [0, 1] \) (recall that a function is differentiable at \( x \) if \( \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \) exists).

Problem 5: Chapter 1.5, Question 44.