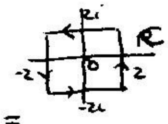
HOMEWORK ASSIGNMENT #8

EX. p. 162, n. 1 (de)



(d)
$$\int_C \frac{\cosh z}{z^4}\,dz = \frac{2\pi i}{3!}\frac{d^3}{dz^3}\left[\cosh z\right]\Big|_{z=0} = \frac{2\pi i}{6}\sinh 0 = 0$$

$\uparrow$ Cauchy's Formula ($z_0 = 0$, $n = 3$)

(e)
$$\int_C \frac{\tan\frac{z}{2}}{(z-x_0)^2}\,dz \qquad -2 < x_0 < 2$$

$\tan\frac{z}{2}$ is analytic as long as $\cos\frac{z}{2} \neq 0$

$\iff z_n \neq \pi + 2n\pi \quad n \in \mathbb{Z}$

none of these pts lies in the region bounded by C

$$\Rightarrow \int_C \frac{\tan\frac{z}{2}}{(z-x_0)^2}\,dz = 2\pi i\,\frac{d}{dz}\tan\frac{z}{2}\Big|_{z=x_0} = 2\pi i\,\frac{1}{2}\sec^2\left(\frac{x_0}{2}\right) = \pi i \sec^2\left(\frac{x_0}{2}\right)$$

$\uparrow$ Cauchy Formula, $z_0 = x_0$ (inside C)

EX. p 164, n. 8

(a)
$$P_n(z) = \frac{1}{n!\,2^n}\frac{d^n}{dz^n}(z^2-1)^n =$$

$$= \frac{1}{n!\,2^n}\frac{d^n}{dz^n}\left[\sum_{j=0}^{n}\binom{n}{j}z^{2j}(-1)^{n-j}\right] =$$

$$= \frac{1}{n!\,2^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\frac{d^n}{dz^n}\left(z^{2j}\right) = \frac{1}{n!\,2^n}\sum_{j\geq\frac{n}{2}}^{n}(-1)^{n-j}\binom{n}{j}\,2j(2j-1)\cdots(2j-n+1)\,z^{2j-n} =$$

$$= \frac{1}{n!\,2^n}\sum_{j\geq\frac{n}{2}}^{n}(-1)^{n-j}\binom{n}{j}\frac{2j!}{(2j-n)!}\,z^{2j-n}$$

this is a polynomial of degree n

since the coefficient corresponding to $j = n$ is $\neq 0$

and $2j - n \leq n \quad (\forall j \geq \frac{n}{2}$ and $j \leq n)$

(b)
$$\frac{1}{n!\,2^n}\frac{d^n}{dz^n}(z^2-1)^n = \left[\frac{n!}{2\pi i}\int_C \frac{(\zeta^2-1)^n}{(\zeta-z)^{n+1}}\right]\cdot\frac{1}{n!\,2^n} = \frac{1}{2^{n+1}\pi i}\int_C \frac{(\zeta^2-1)^n}{(\zeta-z)^{n+1}}\,d\zeta$$

(c) if $z = 1$:
$$\frac{(\zeta^2-1)^n}{(\zeta-1)^{n+1}} = \frac{(\zeta-1)^n(\zeta+1)^n}{(\zeta-1)^{n+1}} = \frac{(\zeta+1)^n}{(\zeta-1)}$$

$$\Rightarrow P(1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(\zeta+1)^n}{(\zeta-1)}\,d\zeta = \frac{1}{2^n}\cdot(\zeta+1)^n\Big|_{\zeta=1} = \frac{1}{2^n}2^n = 1$$

$\uparrow$ Cauchy formula

similarly:

If $z=-1$: $$\frac{(\zeta^2-1)^m}{(\zeta+1)^{m+1}} = \frac{(\zeta-1)^m(\zeta+1)^m}{(\zeta+1)^{m+1}} = \frac{(\zeta-1)^m}{\zeta+1}$$

$$\Longrightarrow \quad P_m(-1) = \frac{1}{2^{m+1}\pi i}\int_C \frac{(\zeta-1)^m}{(\zeta+1)}\,d\zeta = \frac{1}{2^m}\cdot(\zeta-1)^m\Big|_{\zeta=-1} = (-1)^m.$$

$\boxed{\text{EX PAG 172}}$ $\boxed{\text{n° 4}}$

$f \in C(R)$ R closed bounded region

& f analytic non constant in $\mathring{R}$.

Moreover assume $f(z)\neq 0$ in R.

