HOMEWORK ASSIGNMENT #5

EX. PAG 94, N. 6

$(*)\quad e^{\log z} = z \longrightarrow$ (chain rule) $\dfrac{d}{dz} e^{\log z} = e^{\log z} \cdot \dfrac{d \log z}{dz} \overset{(*)}{=} z \dfrac{d}{dz} \log z$

$$\overset{(*)}{\Vert}$$

$$\frac{d}{dz} z = 1$$

$\Longrightarrow$ for $z \neq 0$ $\quad \dfrac{d}{dz} \log z = \dfrac{1}{z}$

EX PAG 96, N. 2

Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$: $\quad \mathrm{Log}(z_1 z_2) = \underbrace{\log|z_1 z_2|}_{\Vert} + \mathrm{Arg}(z_1 \cdot z_2)\, i =$

$$\log(|z_1| \cdot |z_2|)$$
$$\Vert$$
$$\log|z_1| + \log|z_2|$$

$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 + 2\pi N$

$N = 0, \pm 1$

(in fact $\mathrm{Arg}\, z_1 z_2 \in (-\pi, \pi]$ while $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$ may not!)

$$= \log|z_1| + \log|z_2| + i\,\mathrm{Arg}\, z_1 + i\,\mathrm{Arg}\, z_2 + 2\pi i N$$

$$= \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2 + 2\pi i N$$

EX. PAG. 116, N. 7

Show that for all $-1 \leq x \leq 1$, $\quad |P_m(x)| = \left| \dfrac{1}{\pi} \displaystyle\int_0^{\pi} (x + i\sqrt{1-x^2}\cos\theta)^m \, d\theta \right| \leq 1 \qquad (m = 0, 1, 2, \dots)$

In fact

$$|P_m(x)| \overset{\text{ineq.(5)}}{\leq} \frac{1}{\pi} \int_0^{\pi} |x + i\sqrt{1-x^2}\cos\theta|^m \, d\theta \leq \frac{1}{\pi} \int_0^{\pi} (x^2 + (1-x^2)\cos^2\theta)^{\frac{m}{2}} \, d\theta \leq$$

$$\underset{|\cos\theta| \leq 1}{\leq} \frac{1}{\pi} \int_0^{\pi} \cancel{x^2} + (1 - \cancel{x^2}) \, d\theta = \frac{\pi}{\pi} = 1$$

- Show in addition that P_m is a polynomial of degree m in x.

$$P_m(x) = \frac{1}{\pi} \int_0^{\pi} (x + i\sqrt{1-x^2}\cos\theta)^m \, d\theta = \frac{1}{\pi} \int_0^{\pi} \sum_{k=0}^{m} \binom{m}{k} x^{m-k} i^k (1-x^2)^{\frac{k}{2}} \cos\theta^k \, d\theta$$

$$= \frac{1}{\pi} \sum_{k=0}^{m} \binom{m}{k} x^{m-k} i^k (1-x^2)^{\frac{k}{2}} \int_0^{\pi} (\cos\theta)^k \, d\theta =$$

$\boxed{\text{If } k \text{ is odd then } \int_0^{\pi} \cos^k\theta \, d\theta = 0}$

$$= \frac{1}{\pi} \sum_{\substack{k=0 \\ k \text{ even}}} \binom{m}{k} x^{m-k} (1-x^2)^{\frac{k}{2}} \underbrace{\left(\int_0 \cos\theta \, d\theta\right)}_{\substack{\shortparallel \\ c_k \in \mathbb{R}}} i^{k} =$$ (2)

$$= \frac{1}{\pi} \sum_{j=0}^{m_m} \binom{m}{2j} x^{m-2j} (1-x^2)^j \, c_{2j} \, (-1)^j \qquad \text{where } m_m = \begin{cases} \frac{m}{2} & \text{if } m \text{ is even} \\ [\frac{m}{2}] & \text{if } m \text{ is odd} \end{cases}$$

it's a polynomial in $\mathbb{R}[x]$, with degree m.

