PAG. 59-60

EX 1:

(a) $f(z) = 3z^2 - 2z + 4 \longrightarrow f'(z) = \frac{d}{dz} f(z) = \frac{d}{dz}\left(3z^2 - 2z + 4\right) \overset{(3)\text{ PAG.58}}{=} \frac{d}{dz}(3z^2) + \frac{d}{dz}(-2z) + \frac{d}{dz}(4) =$

$$\overset{(1)-(2)\text{ PAG 57}}{=} 3\underbrace{\frac{d(z^2)}{dz}}_{2z} - 2\underbrace{\frac{dz}{dz}}_{=1} + \underbrace{\frac{d}{dz}(4)}_{=0} = 6z - 2$$

(b) $f(z) = (1-4z^2)^3 \longrightarrow f'(z) \overset{\text{chain rule (6) PAG 58}}{=} \left[3\cdot(1-4z^2)^2\right]\cdot(-8z) = -24z\,(1-4z^2)^2$

(c) $f(z) = \frac{z-1}{2z+1} \quad (z \neq -\frac{1}{2}) \longrightarrow f'(z) \overset{(5)\text{ PAG 58}}{=} \frac{1\cdot(2z+1) - (z-1)\cdot 2}{(2z+1)^2} = \frac{\cancel{2z} + 1 - \cancel{2z} + 2}{(2z+1)^2} = \frac{3}{(2z+1)^2}$

(d) $f(z) = \frac{(1+z^2)^4}{z^2} \quad (z \neq 0) \longrightarrow f'(z) = \frac{4(1+z^2)^3 2z(z^2) - (1+z^2)^4 \cdot 2z}{z^4} =$

$$= \frac{2z(1+z^2)^3\left[4z^2 - 1 - z^2\right]}{z^4} = \frac{2(1+z^2)^3(3z^2-1)}{z^3}$$

EX.2

(a) $P(z)$ is sum of $(m+1)$ functions $g_k(z) = a_k z^k \quad (k = 0 \dots m)$, THAT ARE differentiable at every z (see (2) pag 57) $\Longrightarrow P(z) = \sum_{k=0}^{m} g_k(z)$ is differentiable at every z (see (3) pag 58)

Moreover, applying (1), (2), (3) in section (19) we get:

$$P'(z) = \frac{d}{dz}\left[\sum_{k=0}^{m} g_k(z)\right] = \sum_{k=0}^{m} g_k'(z) = \sum_{k=0}^{m} \frac{d}{dz}(a_k z^k) = \sum_{k=1}^{m} k a_k z^{k-1} =$$

$$= a_1 + 2a_2 z + \dots + m a_m z^{m-1}.$$

(b) obviously $a_0 = P(0)$ and $a_1 = P'(0)$ (see expressions above)

~~[illegible]~~

by ~~[illegible]~~

~~[illegible]~~

Let's prove that $\forall\ 0 \le k \le m$, we have: (2)

$$P^{(k)}(z) = k!\,a_k + [(k+1)\cdots 2]\,a_{k+1} z + \dots + m\cdot(m-1)\cdots(n-k+1) z^{m-k} a_m =$$

$$= \sum_{j \ge k}^{m} a_j \frac{j!}{(j-k)!} z^{j-k}$$

base of induction: $k=0$ $\quad P^{(0)}(z) = P(z) = \sum_{j=0}^{m} a_j \frac{\cancel{j!}}{\cancel{j!}} z^j$ □

suppose this is true for $k-1$ and show it for k:

$$P^{(k)}(z) = \frac{d}{dz} P^{(k-1)}(z) = \frac{d}{dz}\left[\sum_{j=k-1}^{m} a_j \frac{j!}{(j-k+1)!} z^{j-k+1}\right] = \sum_{j=k-1}^{m} \frac{d}{dz}\left(a_j \frac{j!}{(j-k+1)!} z^{j-k+1}\right) =$$

$$= \sum_{j=k}^{m} a_j \frac{j!}{\underbrace{\cancel{(j-k+1)!}}_{(j-k)!}} \cancel{(j-k+1)}\, z^{j-k} = \sum_{j=k}^{m} a_j \frac{j!}{(j-k)!} z^{j-k}$$ □

From the expression above, it follows immediately: $P^{(k)}(0) = k!\,a_k \iff a_k = \frac{P^{(k)}(0)}{k!}$ □

