MAT 320: PROBLEM SET 3

DUE MONDAY SEPTEMBER 27

Problem 1: Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be positive sequences.

(i) Show that

$$\limsup_{n\to\infty} a_n b_n \leq \limsup_{n\to\infty} a_n \limsup_{n\to\infty} b_n.$$

(ii) Give an example where the inequality in (i) is a strict inequality.

(iii) Show that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} a_n b_n = ab$.

Problem 2: Let $a_n = cr^n$, where $c, r \in \mathbb{R} \setminus \{0\}$. Show that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $|r| < 1$.

Problem 3: Tonelli's theorem states: Let $\{a_{n,m}\}_{n,m\in\mathbb{N}\times\mathbb{N}}$ be a sequence indexed by $\mathbb{N}\times\mathbb{N}$. If $a_{n,m} \geq 0$, then

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{n,m}\right) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{n,m}\right).$$

Show that the equality does not necessarily hold if the terms $a_{n,m}$ are not assumed to be non-negative.

Problem 4: Let $E \subset \mathbb{R}$. A function $f\colon E \to \mathbb{R}$ is called Lipschitz if there exists a constant $C \geq 0$ so that for all $x, y \in E$, the function satisfies $|f(x) - f(y)| \leq C|x - y|$.

(i) Show all Lipschitz functions are continuous.

(ii) Show that $f(x) = \sqrt{x}$ defined on $(0, 1)$ is not Lipschitz.

(iii) Let $f\colon [0,1] \to \mathbb{R}$ be a function. Show that if there exists a constant $C \geq 0$ so that for all $x, y \in [0, 1]$

$$|f(x) - f(y)| \leq C|x - y|^2,$$

then f is constant. **Hint:** Show that f is differentiable at every point in $[0, 1]$ (recall that a function is differentiable at x if $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists).

Problem 5: Chapter 1.5, Question 44.