$\boxed{\text{Ex. PAG. 121, N 5}}$

$f(z) = u(x,y) + i\,v(x,y)$ $\Rightarrow$ $w(t) = f(z(t)) = u(x(t), y(t)) + i\,v(x(t), y(t))$

$z(t) = x(t) + i\,y(t) \qquad (a \le t \le b)$

$\Rightarrow$ (applying chain rule)

$$w'(t) = \left(u_x \cdot x'(t) + u_y\, y'(t)\right) + i\left(v_x\, x'(t) + v_y\, y'(t)\right) =$$

$$= (u_x + i v_x)\, x'(t) + i\,\underbrace{(v_y - u_y)}_{\substack{\shortparallel \leftarrow \text{Cauchy-Riem} \\ u_x + i v_x}}\, y'(t) =$$

$$= (u_x + i v_x)(x'(t) + i\,y'(t)) = f'(z(t))\, z'(t) \qquad \square$$

$\boxed{\text{EX- PAG 130, N. 10}}$ $\qquad C_0 = \{z_0 + Re^{i\theta} \mid -\pi \le \theta \le \pi\}$

(a) $$\int_{C_0} \frac{dz}{z-z_0} = \int_{-\pi}^{\pi} \frac{\cancel{R}\, i\, \cancel{e^{i\theta}}\, d\theta}{\cancel{R e^{i\theta}}} = i 2\pi$$

$|m| \ge 1$

(b) $$\int_{C_0} (z-z_0)^{m-1} dz = \int_{-\pi}^{\pi} (Re^{i\theta})^{m-1} \cdot R i e^{i\theta}\, d\theta = \int_{-\pi}^{\pi} R^m i\, e^{im\theta} d\theta = R^m i \left[\frac{e^{im\theta}}{im}\right]_{-\pi}^{\pi} = 0$$ ✓

$(m = \pm 1, \pm 2, \dots)$

EX. PAG. 134 N. 6

$C_\rho = \{ |z| = \rho \}$ $\quad 0<\rho<1$ $\quad$ oriented in the counterclockwise direction

$f(z)$ analytic in $|z|\le 1$.

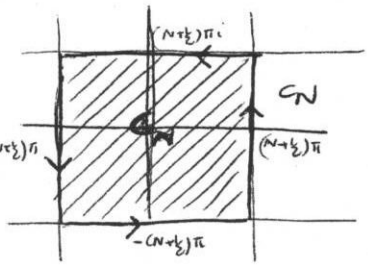
(f is analytic on $|z|\le 1$ (compact set) $\Rightarrow \exists\, M \ge 0$ st $|f(z)|\le M \;\; \forall\, |z|\le 1$)

$$\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \le \int_{C_\rho} |z|^{-1/2} |f(z)|\,|dz| \le M \int_{C_\rho} |z|^{-1/2} |dz| =$$

$$= M \rho^{-1/2}\, 2\pi\rho = 2\pi M \sqrt{\rho}$$

(M doesn't depend on ρ!)

EX. 7 PAG. 134



$N>0$ integer

a)

- Show $|\sin z| \ge 1$ on vertical lines: $\quad |\sin z| \ge |\sin x| = |\sin \pm(N+\tfrac{1}{2})\pi| = 1$

  to be proven in the following

- " $\quad |\sin z| > \sinh(\tfrac{\pi}{2})$ on the horizontal sides: $\quad |\sin z| \ge |\sinh(y)| = |\sinh \pm(N+\tfrac{1}{2}\pi)| \overset{N>0}{=}$

$$= \sinh (N+\tfrac{1}{2})\pi > \sinh \tfrac{\pi}{2}$$

(sinh is strictly increasing.)

Hence for any $z \in C_N$, $|\sin z| \ge A$ (independent of N)

(Pick $A = \min\{1, \sinh \tfrac{\pi}{2}\} = 1$)

b)

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \le \int_{C_N} \frac{|dz|}{|z|^2 |\sin z|} \le \frac{1}{A} \int_{C_N} \frac{|dz|}{|z|^2} \le \frac{1}{A} \cdot \frac{\text{length}(C_N)}{[(N+\tfrac{1}{2})\pi]^2} =$$

$|z| \ge (N+\tfrac{1}{2})\pi$

It is the distance from the origin

$$= \frac{1}{A} \frac{4(2N+1)\pi}{(N+\tfrac{1}{2})^2 \pi^2} = \frac{1}{A} \frac{4\cancel{(2N+1)\pi}\, 4}{(2N+1)^2 \pi^2} = \frac{16}{(2N+1)\pi A}$$