$\to$ consider $g(z)=\frac{1}{f(z)}$: it's analytic in $\mathring{R}$ and continuous in R

moreover $g \not\equiv$ const (since $f\not\equiv$ const) $\Longrightarrow$ $\max_R |g(z)| = \max_{\partial R}|g(z)|$

$$\Longleftrightarrow \quad \max_R \frac{1}{|f(z)|} = \max_{\partial R}\frac{1}{|f(z)|} \quad \Longleftrightarrow \quad \min_R |f(z)| = \min_{\partial R}|f(z)| \quad \square$$

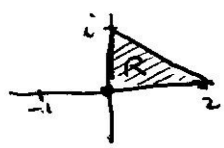
NOTE If we drop the hp that $f(z)\neq 0$, this is not true anymore.

(Ex: $f(z)=z$ in any bounded domain D containing the origin in the interior!) $\min_D |z| = 0 \neq \min_{\partial D}|z|$

$\boxed{\text{n° 5}}$ $f(z)=z$ $R=\{|z|\leq 1\}$.

$$\min_R |f(z)| = 0 \neq 1 = \min_{\partial R}|z|.$$

$\boxed{\text{n° 6}}$ $f(z)=(z+1)^2$



$|f(z)| = |z+1|^2 = |z-(-1)|^2$

is the square of the distance from $z_0=-1$

$$\Longrightarrow \quad \min_R |f(z)| = \left(\min_R d(z,-1)\right)^2 = 1 \qquad \text{it's attained at } z=0$$

$$\max_R |f(z)| = \left(\max_R d(z,-1)\right)^2 = 9 \qquad \text{" " at } z=2$$

EX. PAG 172, n°10

$P(z) = a_0 + a_1 z + \dots + a_m z^m$ & $z_0 \in \mathbb{C}$: $P_m(z_0) = 0$
$(m \geq 1 \quad a_m \neq 0)$

ⓐ verify $z^k - z_0^k = (z-z_0)(z^{k-1} + z^{k-2} z_0 + \dots + z z_0^{k-2} + z_0^{k-1})$ $\quad k = 2, 3, \dots$

by induction: $k=2$: $z^2 - z_0^2 = (z-z_0)(z+z_0)$

suppose it holds for k and show for $k+1$:

$$z^{k+1} - z_0^{k+1} = z^{k+1} + z_0^k z - z_0^k z - z_0^{k+1} = z(z^k - z_0^k) - z_0^k (z - z_0)$$

$$= z(z-z_0)(z^{k-1} + z^{k-2} z_0 \dots + z z_0^{k-2} + z_0^{k-1}) +$$

$$+ z_0^k (z-z_0) =$$

$$= (z-z_0)(z^k + z^{k-1} z_0 + \dots + z^2 z_0^{k-2} + z_0^{k-1} z - z_0^k)$$

$$= (z-z_0)\underbrace{(z^{(k+1)-1} + z^{(k+1)-2} z_0 + \dots + z_0^{(k+1)-2} z - z_0^{(k+1)-1})}_{\Gamma_{k+1}(z, z_0)} \quad \circ$$

ⓑ $P(z) - \overset{0}{P(z_0)} =$

$$= \cancel{a_0 - a_0} + a_1 (z - z_0) + \dots + a_m (z^m - z_0^m) \overset{ⓐ}{=} (z-z_0) \sum_{k=1}^{m} a_k \Gamma_k(z, z_0)$$

$$= (z-z_0) Q(z) \quad \Rightarrow P(z) = (z-z_0) Q(z)$$

$Q(z)$ has degree $m-1$.

EX. 181, N°3

if $\lim_{m\to\infty} z_m = z \Rightarrow \forall \varepsilon > 0 \; \exists N_0 = N_0(\varepsilon) : n \geq N_0 : |z_m - z_0| < \varepsilon$

but $\big||z_m| - |z|\big| \leq |z_m - z_0| < \varepsilon \quad \Rightarrow \forall \varepsilon > 0$ if $m \geq N_0(\varepsilon)$ $\big||z_m| - |z|\big| < \varepsilon$

triangle ineq

$\Rightarrow \lim_{m\to\infty} |z_m| = |z|$

EX p 181 N°4)

$$\sum_{n=0}^{\infty} z^n \overset{z=re^{i\theta}}{=} \sum_{n=0}^{\infty} r^n e^{in\theta} = \sum_{n=0}^{\infty} r^n(\cos n\theta + i \sin n\theta) =$$

$$= \sum_{n=0}^{\infty} r^n \cos n\theta + i \sum_{n=0}^{\infty} r^n \sin n\theta$$

$$\|$$

$$\frac{1}{1-z}$$

$$\|$$

$$\frac{1}{1-re^{i\theta}} = \frac{1-re^{-i\theta}}{|1-re^{i\theta}|^2} = \frac{1-r\cos\theta + i\sin\theta\, r}{(1-r\cos\theta)^2 + (r\sin\theta)^2} =$$

