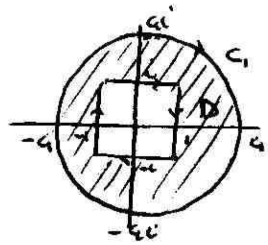
HOMEWORK ASSIGNMENT # 7

Ex pag. 153, m. 2

C_1 and C_2 are positively oriented simple curve contours. It suffices to show that f is analytic in the closed region D:



ⓐ $f(z) = \frac{1}{3z^2+1}$

singularity at: $3z^2+1=0 \quad z^2 = -\frac{1}{3} \Rightarrow z_{1,2} = \pm\frac{i}{\sqrt{3}} \notin D$

ⓑ $f = \frac{z+2}{\sin(z/2)}$. Singularity at $\sin\frac{z}{2}=0 \iff \frac{z}{2}=m\pi \quad n\in\mathbb{Z} \iff z_m = 2m\pi \quad (m\in\mathbb{Z})$

$z_0 = 0 \notin D$

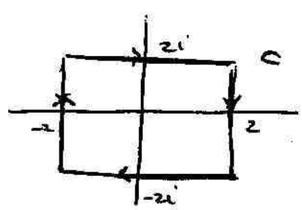
$|m|\geq 1 \quad z_m = 2m\pi : \quad |z_m| > 2\pi > 4 \quad \Rightarrow z_m \notin D \quad (n\neq 0)$

ⓒ $f(z) = \frac{z}{1-e^z}$. Singularity at $1-e^z=0 \iff e^z = 1 \iff z_m = 2\pi i m$

(see above why $z_m \notin D$)

[observe: for $m=0$ $\quad z_0=0$ is not a pole, but a removable singularity, i.e. the function can be extended to an analytic function at 0].

Ex. pag 162, m. 1 (a-c)



a) $\int_C \frac{e^{-z}}{z-\frac{\pi i}{2}}\,dz = 2\pi i \underbrace{e^{-\frac{\pi}{2}i}}_{-i} = 2\pi$

Cauchy Formula $f = e^{-z}$ and $z_0 = \frac{\pi}{2}i$
(e^{-z} is analytic everywhere)
and z_0 is inside C

ⓑ $\int_C \frac{\cos z}{z(z^2+8)}\,dz = 2\pi i \frac{\cos 0}{8} = \frac{1}{8}\cdot 2\pi i = \frac{\pi}{4}i$

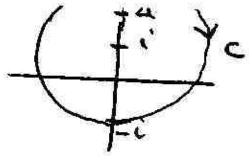
Cauchy Formula
for $f = \frac{\cos z}{z^2+8}$ and $z_0=0$
It is analytic inside and on C
(the only singularities are at $\pm 2\sqrt{2}\,i$)
and z_0 is inside C

ⓒ $\int_C \frac{z\,dz}{2z+1} = \frac{1}{2}\int_C \frac{z}{z+\frac{1}{2}}\,dz =$

$= 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi i}{2}$

Cauchy formula
w/ $f(z) = \frac{z}{2}$ and $z_0 = -\frac{1}{2}$
(f is analytic everywhere and z_0 is inside C)

Ex pag 162 n. 2



(a)

$$g(z) = \frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$$

apply Cauchy formula w/ $f(z) = \frac{1}{z+2i}$ (it's analytic inside and on C)
and $z_0 = +2i$ (inside C)

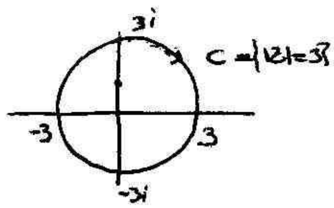
$$\Rightarrow \int_C g(z)dz = \int_C \frac{f(z)}{z-2i}dz = 2\pi i f(2i) = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$$

(b) $g(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z-2i)^2(z+2i)^2}$

apply Cauchy formula for $f(z) = \frac{1}{(z+2i)^2}$ and $z_0 = 2i$ $\Rightarrow f'(z) = \frac{-2}{(z+2i)^3}$
(for the derivative of an analytic function)

$$\int_C g(z)dz = 2\pi i\, f'(2i) = 2\pi i \left[\frac{-2}{(4i)^3}\right] = \frac{-4\pi i}{64(-i)} = \frac{\pi}{16}$$

Ex pag 162 n. 3



$$g(\omega) = \int_C \frac{2z^2-z-2}{z-\omega} \qquad |\omega| \neq 3$$

$$g(2) = \int_C \frac{2z^2-z-2}{z-2} \underset{\uparrow}{=} 2\pi i\, g(2) = 2\pi i\,(4\cdot 2-2-2) = 8\pi i$$

Cauchy formula
$z_0 = 2$
$f(z) = 2z^2-z-2$

if $|\omega|>3$ $\Rightarrow g(\omega)=0$ since $\frac{2z^2-z-2}{z-\omega}$ is an analytic function inside and on C
and I can apply Cauchy-G. theorem.