□

Let's prove the two inequalities above:

.) $|\sin z| \geq |\sin x|$ & ..) $|\sin z| \geq |\sinh y|$

~~Using expressions (1.5)~~

$$|\sin z|^2 = |\sin(x+iy)|^2 = |\sin x \cos iy + \sin iy \cos x|^2 =$$

$\cos iy = \cosh y$

$\sin iy = i\sinh y$

they come directly from the definition, using the exponential map.

$$= |\sin x \cosh y + i\sinh y \cos x|^2 =$$

$$= \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x$$

$$\Longrightarrow \quad |\sin z|^2 \geq \sin^2 x \cosh^2 y \underset{\substack{\uparrow \\ \cosh y \geq 1}}{\geq} \sin^2 x \quad \longrightarrow \quad |\sin z| \geq |\sin x|$$

~~Analogously: $|\sin z|^2 \geq \cos^2 x \sinh^2 y$~~

observe: $|\sin z|^2 = \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x = \sin^2 x(1+\sinh^2 y) + \sinh^2 y \cos^2 x =$

$$= \sin^2 x + \sinh^2 y$$

$$\Longrightarrow \quad |\sin z| \geq |\sinh y| \quad \square$$

$\boxed{\text{EX. PAG 142 N.3}}$ $\quad f(z) = (z-z_0)^{n-1}$ defined in $D = \begin{cases} \mathbb{C} & \text{if } n-1>0 \text{ (ie } n \geq 1) \\ \mathbb{C}\setminus\{z_0\} & \text{if } n-1<0 \text{ (ie } n \leq -1) \end{cases}$

~~$f(z) = \frac{d}{dz}(z-z_0)^{n+1}$ is defined in $D = \mathbb{C}\setminus\{z_0\}$ $(n = \pm1, \pm2, \dots)$~~

Thus f has an antiderivative in D: $F(z) = \dfrac{(z-z_0)^n}{n}$ $\quad (n = \pm1, \pm2, \dots)$

$\Longrightarrow$ Applying the thm in section 4.2, the integral of $f(z)$ around closed contours lying entirely in D is zero!

<u>NOTE</u>: for $n=0$ the above statement doesn't hold

$$\int_{S^1} \frac{1}{z}\,dz = 2\pi i \qquad \left(\tfrac{1}{z} \text{ hasn't any antiderivative in } \mathbb{C}\setminus\{0\}\right)$$

1. C = closed positively oriented ~~[illegible]~~ contour along $|z|=1$

prove that for any function $g(z)$: $\overline{\int_C g(z)\,dz} = -\int_C \frac{\overline{g(z)}}{z^2}\,dz$

<u>proof</u>:

$$C = \{ z = e^{i\theta} \qquad \theta_0 \le \theta \le \theta_1 \}$$

$$\Longrightarrow \overline{\int_C g(z)\,dz} = \overline{\int_{\theta_0}^{\theta_1} g(e^{i\theta})\, i e^{i\theta} d\theta} = \int_{\theta_0}^{\theta_1} -\overline{g(e^{i\theta})}\, i\, e^{-i\theta}\, d\theta =$$

$$= \int_{\theta_0}^{\theta_1} -i\,\overline{g(e^{i\theta})}\, e^{-i2\theta} e^{i\theta} d\theta = \int_{\theta_0}^{\theta_1} -\frac{\overline{g(e^{i\theta})}}{e^{i2\theta}}\, i e^{i\theta} d\theta = -\int_C \frac{\overline{g(z)}}{z^2}\, dz$$

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2. C contour in $\mathbb{C}\setminus\{0\}$ $\qquad z = z(s) \quad 0 \le s \le T$

$\forall\ 0 \le t \le T$ contour $C_t = \{ z = z(s) : 0 \le s \le t \}$ & define $I(t) = \int_{C_t} \frac{dz}{z}$.

(a) show: $g(t) = \frac{e^{I(t)}}{z(t)}$ is st $g'(t) = 0$ (ie is constant).

