Characterization of Branched Covers with Simplicial Branch Sets

Eden Prywes

Princeton University

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Joint work with Rami Luisto

Definition

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- At most points f is a local homeomorphism. The *branch set* of f, denoted B_f, is the set of points where f fails to be a local homeomorphism.
- Branched covers are topological generalization of quasiregular maps.

Branched Covers in Dimension Two

In two dimensions the typical example of a branched cover is a rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

- The branch set is the finite set of critical points of f.
- Near the branch points, f behaves like the map z^d, where d is the degree of the map.

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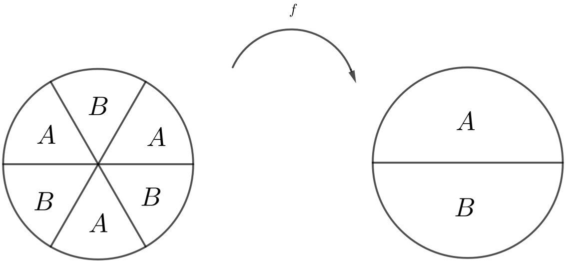
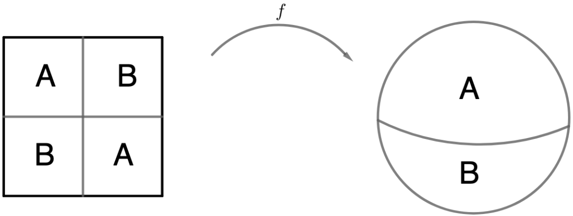
- The branch set is the finite set of critical points of f.
- Near the branch points, f behaves like the map z^d, where d is the degree of the map.
- Topologically, this map is equivalent to a winding map: $(r, \theta) \mapsto (r, d\theta)$.



Example

Define $f\colon \mathbb{C}/\mathbb{Z}^2 \to \widehat{\mathbb{C}}$ as:



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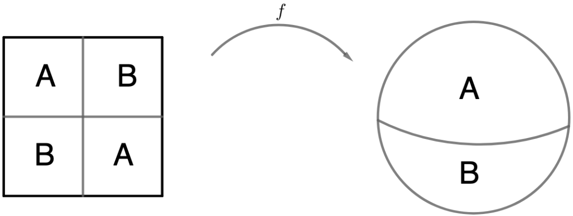

- This is topologically the same as the Weierstrass p-function.
- Locally near the branch point the maps behaves like a winding map.
- Can extend this to a PL-map $F\colon \mathbb{R}^{2n} \to S^2 \times \cdots \times S^2$.

Up to homeomorphism, rational maps characterize every branched cover.

Theorem (Stoïlow)

Let $f: S^2 \to \widehat{\mathbb{C}}$ be a branched cover. Then there exists a homeomorphism $h: \widehat{\mathbb{C}} \to S^2$ so that $f \circ h$ is a rational map.

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Corollary

Every branched cover from $S^2 \to S^2$ is equivalent up to a homeomorphism to a piecewise linear (PL) map.

Motivation

Definition

A map $f : \Omega \to \mathbb{R}^n$ is *K-quasiregular* if $f \in W^{1,n}_{\text{loc}}(\Omega)$ and for almost every $x \in \Omega$,

$$\|Df\|^n \leq KJ_f,$$

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- By a theorem due to Reshetnyak, quasiregular maps are branched covers.
- The converse is false in general and it is difficult to construct quasiregular maps.
- PL maps are typically quasiregular.

Quasiregular Ellipticity

An n-dimensional manifold M is *Quasiregularly Elliptic* if there exists a quasiregular map $f\colon \mathbb{R}^n \to M$.
In dimension 2, M is homeomorphic to $\mathbb{C}, \widehat{\mathbb{C}}, S^1 \times \mathbb{R}$ or $S^1 \times S^1$.
In dimension 3, closed quasiregularly elliptic manifolds are quotients of either $S^3, S^1 \times S^1 \times S^1$ or $S^2 \times S^1$.

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Theorem (P., '19)

If M is a closed, orientable Riemannian manifold of dimension d that admits a quasiregular map from $\mathbb{R}^d$, then $\dim H^\ell(M) \leq \binom{d}{\ell}$. If $\ell = d/2$, then $b^+_{d/2}(M), b^-_{d/2}(M) \leq \frac{1}{2}\binom{d}{d/2}$.

Flat Branch Sets

The geometry of the branch set can give information on the behavior of the map.

Theorem (Church and Hemmingsen, '60)

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover, where Ω is a domain in $\mathbb{R}^n$. If $f(B_f)$ can be embedded into a codimension 2 subspace, then f is topologically equivalent to a winding map.

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- The k-winding map for $k \in \mathbb{N}$ is $w_k(r, \theta, x_2, \ldots, x_n) := (r, k\theta, x_2, \ldots, x_n)$.
- By a theorem due to Černavskiĭ and Väisälä, B_f and $f(B_f)$ have topological dimension less than or equal to $n - 2$.
- In dimension 2 this hypothesis is always satisfied, but it is not always satisfied in higher dimensions.

