HOMEWORK ASSIGNMENT #9

EX. PAG 188, N. 1

$$z\cosh(z^2) = z\cdot\sum_{m=0}^{\infty}\frac{(z^2)^{2m}}{(2m)!} = \sum_{n=0}^{\infty}\frac{z^{4m+1}}{(2m)!} \qquad |z|<\infty$$

$|z|<\infty$

EX. PAG 188, N. 3

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\cdot\frac{1}{1+\frac{z^4}{9}} = \frac{z}{9}\sum_{n=0}^{\infty}(-1)^m\left(\frac{z^4}{9}\right)^m = \sum_{n=0}^{\infty}\frac{(-1)^m}{3^{2m+2}}z^{4m+1}$$

$\left|\frac{z^4}{9}\right|<1$

$|z|<\sqrt{3}$

EX. PAG. 188, N. 4

$f(z) = \sin z \quad \rightarrow \quad f^{(2m)}(z) = (-1)^m \sin z \quad \longrightarrow \quad f^{(2m)}(0) = 0$

$f^{(2m+1)}(z) = (-1)^m \cos z \quad \longrightarrow \quad f^{(2m+1)}(0) = (-1)^m$

NOT ASSIGNED.!

$$\Rightarrow \quad \sin z = \sum_{n=0}^{\infty}\frac{f^{(m)}(0)}{m!}z^m = \sum_{n=0}^{\infty}\frac{(-1)^m z^{2m+1}}{(2m+1)!}$$

It's entire

$\rightarrow \; |z|<\infty$

EX. PAG. 188, N. 10

$f(z) = \tanh z = \frac{\sinh z}{\cosh z} \quad \longrightarrow$ it's not analytic at $z_m = i\left(\frac{\pi}{2}+\pi m\right) \quad n\in\mathbb{Z}$

since $\cosh z_m = 0$

$\longrightarrow$ the largest circle within the Maclaurin series converges has $R = \frac{\pi}{2}$.

In particular, for $|z|<\pi$:

$$f(z) = \frac{\sinh z}{\cosh z} = \frac{z+\frac{z^3}{6}+o(z^4)}{1+\frac{z^2}{2}+o(z^3)} =$$

$$= \left[z+\frac{z^3}{6}+o(z^4)\right]\frac{1}{1+\left(\frac{z^2}{2}+o(z^3)\right)} =$$

$$= \left[z+\frac{z^3}{6}+o(z^4)\right]\left[1-\frac{z^2}{2}+o(z^3)\right]$$

$$= \left[z+\frac{z^3}{6}-\frac{z^3}{2}+o(z^4)\right] = z+\frac{+1-3}{6}z^3+o(z^4) =$$

$$= z-\frac{z^3}{3}+o(z^4)$$

$\boxed{\text{EX PAG 188, N. 11}}$

a) $\dfrac{e^z}{z^2} = \dfrac{1}{z^2} \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = \sum_{n=0}^{\infty} \dfrac{z^{n-2}}{n!} = \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{2!} + \dfrac{z}{3!} + \dfrac{z^2}{4!} + \dots \qquad (z \neq 0)$

b) $\dfrac{\sin(z^2)}{z^4} = \dfrac{1}{z^4} \sum_{n=0}^{\infty} \dfrac{(z^2)^{2n+1}(-1)^n}{(2n+1)!} = \sum_{n=0}^{\infty} z^{\overbrace{4n+2-4}^{=4n-2}} \dfrac{(-1)^n}{(2n+1)!} =$

$= \dfrac{1}{z^2} - \dfrac{z^2}{\underbrace{6}_{=3!}} + \dfrac{z^6}{5!} + \dots \qquad (z \neq 0)$

$\boxed{\text{EX. PAG 188, N. 12}}$

a) $\dfrac{\sinh z}{z^2} = \dfrac{1}{z^2} \sum_{n=0}^{\infty} \dfrac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \dfrac{z^{2n-1}}{(2n+1)!} = \dfrac{1}{z} + \dfrac{z}{3!} + \dots =$

$= \dfrac{1}{z} + \sum_{n=1}^{\infty} \dfrac{z^{2n-1}}{(2n+1)!} \underset{\substack{\uparrow \\ m=n-1 \\ n=m+1}}{=} \dfrac{1}{z} + \sum_{m=0}^{\infty} \dfrac{z^{2m+1}}{(2m+3)!} \qquad 0<|z|<\infty$

b) $z^3 \cos\left(\dfrac{1}{z}\right) = z^3 \cdot \sum_{n=0}^{\infty} \dfrac{(1/z)^{2n}}{(2n)!} =$