EX pag 162, n. 5

f analytic within and on C, $z_0 \notin C$ $\implies$ $\displaystyle\int_C \frac{f'(z)\,dz}{z-z_0} = \int_C \frac{f(z)}{(z-z_0)^2}\,dz$

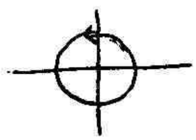
• If z_0 is in the unbounded connected component of $\mathbb{C}\setminus C$ $\implies$ $\dfrac{f'}{z-z_0}$ and $\dfrac{f}{(z-z_0)^2}$ are both analytic within and on C $\implies$ (Cauchy - G. theorem):

$$\int_C \frac{f'(z)}{z-z_0} = 0 = \int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

• suppose z_0 is within C:

$$\int_C \frac{f'(z)}{z-z_0}\,dz \underset{\text{Cauchy formula}}{=} 2\pi i\, f'(z_0) \underset{\text{Cauchy formula for the derivative}}{=} \int_C \frac{f(z)}{(z-z_0)^2}\,dz \qquad \square$$

EX. pag 162, n.7



$C = \{ z = e^{i\theta} \quad -\pi \le \theta \le \pi \}$

$$\int_C \frac{e^{az}}{z}\,dz \underset{\substack{\text{Cauchy formula} \\ f(z)=e^{az} \text{ is entire}}}{=} 2\pi i \cdot e^{a\cdot 0} = 2\pi i \qquad \forall a \in \mathbb{R} \qquad (*)$$

$$\int_C \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} e^{ae^{i\theta}} e^{-i\theta}\, i e^{i\theta}\,d\theta = i\int_{-\pi}^{\pi} e^{ae^{i\theta}}\,d\theta = i\int_0^{\pi} \left(e^{ae^{i\theta}} + e^{ae^{-i\theta}}\right) d\theta =$$

$$= i\int_0^{\pi} \left(e^{a\cos\theta + ia\sin\theta} + e^{a\cos\theta - ia\sin\theta}\right) d\theta = i\int_0^{\pi} e^{a\cos\theta}\left(e^{ia\sin\theta} + e^{-ia\sin\theta}\right) d\theta$$

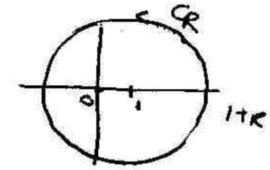
$$= 2i\int_0^{\pi} e^{a\cos\theta}\cos[a\sin\theta]\,d\theta \overset{(*)}{=} 2\pi i$$

$$\implies \boxed{\int_0^{\pi} e^{a\cos\theta}\cos[a\sin\theta]\,d\theta = \pi}$$

(4)

EX 2 Show, using Cauchy's thm and its variants in § 4.6, that $g(z) = \frac{1}{z(z-1)}$ has antiderivative in $\mathbb{C} \setminus \{0 \le z \le 1\}$

NOTE: consider $C_R = \{|z-1| = R\}$ $\quad R > 1$



and :
$$\int_{C_R} g(z)\,dz = \int_{C_R} \frac{1}{z} - \int_{C_R} \frac{1}{z-1}\,dz =$$

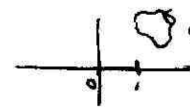
$$= 2\pi i - 2\pi i = 0$$

$\uparrow$

Cauchy formula
con $f(z) \equiv 1$

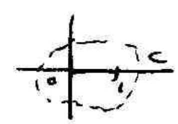
Let's show that $\forall$ C closed curve in $\mathbb{C} \setminus [0,1]$ (simple curve) $\int_C g(z) = 0$

. If C doesn't wind around the segment $[0,1]$



$$\Rightarrow \int_C g(z)\,dz = 0 \qquad \text{(by Cauchy thm)}$$

. suppose C is a positive oriented simple closed curve, wrapping around $[0,1]$:



$\exists R \gg 1$ st C and C_R satisfy hp of corollary 2 pag 151

(taking $C_1 = C_R$ and $C_2 = C$)

$$\Rightarrow \int_C g(z) = \int_{C_R} g(z)\,dz = 0$$

$\Rightarrow$ the integral of g along any closed curve in $\mathbb{C} \setminus [0,1]$ is 0
(we have shown only for positive oriented simple curves, but it follows easily that this implies the result for generic curves).

$$\Rightarrow \int_C g(z) = 0 \qquad \forall C \text{ closed curve in } \mathbb{C} \setminus [0,1]$$