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- Note that $\pi_1(S^5 \setminus \Sigma\Sigma\pi(B_{\Sigma\Sigma\pi}))$ has order 120.

The natural choice for an open neighborhood of a point in $\Sigma\Sigma\pi(B_{\Sigma\Sigma\pi})$ has boundary that is homeomorphic to ΣP not S^4.

Generalizing Church and Hemmingsen

Theorem (Martio and Srebro, '79)

Let $f : \Omega \to \mathbb{R}^3$ be a branched cover and $x_0 \in B_f$. If there exists and open neighborhood V of x_0 so that the image of the branch set $f(B_f \cap V)$ can be embedded into a union of finitely many line segments originating from $f(x_0)$, then f is topologically equivalent on V to a cone of a rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

- A cone of a map g is the map

$$g \times \text{id}\colon\ \text{cone}(\widehat{\mathbb{C}}) \to \text{cone}(\widehat{\mathbb{C}}),$$

$$\text{cone}(\widehat{\mathbb{C}}) = \frac{\widehat{\mathbb{C}} \times [0,1]}{\{(z,0) \sim (w,0)\}}$$

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- This implies that f is topologically equivalent to a PL map.

Main Result

Theorem (Luisto and P., '19)

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in B_f$. If there exists an open neighborhood V of x_0 so that the image of the branch set $f(B_f \cap V)$ can be embedded into an $(n-2)$-simplicial complex, then f is topologically equivalent on V to a cone of a PL map $g : S^{n-1} \to S^{n-1}$.

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- This implies that f is topologically equivalent to a PL map.
- This theorem also extends to a global result for $f: S^n \to S^n$.

Construction of a QR map

We can use this result to construct quasiregular maps.

Corollary

For each $n \in \mathbb{N}$ there exists a quasiregular map $f : \mathbb{R}^{2n} \to \mathbb{CP}^n$.

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For each $n \in \mathbb{N}$ there exists a quasiregular map $f \colon \mathbb{R}^{2n} \to \mathbb{CP}^n$.

- There exists a quasiregular map from $\mathbb{R}^{2n} \to (\mathbb{CP}^1)^n$.
- The map

$$([z_1 : w_1], \cdots, [z_n : w_n]) \mapsto \\ [z_1 \cdots z_n : \sum_{i=1}^{n} z_1 \cdots \widehat{z_i} \cdots z_n w_i : \cdots : w_1 \cdots w_n]$$

is a branched cover from $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1 \to \mathbb{CP}^n$.

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The map can be thought of as the coefficients of the polynomial

$$p(u, v) = (z_1 u + w_1 v) \cdots (z_n u + w_n v).$$

So the branch set is

$$\{([z_1 : w_1], \cdots, [z_n : w_n]) \in (\mathbb{CP}^1)^n : [z_i : w_i] = [z_j : w_j] \text{ for some } i \neq j\}.$$

The image of this can be given a simplicial structure and so there is a PL version of the map.

For dimension 4:

Theorem (Piergallini and Zuddas, '18)

If M is of the form $\#_m \mathbb{CP}^2 \#_n \overline{\mathbb{CP}^2}$ or $\#_n(S^2 \times S^2)$, then N admits a PL (and quasiregular) map from $\mathbb{R}^4$ when $b_2^+(M), b_2^-(M) \leq 3$.

If M is quasiregularly elliptic, then $b_2^+(M), b_2^-(M) \leq \frac{1}{2}\binom{n}{n/2}$.

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- Is there a manifold that admits a quasiregular (PL) map from $\mathbb{R}^d$, but does not admit a quasiregular (PL) map from T^d?
- All the above examples factor through the torus.

Normal Neighborhoods

Theorem (Luisto and P., '19)

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in B_f$. If there exists an open neighborhood V of x_0 so that the image of the branch set $f(B_f \cap V)$ can be embedded into an $(n-2)$-simplicial complex, then f is topologically equivalent on V to a cone of a PL map $g : S^{n-1} \to S^{n-1}$.

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in \Omega$ be a point. There exists a radius $r_0 > 0$ and a family of neighborhoods, denoted $U(x_0, r)$, such that for $0 < r \leq r_0$

- $x_0 \in U(x_0, r)$
- $f(U(x_0, r)) = B(f(x_0), r)$
- $f(\partial U(x_0, r) = \partial B(f(x_0), r)$
- $f^{-1}\{f(x_0)\} \cap U(x_0, r) = \{x_0\}$

Outline of Proof

- Suppose that near x_0, $\partial U(x_0, r)$ is homeomorphic to S^{n-1}.
- It is a fact that restricted to $\partial U(x_0, r)$, f is still a branched cover. So if we induct on the dimension, $f\colon \partial U(x_0, r) \to S^{n-1}$ is equivalent to a PL map.