$= \sum_{n=0}^{\infty} \dfrac{1}{z^{2n-3}} \dfrac{1}{(2n)!} = z^3 + \dfrac{z}{2} + \sum_{n=2}^{\infty} \dfrac{1}{z^{(2n-3)}} \dfrac{1}{(2n)!} =$

$\underset{\substack{\uparrow \\ m=n-1 \\ n=m+1}}{=} z^3 + \dfrac{z}{2} + \sum_{m=1}^{\infty} \dfrac{1}{(2m+2)!} \dfrac{1}{z^{2m-1}} \qquad 0<|z|<\infty$

$\boxed{\text{EX. PAG 198, N. 1}}$

$f(z) = z^2 \sin\left(\dfrac{1}{z^2}\right) = z^2 \sum_{n=0}^{\infty} \left(\dfrac{1}{z^2}\right)^{2n+1} \dfrac{(-1)^n}{(2n+1)!} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} \dfrac{1}{z^{4n}} \qquad 0<|z|<\infty$

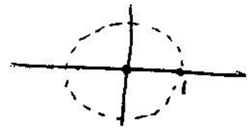
$\boxed{\text{EX. PAG. 198, N. 3}}$

$f(z) = \dfrac{1}{1+z} = \dfrac{1}{z} \cdot \dfrac{1}{1+\frac{1}{z}} \underset{\substack{\uparrow \\ |z|>1 \\ \Rightarrow \frac{1}{|z|}<1}}{=} \dfrac{1}{z} \cdot \sum_{n=0}^{\infty} (-1)^n \dfrac{1}{z^n} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{z^{n+1}} \underset{\substack{\uparrow \\ m=n+1}}{=} \sum_{m=1}^{\infty} \dfrac{(-1)^{m+1}}{z^m}$

$[(-1)^{m-1} = (-1)^{m+1}]$

$\boxed{\text{EX. PAG. 198 N. 4}}$

$f(z) = \dfrac{1}{z^2(1-z)}$ it's analytic in $\mathbb{C} \setminus \{0,1\}$



consider $D_1 = \{0<|z|<1\}$ & $D_2 = \{|z|>1\}$

- find an expansion in D_1:

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2} \cdot \frac{1}{1-z} \underset{|z|<1}{=} \frac{1}{z^2} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n$$

- in D_2: $\frac{1}{|z|}<1$:

$$\frac{1}{z^2} \cdot \frac{1}{1-z} = \frac{-1}{z^3} \frac{1}{1-\frac{1}{z}} = \frac{-1}{z^3} \sum_{n=0}^{\infty} \frac{1}{z^n} = - \sum_{n=0}^{\infty} \frac{1}{z^{n+3}} =$$

$$= -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \dots = - \sum_{n=3}^{\infty} \frac{1}{z^n}$$

$\boxed{\text{EX. PAG 198. N. 9}}$

Suppose that a series $\displaystyle\sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$ ——→ converges to an analytic function in $R_1<|z|<R_2$.

$\parallel$

$X(z)$ ← z transform of $x[n]$

suppose $R_1<1<R_2$.

by uniqueness theorem, $x[n]$ are the Laurent's coefficients of $X(z)$

$$\Rightarrow x[-n] = \frac{1}{2\pi i} \int_C \frac{X(z)}{z^{n+1}}$$

where C is a simple closed curve (positively oriented) contained in the annulus

$n \in \mathbb{Z}$

Since this integral is independent of C, we may choose C = unit circle

$$\to x[-n] = \frac{1}{2\pi i} \int_{-\pi}^{\pi} X(e^{i\theta})\, e^{-(n+1)\theta} e^{i\theta} i \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\theta}) e^{-in\theta} d\theta \qquad n \in \mathbb{Z}$$

$$\Rightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\theta})\, e^{in\theta} d\theta$$