Fix $z_0 \in \mathbb{C} \setminus [0,1]$ and define $F(z) = \int_{\gamma(z_0,z)} g(\xi)\,d\xi$. [$\gamma$ is a generic path joining z_0 in z that lies in $\mathbb{C} \setminus [0,1]$]

F is well defined $\forall z \in \mathbb{C} \setminus [0,1]$ (doesn't depend on the choice of γ) and it can be easily shown that $F'(z) = g(z)$. $\square$ (see next exercise for more details)

Ex. 4

g analytic in $\mathbb{C} \setminus \{z_1 .. z_m\}$ and $\rho = \min_{i \neq j} \{|z_i - z_j|\}$

$C_i := \{|z - z_i| = \rho/3\}$ & $\int_{C_i} g \, dz = 0 \quad \forall i = 1 \dots m$.

Let's show several claims:

① $\forall i$, if I consider $r \leq \frac{\rho}{3}$: $\int_{|z-z_i|=r} g = 0$. In fact, this follows immediately from corollary §46 ($C_1 = C_i$ and $C_2 := |z - z_i| = r$) observing the only singularity point within C_i is z_i.

② let $\tilde{C}$ be a closed simple curve (positive oriented, for instance) in $\mathbb{C} \setminus \{z_1 .. z_m\}$

$\Longrightarrow \int_{\tilde{C}} g(z) = 0$

In fact: let $\{z_{i_1} \dots z_{i_k}\}$ be the singularity points inside the region bounded by $\tilde{C}$.
(if this set is empty, the thesis comes from Cauchy - Goursat thm)

consider $r = \min_j \{ \text{dist}(z_{i_j}, \tilde{C}), \frac{\rho}{3} \}$

and consider $\Gamma = \bigcup_{j=1}^{k} \{ |z - z_{i_j}| = r \}$ positive oriented

$\Rightarrow$ by corollary §46: $\int_{\tilde{C}} g \, dz = \int_{\Gamma} g \, dz = \sum_{j=1}^{k} \int_{|z - z_{i_j}| = r} g \, dz \overset{①}{=} 0$

③ $\forall C$ closed curve in $\mathbb{C} \setminus \{z_1 .. z_m\} \Longrightarrow \int_C g(z) \, dz = 0$

④ define $G(z) = \int_{\gamma(z_0, z)} g(\xi) d\xi$ where $z_0 \in \mathbb{C} \setminus \{z_1 .. z_m\}$ fixed and $\gamma(z_0, z)$ is a generic path joining z_0 to z, that lies in $\mathbb{C} \setminus \{z_1 .. z_m\}$

- G is analytic and well defined in $\mathbb{C} \setminus \{z_1 .. z_n\}$
  (obviously it doesn't depend on γ: if we choose another $\tilde{\gamma}(z_0, z)$ and consider $C = \gamma - \tilde{\gamma} \longrightarrow$ by ③ $\int_C g = 0 \iff \int_{\gamma} - \int_{\tilde{\gamma}} = 0$)

- Let's show that $G'(z) = g(z)$ in $\mathbb{C} \setminus \{z_1 .. z_n\}$ $\longrightarrow$

let's show that $\forall z \in \mathbb{C}\setminus\{z_1 \dots z_m\}$ $\lim_{\Delta z \to 0} \frac{G(z+\Delta z) - G(z)}{\Delta z} = g(z)$

(6)

$$G(z+\Delta z) - G(z) = \int_{\gamma(z_0, z+\Delta z)} - \int_{\gamma(z_0, z)} = \int_{\substack{\text{segment} \\ z \text{ to } z+\Delta z}} g(\xi)\, d\xi$$

since G doesn't depend on γ, I can acquire $\gamma(z_0, z+\Delta z) = \gamma(z_0, z) * \{\text{segment joining } z \text{ to } z+\Delta z\}$

If $\Delta z \ll 1$ this segm lies in $\mathbb{C}\setminus\{z_1 \dots z_m\}$ (by openness of $\mathbb{C}\setminus\{z_1 \dots$

$$\Rightarrow \left| \frac{G(z+\Delta z) - G(z)}{\Delta z} - g(z) \right| =$$

$$= \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} g(\zeta)\, d\zeta - \frac{1}{\Delta z} \int_z^{z+\Delta z} g(z)\, d\zeta \right| \leq \frac{1}{\Delta z} \int_z^{z+\Delta z} |g(\zeta) - g(z)|\, d\zeta$$

$$\leq \frac{1}{\Delta z} \cdot \Delta z \cdot \varepsilon = \varepsilon$$

g is continuous at z hence $\forall \varepsilon > 0$ if Δz is small enough $|g(\zeta) - g(z)| \leq \varepsilon$

since ε can be arbitrary small

$$\Rightarrow G'(z) = g(z) \quad \square$$