HOMEWORK ASSIGNMENT #6

EX PAG 161, N.2

(a) $\displaystyle\int_i^{i/2} e^{\pi z}\,dz = \frac{1}{\pi}\left[e^{i\frac{\pi}{2}} - e^{i\pi}\right] = \boxed{\frac{1}{\pi}(i+1)}$

antideriv. $F(z) = \dfrac{e^{\pi z}}{\pi}$

(b) $\displaystyle\int_0^{\pi+2i} \cos\frac{z}{2}\,dz = 2\left[\sin\left(\frac{\pi+2i}{2}\right)\right] =$

antid. $F(z) = 2\sin\dfrac{z}{2}$

$= 2\left[\dfrac{e^{i(\frac{\pi}{2}+i)} - e^{-i(\frac{\pi}{2}+i)}}{2i}\right] =$

$= \dfrac{1}{i}\left[\underbrace{e^{\frac{\pi}{2}i}}_{i}e^{-1} - \underbrace{e^{-\frac{\pi}{2}i}}_{-i}e^{1}\right] =$

$= \boxed{e + \frac{1}{e}}$

(c) $\displaystyle\int_1^3 (z-2)^3\,dz = \left.\frac{(z-2)^4}{4}\right|_1^3 = \frac{1}{4} - \frac{1}{4} = \boxed{0}$

EX. N.1 pag 153

(a) $f(z) = \dfrac{z^2}{z-3}$ is analytic in $\mathbb{C}\setminus\{3\}$ (it has a pole at this point)

$\Rightarrow$ it's analytic at all points interior to and on the contour $C = \{|z| = 1\}$

$\Rightarrow \displaystyle\int_C f(z)\,dz = 0$

(b) $f(z) = z e^{-z}$ is an entire function

(c) $f(z) = \dfrac{1}{z^2+2z+2}$

$z^2+2z+2=0$ iff $z = -1 \pm \sqrt{1-2} = -1 \pm i$

f is analytic in $\mathbb{C}\setminus\{-1\pm i\}$

(d) $f = \operatorname{sech} z = \dfrac{1}{\cosh z}$

$\cosh z = 0$ iff $|\cosh z| = 0$ iff $|\cosh z|^2 = (\sinh x)^2 + \cos^2 y = 0$

$\Longleftrightarrow \quad x = 0 \quad \& \quad y = \dfrac{\pi}{2} + \pi m \quad (m \in \mathbb{Z})$

$\Rightarrow$ sech z is analytic on $\mathbb{C}\setminus\left\{\left(\frac{\pi}{2}+\pi m\right)i \;/\; m\in\mathbb{Z}\right\}$

(e) $f = \tan z = \dfrac{\sin z}{\cos z}$ is analytic in $\mathbb{C}\setminus\{\frac{\pi}{2}+2\pi m \mid m\in\mathbb{Z}\}$

since $\cos z = 0$ iff $z = \frac{\pi}{2} + \pi m$ (in fact $|\cos z|^2 = \cos^2 x + \sinh^2 y$)

(f) $f = \operatorname{Log}(z+2)$

I can define an analytic branch of this function in $\mathbb{C}\setminus(-\infty, -2]$

$\boxed{\text{Ex pag 156 n°7}}$

C = positively oriented simple closed contour

show: Area of the region enclosed by C, can be written: $\frac{1}{2i}\int_C \bar{z}\,dz$

$$\frac{1}{2i}\int_C (x-iy)(dx+i\,dy) = \cancel{\frac{1}{2i}\int_C x\,dx + iy\,dy + ix\,dy + iy\,dy}$$

$$= \frac{1}{2i}\int_C (x-iy)\,dx + i(x-iy)\,dy \overset{\text{STOKES}}{=} \frac{1}{2i}\iint_R \left[\frac{\partial}{\partial y}(x-iy) + \frac{\partial}{\partial x}\big(i(x-iy)\big)\right] dx\wedge dy =$$

let R denote the region enclosed by C

$$= \frac{1}{\cancel{2i}}\iint_R \cancel{2i}\,dx\wedge dy = \iint_R dx\,dy = \text{Area}(R)$$

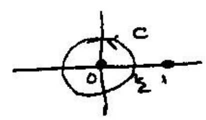
EXERCISE N°2

$$g(z) = \frac{1}{z(z-1)}$$

(a) This is the quotient of two polynomials $\Rightarrow$ it's analytic in $\mathbb{C}\setminus\{z(z-1)=0\}$ =