EX 5

$$\frac{f(z)}{g(z)} = \frac{f(z) - \overset{0}{f(z_0)}}{g(z) - \underset{0}{g(z_0)}} = \frac{f(z) - f(z_0)}{g(z) - g(z_0)} \cdot \underbrace{\frac{z-z_0}{z-z_0}}_{=1} =$$

$$= \frac{f(z) - f(z_0)}{z - z_0} \Big/ \frac{g(z) - g(z_0)}{z - z_0} \xrightarrow{z \to z_0} \frac{f'(z_0)}{g'(z_0)}$$ □ (since both limits exist and $g'(z_0) \neq 0$)

EX. 6

(a) BASE OF INDUCTION $\quad \frac{d}{dz}(z^1) = \frac{d}{dz}(z) = 1 = 1\cdot z^0$ ✓ $(m=1)$

suppose this holds for $m-1$ and show it for n:

$$\frac{d}{dz}(z^m) = \underset{\uparrow \text{ product rule}}{\frac{d}{dz}(z^{m-1} z) =} \frac{d}{dz}(z^{m-1})\cdot z + z^{m-1} \underbrace{\frac{dz}{dz}}_{\substack{\text{base of induct.}\\ =1}} \overset{\text{ind. hp.}}{=} (m-1) z^{m-2}\cdot z + z^{m-1} =$$

$$= (m-1) z^{m-1} + z^{m-1} = m z^{m-1}$$ □

(b) $$\frac{d}{dz}(z^m) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^m - z^m}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sum_{k=0}^{m} \binom{m}{k} z^{n-k} \Delta z^k - z^m}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sum_{k=1}^{m} \binom{m}{k} z^{n-k} \Delta z^k}{\Delta z} =$$

$$= \lim_{\Delta z \to 0} \left[\underbrace{\binom{m}{1}}_{=m} z^{m-1} + \sum_{k=1}^{m-1} \binom{m}{k+1} z^{m-(k+1)} \underbrace{\Delta z^k}_{\Delta z \to 0:\ 0}\right] = m z^{m-1}$$ □

EX. 1:

~~[illegible]~~

(a) $f(z) = \bar{z} = x - iy \Rightarrow u(x,y) = x \quad v(x,y) = -y$

hence: $u_x(x,y) = 1 \quad v_x(x,y) = 0$

$u_y(x,y) = 0 \quad v_y(x,y) = -1$

cauchy-Riemann eq.'s don't hold at any point

$\Rightarrow f'(z)$ doesn't exist at any point.

(b) $f(z) = z - \bar{z} \Rightarrow u(x,y) = 0 \quad v(x,y) = 2y$

$= 2i \, \mathrm{Im}\, z$

hence: $u_x(x,y) = 0 \quad v_x(x,y) = 0$

$u_y(x,y) = 0 \quad v_y(x,y) = 2$

CR eq's don't hold at any pt.

(c) $f(z) = 2x + ixy^2 \Rightarrow u(x,y) = 2x \quad v(x,y) = xy^2$

hence: $u_x(x,y) = 2 \quad v_x(x,y) = y^2$

$u_y(x,y) = 0 \quad v_y(x,y) = 2xy$

CR eq's are not valid!

(d) $f(z) = e^x \cdot e^{-iy} = e^x(\cos y - i \sin y) \Rightarrow u(x,y) = e^x \cos y \quad v(x,y) = -e^x \sin y$

hence: $u_x(x,y) = e^x \cos y \quad v_x(x,y) = -e^x \sin y$

$u_y(x,y) = -e^x \sin y \quad v_y(x,y) = -e^x \cos y$

if $u_x = v_y \underset{e^x > 0}{\Rightarrow} \cos y = -\cos y \Rightarrow \cos y = 0 \Rightarrow y = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$

if $u_y = -v_x \Rightarrow \sin y = -\sin y \Rightarrow \sin y = 0 \Rightarrow y = k\pi \quad k \in \mathbb{Z}$

Therefore these conditions cannot be satisfied at the same points!