$\square$

$\boxed{\text{EX. PAG 214, N. 8}}$

Suppose f is analytic at z_0 and $f(z_0) = f'(z_0) = \dots = f^{m}(z_0) = 0$

$\Longrightarrow$ $\exists\, \delta > 0$ st in $|z-z_0| < \delta$ we have $f(z) = \sum_{n=m+1}^{\infty} \frac{f^{m}(z_0)}{m!}(z-z_0)^{m}$

consider the function: $h(z) = \frac{f(z)}{(z-z_0)^{m+1}} \underset{\substack{\uparrow \\ 0<|z-z_0|<\delta}}{=} \frac{1}{(z-z_0)^{m+1}} \sum_{n=0}^{\infty} \frac{f^{(n+m+1)}(z_0)}{(n+m+1)!}(z-z_0)^{n+m+1} =$

$$= \frac{f^{(m+1)}(z_0)}{(m+1)!} + \frac{f^{m+2}(z_0)}{(m+2)!}(z-z_0) + o(z-z_0) \qquad (*)$$

h is analytic in $0 < |z-z_0| < \delta$

define $g(z)$ as in the book: $g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^{m+1}} & z \neq z_0 \\ \frac{f^{(m+1)}(z_0)}{(m+1)!}, & z = z_0 \end{cases}$

g is analytic at $0 < |z-z_0| < \delta$

to show that g is analytic at z_0 we need to show:

① g is continuous at z_0: $\underset{\substack{z\neq z_0 \ \| \\ g(z)}}{h(z)} \xrightarrow[(*)]{z\to z_0} \frac{f^{(m+1)}(z_0)}{(m+1)!} = g(z_0)$

② g is differentiable at z_0:

$(z \neq z_0)$ $\frac{g(z)-g(z_0)}{z-z_0} \underset{\substack{\uparrow \\ (*)}}{=} \frac{\cancel{\frac{f^{(m+1)}(z_0)}{(m+1)!}} + \frac{f^{(m+2)}(z_0)}{(m+2)!}(z-z_0) + o(z-z_0) - \cancel{\frac{f^{(m+1)}(z_0)}{(m+1)!}}}{z-z_0} =$

$$= \frac{f^{(m+2)}(z_0)}{(m+2)!} + o(1) \xrightarrow{z\to z_0} \frac{f^{m+2}(z_0)}{(m+2)!}$$

[obviously from (*) : $g'(z) \underset{\substack{\uparrow \\ z\neq z_0}}{=} \frac{f^{(m+2)}(z_0)}{(m+2)!} + o(1)$ hence g' is continuous at z_0]

EX. PAG 219, N. 3

~~$\frac{1}{\exp(z)-1} = \frac{1}{1+z+\frac{z^2}{2}+\frac{z^3}{6}+o(z^3)-1} = \frac{1}{z}$~~

$$\frac{1}{e^z-1} = \frac{1}{\cancel{1}+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+o(z^4)-\cancel{1}} = \frac{1}{z}\left[\frac{1}{1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+o(z^3)}\right] =$$

(#)

$$= \frac{1}{z}\left[\sum_{n=0}^{\infty}\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+o(z^3)\right)^n(-1)^n\right] =$$

$0<|z|<2\pi$

$$= \frac{1}{z}\left[1-\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}\right)+\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}\right)^2-\left(\frac{z}{2}+\frac{z^2}{2}+\frac{z^3}{24}\right)^3+o(z^3)\right]$$

$$= \frac{1}{z}\left[1-\frac{z}{2}+z^2\underbrace{\left(-\frac{1}{6}+\frac{1}{4}\right)}_{=\frac{-2+3}{12}}+z^3\underbrace{\left(-\frac{1}{24}+2\cdot\frac{1}{2}\frac{1}{6}-\frac{1}{8}\right)}_{=\frac{-1+4-3}{24}=0}+o(z^3)\right]$$

$$= \frac{1}{z}\left[1-\frac{z}{2}+\frac{z^2}{12}+z^3\cdot 0+\dots\right]$$

$$= \frac{1}{z}-\frac{1}{2}+\frac{z}{12}+\dots$$

(I could have gotten more terms, if I had considered more terms in (#) above)

EX. PAG 219, N. 8

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty}\frac{E_n}{n!}z^n \qquad |z|<\pi/2$$

this representation makes sense in this disk, since $\frac{1}{\cosh z}$ is analytic in $\mathbb{C}\setminus\left\{\frac{\pi}{2}i+\pi i n\right\}$ $= \frac{(2n+1)}{2}\pi i$

→ the largest disk on which this sum converges has radius $\pi/2$.