$\Rightarrow$ " " in $\mathbb{C}\setminus\{0,1\}$

in $z=0$ and $z=1$ the function has a pole $\lim_{z\to 0\,(1)} g(z) = \infty$



(b) C = positive oriented contour on $|z| = \frac{1}{2}$

$$\cancel{\int_C g(z)\,dz = \int_{\theta_0}^{\theta_1} g\left(\tfrac{1}{2}e^{i\theta}\right)\tfrac{i}{2}e^{i\theta}\,d\theta =}$$

$$\cancel{= \int_{\theta_0}^{\theta_1} \frac{1}{z(z-1)}\Big|_{z=\frac{e^{i\theta}}{2}} \tfrac{i}{2}e^{i\theta}\,d\theta = \int_{\theta_0}^{\theta_1}\left(-\frac{1}{\frac{1}{2}e^{i\theta}}\,\frac{i}{2}e^{i\theta} + \frac{1}{\frac{1}{2}e^{i\theta}-1}\right)d\theta}$$

$$\cancel{-\frac{1}{z} + \frac{1}{z-1}}$$

by Cauchy thm

$$\int_C g(z)\,dz = \int_C \left(\frac{1}{z-1} - \frac{1}{z}\right)dz = \underbrace{\int_C \frac{dz}{z-1}}_{=0} - \int_C \frac{1}{z}\,dz = -\int_C \frac{1}{z}\,dz =$$

$$= \cancel{\ldots}\ 2\pi i m \qquad \exists\, m\in\mathbb{Z} \qquad m = -\mathrm{ind}_0(C)$$

(c) g cannot have an antiderivative in $\mathbb{C}\setminus\{0,1\}$, since $\int_C g(z)\,dz$ may be $\neq 0$

(d) Let's start finding an antiderivative of g in $\mathbb{C}\setminus(-\infty,1]$

Here I can define both $\mathrm{Log}\, z$ and $\mathrm{Log}(z-1)$

$$\Rightarrow \quad g(z) = \frac{1}{z-1} - \frac{1}{z} \longrightarrow \underset{\substack{\uparrow \\ \text{antideriv}}}{\tilde{G}(z)} = \mathrm{Log}(z-1) - \mathrm{Log}\, z$$

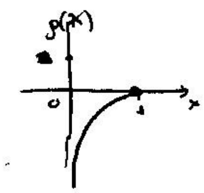
This suggests, we can define an antideriv. $G(z)$ for g in $\mathbb{C}\setminus[0,1]$, by:

$$\boxed{G(z) = \mathrm{Log}\left(\frac{z-1}{z}\right)}$$

where Log is the principal argument of the logarithm in $\mathbb{C}\setminus(-\infty,0]$

In fact:

- $\frac{z-1}{z}$ maps $[0,1]$ into $(-\infty,0]$: $\quad \frac{z-1}{z}\Big|_{[0,1]} = \frac{x-1}{x} = 1-\frac{1}{x}\Big|_{x\in[0,1]}$
- $G'(z) = \frac{1}{\frac{z-1}{z}} \cdot \frac{z-(z-1)}{z^2} = \frac{1}{z(z-1)}$



EXERCISE (3)

Let D be a convex subset of $\mathbb{C}$.

Suppose $\int_C f\,dz = 0$ for every triangular contour

$\to$ show that f has an antiderivative.

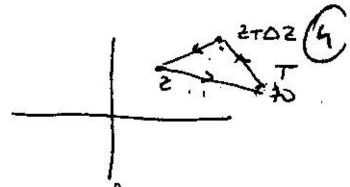
In fact $\quad F(z) = \int_{\gamma(z_0,z)} f\,dz \quad$ where z_0 is a fixed point in D and γ is the segment joining z_0 to z

this function is analytic:

$\cancel{F(z) = \int_0^1 f(z_0 + t(z-z_0))\,dt \to F'(z) =}$

Compute its derivative.

$$\lim_{\Delta z\to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = \lim_{\Delta z\to 0}\left(\int_{\gamma(z_0,z+\Delta z)} f - \int_{\gamma(z_0,z)} f\,dz\right)\frac{1}{\Delta z} =$$



$$= \lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_{\gamma(z,\Delta z)^{z}} f\,dz = \lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_0^1 f(z+t\Delta z)\,\Delta z\,dt$$

using that $\int_T f = 0$

$$= \int_0^1 \lim_{\Delta z\to 0}\frac{f(z+t\Delta z)\cancel{\Delta z}}{\cancel{\Delta z}}\,dt = \int_0^1 f(z)\,dt =$$

$$\Rightarrow \int_{\gamma(z_0,z+\Delta z)} f - \int_{\gamma(z_0,z)} = \int_z$$

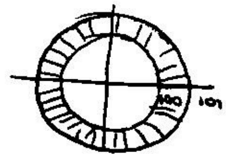
$= f(z)$

$\circ$

Bonus Exercises

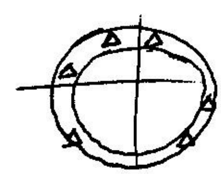
(1) FIRST OBSERVE THAT D IS NOT CONVEX (THIS IS THE REASON, WHY THERE WON'T BE ANY CONTRADICTION w/ (3) OF PART I)

D:



Consider $f(z)=\frac{1}{z}$. Obviously f is analytic in D, but doesn't admit a global antiderivative (b/c it's impossible to define an analytic branch of the log in D).