EX. 2

(a) $f(z) = iz + 2 = i(x+iy)+2 = -y+2+ix$

$\rightarrow u(x,y) = 2-y \qquad v(x,y) = x$

$\Rightarrow u_x(x,y) = 0 = v_y(x,y)$

$u_y(x,y) = -1 = -v_x(x,y)$

hence $f'(z)$ $\exists$ at every point and $f'(z) = i$

Applying the previous theorem to $f'(z) = i \Rightarrow \hat{u}(x,y) = 0 \qquad \hat{v}(x,y) = 1$

$\Rightarrow \hat{u}_x = \hat{v}_y = 0$

$\hat{u}_y = -\hat{v}_x = 0$

$\Rightarrow f''(z)$ exists everywhere and $f''(z) \equiv 0$

(b) $f(z) = e^{-x}e^{-iy} = e^{-(x+iy)} = e^{-z}$

$u(x,y) = e^{-x}\cos y \qquad v(x,y) = -e^{-x}\sin y$

$u_x = -e^{-x}\cos y = v_y$

$u_y = -e^{-x}\sin y = -v_x$

$\Rightarrow \exists f'(z)$ at every point & $f'(z) = u_x + iv_x =$

$= -e^{-x}(\cos y - i\sin y) =$

$= -e^{-z}$

analogously: $f''(z) = e^{-z} = f(z)$

(c) $f(z) = z^3 \Rightarrow f(z) = (x+iy)^3 = \underbrace{(x^3 - 3xy^2)}_{u(x,y)} + i\underbrace{(3x^2y - y^3)}_{v(x,y)}$

$\Rightarrow u_x = 3x^2 - 3y^2 = v_y$

$u_y = -6xy = -v_x$

$\Rightarrow \exists f'(z) = u_x + iv_x = 3x^2 - 3y^2 + 6xyi =$

$= 3(x^2 + 6xyi - y^2) = 3(x+iy)^2 = 3z^2$

similarly $f''(z) = 6z$

(d) $f(z) = \cos x \cosh y - i\sin x \sinh y$

$u(x,y) = \cos x \cosh y$

$v(x,y) = -\sin x \sinh y$

$\rightarrow$

$u_x = -\sin x \cosh y = v_y$

$u_y = \cos x \sinh y = -v_x$

$\rightarrow \exists f'(z) = -\cosh y \sin x - i\cos x \sinh y$

analogously for $f'(z)$ and get:

$f''(z) = -\cos x \cosh y + i\sin x \sinh y = -f(z) \quad \square$

EX. 4

(a) $f(z)=\dfrac{1}{z^4}=\dfrac{1}{(re^{i\theta})^4}=\dfrac{e^{-i4\theta}}{r^4}$ $\quad (r\neq 0)$

$=\dfrac{1}{r^4}(\cos 4\theta - i\sin 4\theta) \Rightarrow u(r,\theta)=\dfrac{\cos 4\theta}{r^4} \quad v(r,\theta)=-\dfrac{\sin 4\theta}{r^4}$

$u_r(r,\theta)=-\dfrac{4\cos 4\theta}{r^5} \qquad v_r(r,\theta)=\dfrac{4\sin 4\theta}{r^5}$

$u_\theta(r,\theta)=-\dfrac{4\sin 4\theta}{r^4} \qquad v_\theta(r,\theta)=-\dfrac{4\cos 4\theta}{r^4}$

$\Rightarrow \quad r u_r = -4\dfrac{\cos 4\theta}{r^4}=v_\theta \quad \& \quad r v_r=\dfrac{4\sin 4\theta}{r^4}=-u_\theta$

$\Rightarrow \exists\, f'(z)=e^{-i\theta}(u_r+iv_r)=e^{-i\theta}\left(-4\dfrac{\cos 4\theta}{r^5}+i\,\dfrac{4\sin 4\theta}{r^5}\right)=\dfrac{-4e^{-i\theta}}{r^5}\,e^{-4i\theta}=$

$=-\dfrac{4e^{-5i\theta}}{r^5}=-\dfrac{4}{z^5}$

(b) $f(z)=\sqrt{r}\,e^{i\theta/2} \quad (r>0,\ \alpha<\theta<\alpha+2\pi) \quad \Rightarrow$ " $f(z)=\sqrt{z}$ "

[NOTE: $(f(z))^2=z$ hence $f(z)$ can be considered as a branch of the function $\sqrt{z}$
This is the reason why we have to restrict our domain to $\mathbb{C}\setminus\{$a semiline$\}$]

$u(r,\theta)=\sqrt{r}\cos\frac{\theta}{2},\ v(r,\theta)=\sqrt{r}\sin\frac{\theta}{2} \Rightarrow u_r(r,\theta)=\dfrac{1}{2\sqrt{r}}\cos\frac{\theta}{2} \quad u_\theta=-\dfrac{\sqrt{r}}{2}\sin\frac{\theta}{2}$