$$= \frac{1-r\cos\theta + i\, r\sin\theta}{1+r^2-2r\cos\theta}$$

$\Longrightarrow$ by thm §52:

$$\sum_{n=0}^{\infty} r^n \cos n\theta = \frac{1-r\cos\theta}{1+r^2-2r\cos\theta} \qquad 0<r<1$$

$$\sum_{n=0}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1+r^2-2r\cos\theta}$$

this also holds when $r=0$:

$$\sum_{n=0}^{\infty} z^n \Big|_{z=0} = 1 + \sum_{n=1}^{\infty} z^n \Big|_{z=0} = 1 =$$

$$\|$$

$$\frac{1-r\cos\theta}{1+r^2-2r\cos\theta} + i\frac{r\sin\theta}{(1+r^2-2r\cos\theta)} \Big|_{r=0}$$

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \sum_{n=0}^{\infty} r^n \cos n\theta - 1$$

$$= \frac{r^2 - r\cos\theta}{1-r^2-2r\cos\theta}$$

EX #2

f is entire & $|f(z)| < A|z|^m + B \quad \exists\ A, B > 0$

$\rightarrow$ let $z_0 \in \mathbb{C}$ and consider $D_R(z_0) = \{|z - z_0| < R\}$

Using Cauchy's formula:

$\forall R > 0$

$$\left|f^{(m+1)}(z_0)\right| = \left| \frac{(m+1)!}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{m+2}}\, d\zeta \right| \le \frac{(m+1)!}{2\pi} (2\pi R) \cdot \frac{A(R+|z_0|)^m + B}{R^{m+2}} =$$

$\uparrow$

$|\zeta - z_0| = R$

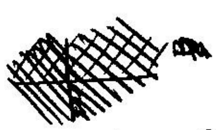
$|f(\zeta)| \le A|\zeta|^m + B \le$

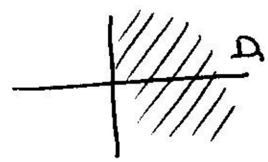
$\le A(R+|z_0|)^m + B$ on ∂D_R

$$= \frac{(m+1)!}{R^{m+1}} \left[A(R+|z_0|)^m + B\right] \xrightarrow{n \to \infty} 0$$

$\Rightarrow \quad |f^{(m+1)}(z_0)| = 0 \quad \forall z_0 \in \mathbb{C} \quad \Rightarrow \quad f^{(m+1)}(z) \equiv 0 \quad \Rightarrow$ f(z) is a polynomial of degree $\le m$

EX #3





(a) i) $D_1 = \{\operatorname{Re} z > 0\}$

$\partial D_1 = \{\text{boundary pts of } D_1\} = \{x = 0\}$

$\uparrow$

imaginary axis

ii) $f\big|_{\partial D_1} = e^{iy}\big|_{y \in \mathbb{R}} \rightarrow \left|f\big|_{\partial D_1}(z)\right| = 1$ hence $|f|$ is bounded on ∂D_1

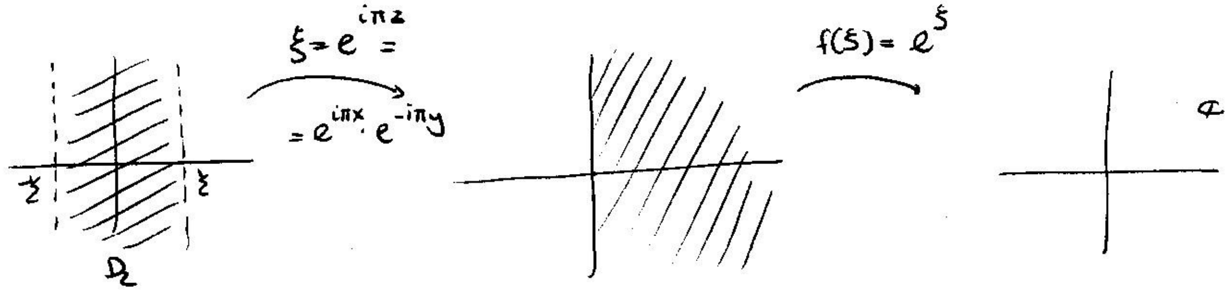
but f is not bounded on D_1: let $z_m = m \quad n \in \mathbb{N}^+ \Rightarrow z_m \in D_1$ &

$f(z_m) = e^m \xrightarrow{n \to +\infty} +\infty$

<u>NOTE</u>: this doesn't contradict the maximum modulus principle, since our domain is not bounded.