Moreover, it's quite clear geometrically (too tedious analytically) that it's impossible to construct in D a triangle winding around the origin: there isn't enough room)
Hence the only triangles that completely lie in D, are contained in domains where it's possible to define an analytic branch of $\log z$ (namely, $\mathbb{C}$ minus a semiline starting at the origin).



$$\Rightarrow \quad \forall T \subset D \quad \int_T \frac{1}{z} = 0 \quad \text{(by Cauchy thm)}$$

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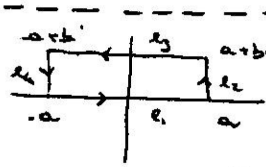
(3) a)



$$\left(\int_{l_1}+\int_{l_3}\right) e^{-z^2}dz = \int_{-a}^{a} e^{-t^2}dt - \int_{-a}^{a} e^{-(t+ib)^2}dt = 2\int_0^a e^{-t^2} - \int_{-a}^{a} e^{-t^2+b^2-2itb}dt =$$

$$= 2\int_0^a e^{-t^2}dt - e^{b^2}\int_0^a e^{-t^2}\left(e^{-2itb}+e^{2itb}\right)dt =$$

$$= 2\int_0^a e^{-t^2}dt - 2e^{b^2}\int_0^a e^{-t^2}\cos 2tb\, dt .$$

$$\left(\int_{\ell_2} + \int_{\ell_4}\right) e^{-z^2} dz = \int_0^b e^{-(a+it)^2} i\,dt - \int_0^b e^{-(-a+it)^2} i\,dt =$$

$$= i\left[\int_0^b e^{-a^2+t^2-2ait} - e^{-a^2+t^2+2ait}\,dt\right] =$$

$$= i\int_0^b e^{-a^2+t^2} \cdot \left(-e^{2ait} - e^{-2ait}\right) dt =$$

$$= 2e^{-a^2}\int_0^b e^{t^2}\,\frac{e^{2ait} - e^{-2ait}}{2i}\,dt = 2e^{-a^2}\int_0^b e^{t^2}\sin(2at)\,dt$$

Cauchy-Goursat theorem: $$\int_R e^{-z^2} dz = 0$$

$$\Rightarrow \left(\int_{+\ell_1} + \int_{+\ell_2} - \int_{+\ell_3} - \int_{+\ell_4}\right) e^{-z^2} dz = 0$$

$$\Rightarrow \cancel{2}\int_0^a e^{-t^2} - \cancel{2}e^{b^2}\int_0^a e^{-t^2}\cos 2tb\,dt = -\cancel{2}e^{-a^2}\int_0^b e^{t^2}\sin(2at)\,dt$$

$$\Rightarrow \boxed{\int_0^a e^{-t^2}\cos(2tb)\,dt = \underbrace{e^{-b^2}\int_0^a e^{-t^2}\,dt}_{=\,A} + \underbrace{e^{-(a^2+b^2)}\int_0^b e^{t^2}\sin 2at\,dt}_{=\,B}}$$

(b) According: $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

and observing: $\left| \int_0^b e^{y^2} \sin(2ay)\,dy \right| < \underbrace{\int_0^b e^{y^2}\,dy}$, show the rest.

it's a constant independent of a!

$$\underbrace{\lim_{a\to\infty} \int_0^a e^{-x^2}\cos(2bx)\,dx}_{\parallel} = \lim_{a\to\infty} (1) + \lim_{a\to+\infty} (2) =$$

$$\int_0^{\infty} e^{-x^2}\cos(2bx)\,dx \quad = \underbrace{\int_0^{\infty} e^{-t^2}\,dt}_{\frac{\sqrt{\pi}}{2}e^{-b^2}} + \underbrace{\lim_{a\to\infty} (2)}_{0}$$

this integral exists since the function is in L^1.

in fact

$$0 \le \left| e^{-(a^2+b^2)} \int_0^b e^{x^2} \sin 2ax \right| \le e^{-a^2} \underbrace{e^{-b^2}\int_0^b e^{x^2}\,dx}_{\text{Const}} \xrightarrow{a\to\infty} 0$$

$\downarrow a\to\infty$

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