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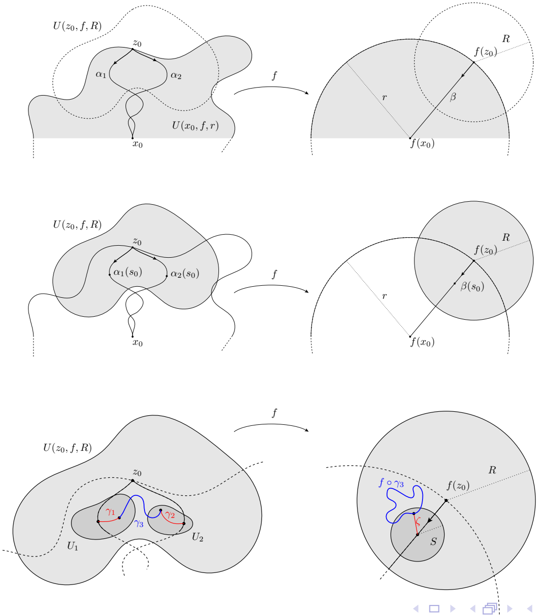
- By a path lifting argument we show that f behaves the same way topologically on the boundaries of $U(x_0, r)$ for all sufficiently small r.
- So f is equivalent to a cone of a PL map.
- It is not clear that $\partial U(x_0, r) \simeq S^{n-1}$, in fact it may not even be a manifold.

Path Lifting



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In dimensions two and three the proof simplifies.

- In dimension two, f is locally injective on $\partial U(x_0, r)$ and so $\partial U(x_0, r)$ is a manifold and is homeomorphic to S^1.

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- In dimension three Martio and Srebro first show that $\partial U(x_0, r)$ is a manifold.
  - Like in dimension 2, $\partial U(x_0, r)$ is a manifold away from the branch set of f.
  - The image of the branch set is "ray-like" by assumption and so intersects $B(f(x_0), r)$ at a discrete set of points.
  - Topologically f behaves like the power map $z \mapsto z^d$ on $\partial U(x_0, r)$ near the intersection by Church and Hemmingsen's theorem and so still can be used to define a chart for $\partial U(x_0, r)$,

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- $\partial U(x_0, r)$ is homeomorphic to S^2 if it is simply connected. This follows because $U(x_0, r)$ is contractible.

$\partial U(x_0, r)$ is a Manifold

In the general case when $f\colon \Omega \to \mathbb{R}^n$,

- f restricted to $\partial U(x_0, r)$ is a branched cover and away from the branch set is a covering map. So $\partial U(x_0, r) \setminus B_f$ is a manifold.
- As in dimension 3, if $x \in \partial U(x_0, r) \cap B_f$, then we consider the map f restricted to a normal neighborhood of x in $\partial U(x_0, r)$.
- We continue to go down in dimension considering more and more nested normal neighborhoods.

$\partial U(x_0, r)$ is homeomorphic to a sphere

We show a stronger fact that for any point x, if we consider normal neighborhoods of dimension $k+1$, then their boundaries will be homeomorphic to S^k when taken sufficiently close to x.

- By the path lifting argument, normal neighborhoods will have a cone structure. So if U is a normal neighborhood of a point x, then $U \simeq \text{cone}(\partial U)$.

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- By the path lifting argument, normal neighborhoods will have a cone structure. So if U is a normal neighborhood of a point x, then $U \simeq \text{cone}(\partial U)$.
- Let $\gamma\colon S^\ell \to \partial U \simeq U \setminus \{x_0\}$. There is a homotopy sending γ to a point in $U \times (0,1)^{n-k-1}$. It can be chosen to avoid $\{x_0\} \times (0,1)^{n-k-1}$ when $1 \leq \ell < k$. So $\pi_\ell(V) = 0$.

Converse Result

There is a partial converse to the Martio-Srebro result.

Theorem (Martio and Srebro, '79)

Let $f : \Omega \to \mathbb{R}^3$ be a branched cover so that at $x \in \Omega$ there exists an $r_0 > 0$ with the property that for all $r \leq r_0$, $\partial U(x_0, r)$ is a manifold. Then at x_0, f is equivalent to a path of rational maps.

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We show a corresponding result:

Theorem (Luisto and P, '19)

Let $f : \Omega \to \mathbb{R}^n$ *be a branched cover so that at* $x \in \Omega$ *there exists an* $r_0 > 0$ *with the property that for all* $r \leq r_0$, $U(x_0, r)$ *is a manifold with boundary. Then at* x_0, f *is equivalent to a path of branched covers.*

Converse Result

We can iterate the previous result to get a lower bound on the topological dimension of B_f.

Corollary (Luisto and P, '19)

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover so that for some k, $2 \leq k \leq n-2$, all the normal domains of dimension less than k are manifolds with boundary, then $\dim_{\text{top}}(B_f) \geq n-k$.

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It is not possible to show that if all the normal domains are manifolds, then f is equivalent to a PL-map. Let $w\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a winding map and let $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a homeomorphism that takes the set $B = \{(0, t^2\cos(1/t), t), t \in \mathbb{R}\}$ to the z-axis near 0. Define $f := w \circ h \circ w$. The branch set is a union of the z-axis and B.

Thank you!