MATH 333 - PROBLEM SET I

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Try to solve the following problems without using any references other than your notes, and the textbook. If you find this hard, you can also use other references (with proper attribution!)

1. Part I

Answer all of the following

(1) give a complete proof of the following theorem discussed in class: let $f \in C^\infty([0, 1])$ be such that for every $x \in [0, 1]$, there is some $n$ (which may depend on $x$) so that $f^{(n)}(x) = 0$. Show that $f$ is a polynomial.

(2) Prove that the measure on $[0, 1]$ with density $d\mu = \frac{1}{\log 2(1+x)} dx$ is invariant under the transformation $T(x) = \langle x^{-1} \rangle$ (with $\langle y \rangle = y - [y]$ denoting the fractional part of $y$). This measure is sometimes called the Guass measure (see Stein’s book p.322, problem 8).

(3) Let $X$ be a compact metric space. What are the extreme points of the space of probability measures on $X$? (we talked about this in class, but there is still something to prove).

(4) (a) Let $A \in SL(n, \mathbb{Z})$, i.e $A$ is an $n \times n$ integer matrix with determinant 1. Define the map $T_A$ from $X = \mathbb{R}^n/\mathbb{Z}^n$ (i.e. the unit cube in $\mathbb{R}^n$ with opposite faces identified) to itself by $T_A(x) = Ax \mod 1$ (the mod 1 is taken on each coordinate separately). Finally, define an isometry $\tilde{T}_A$ on the Hilbert space $H_1 = L^2(X, \lambda_X)$ (where $\lambda_X$ Lebesgue measure on $X$) by $f \mapsto f \circ T_A$. Let $B \in SL(m, \mathbb{Z})$, $Y = \mathbb{R}^m/\mathbb{Z}^m$, and define $H_2, T_B$ and $\tilde{T}_B$ as above. Show that if $T_A$ and $T_B$ are both ergodic, the action of $\tilde{T}_A$ on $H_1$ is unitarily equivalent to the action of $\tilde{T}_B$ on $H_2$, i.e. there is a unitary isomorphism $\Theta : H_1 \to H_2$ so that $\Theta \circ \tilde{T}_A = \tilde{T}_B \circ \Theta$.

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(b) find an ergodic measure preserving transformation $T$ on a probability space $(X, \mu)$ so that the action of the corresponding isometry $\tilde{T}$ on $L^2(X, \mu)$ is not unitary equivalent to $\tilde{T}_A$ acting on $H_1$.

**Remark:** the action of $T_A$ on the probability measures space $(X, \lambda_X)$ is in general not isomorphic to the action of $T_B$ on the probability measures space $(Y, \lambda_Y)$. There is an important invariant of measure preserving transformations called *entropy* which on $T_A$ is equal to the sum $h_A = \sum \lambda n \max(\log |\lambda|, 0)$ the summation being on the set of eigenvalues of $A$ and the $n_\lambda$ being the multiplicity of the eigenvalue (complex or real), so if $h_A \neq h_B$ these actions are not isomorphic. If $h_A = h_B$ these two actions are isomorphic, but this is much harder to show.

2. **Part II**

Read carefully all of the problems below. Answer at least some of them (the more, the better).

(1) (a) p.152 problem 2 from Stein’s book.
(b) Use this covering result to prove the following generalization of the maximal inequality on $\mathbb{Z}$: Let $a_i > 0$. Define for any $n$ and $f: \mathbb{Z} \to \mathbb{R}$

$$M(f, a, n) = \sup_{k \geq 1} \left| \sum_{i=n}^{n+k-1} a_i f(i) \right|.$$

Let $A = \{ n : M(f, a, n) > \lambda \}$. Prove that

$$\sum_{n \in A} a_n \leq C\lambda^{-1} \sum_{n} a_n |f(n)|.$$

(2) p. 152 problem 3*. Directions: prove first that for any $n$ there is a $N = N(n)$ so that any $N$ balls $B_1, \ldots, B_N$ with $\bigcap_{i=1}^{N} B_i \neq \emptyset$ there are some $i \neq j$ so that $B_i$ contains the center of $B_j$. Can you find a corresponding maximal inequality as in II.1.a?

(3) Prove that the Guass measure from problem I.2 is mixing.

(4) Show that there are infinitely many $x \in \mathbb{R}$ so that

$$|x - p/q| < q^{-2006} \quad \text{for infinitely many } q$$

and also

$$\left| x + \sqrt{2} - p/q \right| < q^{-2006} \quad \text{for infinitely many } q.$$

(5) Prove the following fundamental theorem (here it is quite crucial that $L^2(X, \mu)$ is the complex Hilbert space):
**Theorem 2.1.** \((X, \mu, T)\) (if you want to be pedantic, add also the sigma algebra of measurable sets to this notation, and of course we assume that \(\mu\) is a probability measure and \(T\) preserves it [otherwise there is no good notion of weakly mixing]...) is weakly mixing if and only if the corresponding unitary operator \(\tilde{T}\) on \(L^2(X, \mu)\) (see I.4) has no eigenfunctions except the constant function.

The only if part is relatively straightforward. To prove the if part, suppose \(f \in L^2(X \times X)\) is \(T \times T\) invariant, nonconstant.

Show first that without loss of generality, \(f(x, y) = \overline{f(y, x)}\) (this is quite easy, but there is a potential pitfall here). Define an operator \(A_f\) on \(L^2(X)\) by

\[
A_f(g)(y) = \int_X g(x)f(x, y).
\]

Show that \(A_f\) is a self adjoint compact operator and conclude that ...

(6) Prove the following:

**Proposition 2.2.** \((X, \mu, T)\) is weakly mixing if and only for every three measurable sets \(A, B, C\) of positive measure, there is an \(n > 0\) so that both \(\mu(T^{-n}B \cap A) > 0\) and \(\mu(T^{-n}C \cap A) > 0\).

Even if you did not do II.5, you may use the theorem quoted there.