Show: $E_{2n+1}=0$

In fact $\frac{1}{\cosh z} = \frac{1}{\sum_{n=0}^{\infty}\frac{z^{2n}}{2n!}}$ hence only even-degree terms will appear! (use division to show this)

$|z|>0$

see also computation below

$\Rightarrow E_{2n+1}=0$

compute the first three terms (nonzero ones)

$$\frac{1}{\cosh z} = \frac{1}{1+\frac{z^2}{2}+\frac{z^4}{4!}+\frac{z^6}{6!}+o(z^7)} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{z^2}{2}+\frac{z^4}{4!}+\frac{z^6}{6!}+o(z^7)\right]^n =$$

$$= 1-\left(\frac{z^2}{2}+\frac{z^4}{4!}+\frac{z^6}{6!}\right)+\left(\frac{z^2}{2}+\frac{z^4}{4!}+\frac{z^6}{6!}\right)^2-\left(\frac{z^2}{2}+\frac{z^4}{4!}+\frac{z^6}{6!}\right)^3+o(z^7) =$$

$$= 1 + z^2\left(-\frac{1}{2}\right) + z^4\underbrace{\left(-\frac{1}{4!}+\frac{1}{4}\right)}_{\frac{-1+6}{24}=\frac{5}{4!}} + z^6\underbrace{\left(-\frac{1}{6!}+\cancel{2}\frac{1}{\cancel{2}}\frac{1}{4!}-\frac{1}{8}\right)}_{\frac{-1+5\cdot 6-6!/8}{6!}=\frac{29-90}{6!}=\frac{-61}{6!}} + o(z^7)$$

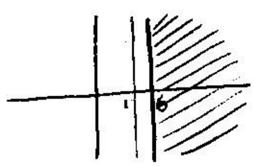
$$= 1-\frac{z^2}{2!}+z^4\frac{5}{4!}-\frac{z^6(61)}{6!}+o(z^7)$$

$$\Rightarrow E_0=1,\quad E_2=-1,\quad E_4=5,\quad E_6=-61$$

$\boxed{\text{EXERCISE \# 3}}$

Riemann zeta function: $\zeta(s)=\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^s}$ $\qquad n^s = e^{s\,\mathrm{Log}\,n}$ $\qquad$ Log analytic branch in $\mathbb{C}\setminus(-\infty,0]$ st $\mathrm{Log}\,1=0$

• Show that this series converges absolutely and uniformly in $\mathrm{Re}(s)\geq\sigma$ $\forall\sigma>1$



$\forall$ $\mathrm{Re}\,s\in[\sigma,+\infty)$

$$|\zeta(s)| = \left|\sum_{n=1}^{\infty}\frac{1}{n^s}\right| \leq \sum_{n=1}^{\infty}\frac{1}{|e^{s\,\mathrm{Log}\,n}|} = \sum_{n=1}^{\infty} e^{-\mathrm{Re}\,s\,\mathrm{Log}\,n} \leq$$

$$\leq \sum_{n=1}^{\infty} e^{-\sigma\,\mathrm{Log}\,n} = \sum_{n=1}^{\infty}\frac{1}{n^\sigma} < \infty \quad \text{since } \sigma>1$$

↑ it's a generalized Harmonic sum.

$\Rightarrow$ this series converges uniformly in $\{\mathrm{Re}\,s\geq\sigma\}$ $\forall\sigma>1$

$$\zeta(s) = \lim_{N \to +\infty} \underbrace{\sum_{n=1}^{N} \frac{1}{n^s}}_{\zeta_N(s)}$$

each of ζ_N is analytic in $\{\operatorname{Re} s > 1\}$. Since the convergence is uniform on every compact, then the limit is still analytic in this domain.

OTHER WAY: • ζ is continuous, since it's uniform limit of continuous function (on every compact set in $\{\operatorname{Re} s > 1\}$)

• let C be a closed contour in D $\rightarrow$ C is contained in $\{\operatorname{Re} z \geq \sigma\}$ $\exists \sigma > 1$

$$\int_C \zeta(z)\, dz = \int_C \lim_{N\to+\infty} \zeta_N(z)\, dz \underset{\substack{\text{uniform}\\ \text{convergence in}\\ \operatorname{Re} z \geq \sigma}}{=} \lim_{N\to+\infty} \underbrace{\int_C \zeta_N(z)\, dz}_{\substack{0\\ \text{by Cauchy}\\ \text{thm}}} = 0$$

$\longrightarrow$ Morera's thm: ζ is analytic throughout $\{\operatorname{Re} z > 1\}$.

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