In fact $g'(t) \overset{(\text{ex p.121 \#5})}{=} \frac{I'(t) e^{I(t)} z(t) + e^{I(t)} z'(t)}{z(t)^2} = e^{I(t)} \frac{I'(t) z(t) - z'(t)}{z(t)^2} = (*)$

and $\frac{d}{dt} I(t) = \frac{d}{dt} \int_0^t \frac{z'(s)}{z(s)}\, ds = \frac{z'(t)}{z(t)}$

Fundam thm of Calculus

$$\Longrightarrow (*) = \frac{e^{I(t)}}{z(t)^2} \cdot \left[\frac{z'(t)}{\cancel{z(t)}} \cancel{z(t)} - z'(t) \right] = 0$$

(b) Since $g(t)$ is constant $\Longrightarrow g(0) = g(T) \Longrightarrow \frac{e^{I(T)}}{z(T)} = \frac{e^{I(0)}}{z(0)}$

$I(0) = 0$

$$\Longrightarrow e^{I(T)} = \frac{z(T)}{z(0)} \Longrightarrow \exp \int_C \frac{dz}{z} = \frac{z_1}{z_0} \qquad \square$$

denote $z_1 = z(T)$ and $z_0 = z(0)$

(c) $\exp \int_C \frac{dz}{z} = \frac{z_1}{z_0} = 1 \implies \int_C \frac{dz}{z} = 2\pi i m \quad \exists\, m \in \mathbb{Z}.$

$$\frac{z_1}{z_0} = \underset{\substack{z_1 = z_0 \\ \text{(since } C \text{ is closed)}}}{\uparrow}\, 1$$

(d) (Bonus)

Consider $C = \{z(s) : s \in [0,T]\}$ and let $0 = s_0 < s_1 \dots < s_N = T$ THE times at which $z(s)$ intersects the negative real semiaxis (ie $z(s_j) \in (-\infty, 0]$)

(up to a reparametrization we can assume $z(0) = z(T) \in (-\infty, 0)$.)

in $D = \{z \in \mathbb{C} : \operatorname{Arg} z \in (-\pi, \pi)\}$ it's possible to define an analytic branch of $\operatorname{Log} z$. (the principal

let $\varepsilon_0 = \min\{s_0, s_1 - s_0, \dots, s_N - s_{N-1}\}$ and choose $\varepsilon < \varepsilon_0$.

Define $C_\varepsilon^j = \{z(s) : s_j + \varepsilon \le s \le s_{j+1} - \varepsilon\} \quad j = 0 \dots N-1$

and observe that $\frac{1}{z}$ has antiderivative $\operatorname{Log} z$ in D. (each C_ε^j is contained in D).

$$\implies \int_{C_\varepsilon^j} \frac{1}{z}\, dz = \underbrace{\operatorname{Log}(z(s_{j+1} - \varepsilon)) - \operatorname{Log} z(s_j + \varepsilon)}_{\downarrow\, \varepsilon \downarrow 0^+} \qquad j = 0 \dots N-1$$

$$[\Delta \operatorname{Arg} z]_{s_j}^{s_{j+1}} = N\pi i$$

where N may be $0, \pm 2$ (it depends on how the curve wraps around the origin)

$$\implies \sum_{j=0}^{N-1} \int_{C_\varepsilon^j} \frac{1}{z}\, dz = \sum_{j=0}^{N-1} \left[\operatorname{Log}(z(s_{j+1} - \varepsilon)) - \operatorname{Log} z(s_j + \varepsilon)\right]$$

$$\downarrow \varepsilon \downarrow 0 \qquad\qquad \downarrow \varepsilon \downarrow 0^+$$

$$\int_C \frac{1}{z}\, dz \qquad\qquad 2\pi M i \quad \exists\, M \in \mathbb{Z}$$

$\square$

In fact
$$\left| \int_C \frac{1}{z}\, dz - \sum_{j=0}^{N-1} \int_{C_\varepsilon^j} \frac{1}{z}\, dz \right| \le \sum_{j=0}^{N-1} \left(\int_{s_j}^{s_j + \varepsilon} + \int_{s_{j+1} - \varepsilon}^{s_{j+1}} \right) \frac{1}{z}\, dz \le C\varepsilon \xrightarrow{\varepsilon \to 0} 0$$

let $C = \max_C \frac{1}{|z|} < \infty$ since C is compact and $0 \notin C$!