(b)

Consider the following composition of maps.



consider $F(z) = e^{e^{i\pi z}}$

$|F|_{\partial D_2}| = |f|_{\partial D_1}| = 1$ but F is not bounded on D_2, since f is not on D_1

since $\partial D_2 \hookrightarrow \partial D_1$

(c) Thm Suppose f is analytic on D_1 and ∂D_1, st $|f(z)| \le M$ on ∂D_1 & $|f(z)| \le A e^{\beta z^{\alpha}}$ $z \in D_1$

$\Longrightarrow$ $|f(z)| \le M$ on D_1

proof
fix $\alpha < \beta < 1$ $\varepsilon > 0$ and consider $\varphi_\varepsilon(z) = e^{-\varepsilon z^{\beta}}$

[Remark in D_1 is possible to define an analytic branch of the principal logarithm.
(since $D_1 \subset \mathbb{C} \setminus (-\infty, 0]$, I can define Log st. Log 1 = 0)]

consider $w \in \mathring{D}_1$. I want to show that $|f(w)| \le M$. (I already know, it happens on ∂D_1)

Prove different facts.

- $\boxed{|\varphi_\varepsilon|_{\partial D_1}| \le 1}$

in fact $\varphi_\varepsilon(iy) = e^{-\varepsilon (iy)^{\beta}} = e^{-\varepsilon e^{\beta(\text{Log}\, iy)}} = e^{-\varepsilon e^{\beta(\log|y| \pm i\frac{\pi}{2})}} = e^{-\varepsilon |y|^{\beta}(\cos\beta\frac{\pi}{2} + i\sin\frac{\pi}{2}\beta)}$

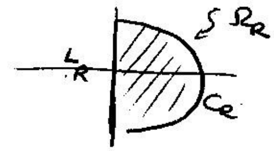
$\Longrightarrow$ $|\varphi_\varepsilon(iy)| = e^{-\varepsilon |y|^{\beta} \cos(\beta\frac{\pi}{2})} \le 1$ (where $\cos\frac{\beta}{2}\pi > 0$, since $0 < \beta < 1$)

Hence $|f\cdot\varphi_\varepsilon| \le M$ on ∂D_1

- Consider $R > |\omega|$ (to be determined later, depending on ε)

and consider $\Omega_R = \dot{D}_1 \cap \{|z| \le R\}$

$\Rightarrow \partial\Omega_R = L_R \cup C_R$



$f\cdot\varphi_\varepsilon$ is analytic on $\mathring{\Omega}_R$ and continuous till $\bar{\Omega}_R$

a) $|f\varphi_\varepsilon| \le M$ on L_R ← Because of our choice of log in D_1

b) φ_ε on C_R: $\varphi_\varepsilon(Re^{i\theta}) = e^{-\varepsilon(Re^{i\theta})^\beta} = e^{-\varepsilon R^\beta e^{i\theta\beta}} \Rightarrow$

$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\Rightarrow |\varphi_\varepsilon(Re^{i\theta})| \le e^{-\varepsilon R^\beta \cos(\beta\frac{\pi}{2})}$$

c) $|f|$ on C_R: $|f(Re^{i\theta})| \underset{\text{by Hp}}{\le} Ae^{B|z|^\alpha} = Ae^{BR^\alpha}$

d) $|f\varphi_\varepsilon|$ on C_R: $|f\varphi_\varepsilon(Re^{i\theta})| \le Ae^{BR^\alpha - \varepsilon R^\beta\cos(\frac{\beta}{2}\pi)}$

observe that the exponent $BR^\alpha - \varepsilon R^\beta\cos(\frac{\beta}{2}\pi) \xrightarrow{R\to+\infty} -\infty$

Therefore, I can choose $R_0 = R_0(\varepsilon) > |\omega|$ st $\forall R > R_0(\varepsilon)$

$$|f\cdot\varphi_\varepsilon|_{C_R}| \le M$$

d + a $\Rightarrow$ (Maximum modulus principle) $|f\cdot\varphi_\varepsilon| \le M \quad \forall z \in \bar{\Omega}_R$

$\Rightarrow |f(\omega)\cdot\varphi_\varepsilon(\omega)| \le M$

This result is true $\forall \varepsilon > 0$ $\Rightarrow$ $\underbrace{\lim_{\varepsilon\to0}|f(\omega)\varphi_\varepsilon(\omega)|}_{= |f(\omega)|} \le M$ ← since $\lim_{\varepsilon\to0}|\varphi_\varepsilon(\omega)| = 1$

$\Rightarrow |f| \le M$ in $\bar{D}_1$ □