$v_r(r,\theta)=\dfrac{1}{2\sqrt{r}}\sin\frac{\theta}{2} \quad v_\theta=+\dfrac{\sqrt{r}}{2}\cos\frac{\theta}{2}$

$\Rightarrow \quad r u_r=\dfrac{\sqrt{r}}{2}\cos\frac{\theta}{2}=v_\theta \qquad \Rightarrow \exists\, f'(z)=e^{-i\theta}\left(\dfrac{1}{2\sqrt{r}}\cos\frac{\theta}{2}+i\,\dfrac{1}{2\sqrt{r}}\sin\frac{\theta}{2}\right)=$

$u_\theta=-\dfrac{\sqrt{r}}{2}\sin\frac{\theta}{2}=-r v_r$

$=\dfrac{e^{-i\theta}}{2\sqrt{r}}e^{i\theta/2}=\dfrac{e^{-i\theta/2}}{2\sqrt{r}} \quad \left(=\text{"}\dfrac{1}{2\sqrt{z}}\text{"}\right)$

$\rightarrow f'(z)=\dfrac{1}{2f(z)}$

(c) $f(z)=e^{-\theta}\cos(\ln r)+ie^{-\theta}\sin(\ln r)= \quad (r>0,\ 0<\theta<2\pi)$

$\Rightarrow \quad u_r=\dfrac{-e^{-\theta}\sin(\ln r)}{r}, \quad u_\theta=-e^{-\theta}\cos(\ln r) \qquad \Rightarrow \qquad r u_r=v_\theta$

$v_r=\dfrac{e^{-\theta}\cos(\ln r)}{r}, \quad v_\theta=-e^{-\theta}\sin(\ln r) \qquad\qquad r v_r=-u_\theta$

$\Rightarrow \exists\, f'(z)=e^{-i\theta}(u_r+iv_r)=e^{-i\theta}\,\dfrac{e^{-\theta}}{r}\big(i\cos(\ln r)-\sin(\ln r)\big)=$

$=\dfrac{e^{-\theta}}{re^{i\theta}}\,i\,\big(\cos(\ln r)+i\sin(\ln r)\big)=i\,\dfrac{f(z)}{z}$ $\square$

EX. 6

(a) $f(z) = \dfrac{2z+1}{z(z^2+1)}$ THIS FUNCTION IS THE QUOTIENT OF TWO POLYNOMIALS, HENCE IT'S ANALYTIC IN ANY DOMAIN THROUGHOUT WHICH $Q(z) \neq 0$

$\rightarrow z(z^2+1) = 0$ iff $z=0$ or $z = \pm i$ (and the numerator doesn't vanish at these points)

$\Rightarrow$ SINGULAR POINTS: $z = 0, \pm i$ (They are poles, ie $\lim\limits_{\substack{z\to 0 \\ (\text{resp } \pm i)}} |f(z)| = +\infty$)

(b) $f(z) = \dfrac{z^3+i}{z^2-3z+2}$ similarly as above, check where the denomin. vanishes:

$z^2-3z+2=0 \iff z = \dfrac{3 \pm \sqrt{9-8}}{2} = \dfrac{3 \pm 1}{2}$ $\nearrow z_1 = 2$, $\searrow z_2 = 1$

and the numerator doesn't vanish at these pts

$\Rightarrow$ singular points $z = 1, 2$ (poles)

(c) $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$

$(z+2)(z^2+2z+2) = 0$ iff $z=-2$ or $z^2+2z+2=0 \iff z = -1 \pm \sqrt{1-2}$
$= -1 \pm i$

$\Rightarrow$ singular points (poles) $z = -2, -1 \pm i$

EX. 7

(a) suppose $f(z) \in \mathbb{R}$ $\forall z \in D$ $\Rightarrow$ $v(x,y) \equiv 0$ on D

$\Rightarrow u_x(x,y) = v_y(x,y) = 0$
$u_y(x,y) = -v_x(x,y) = 0$ on D $\Rightarrow$ $\nabla u(x,y) \equiv 0$ on D

$\Rightarrow u(x,y) \equiv$ constant on D $\Rightarrow$ $f(z) \equiv$ constant on D

ⓑ Suppose $|f(z)| = c \quad \forall z \in D$

If $c=0 \Rightarrow f(z) \equiv 0$ on D, hence it's constant.

If $c \neq 0$: $|f(z)|^2 = c^2 \iff f(z)\overline{(f(z))} = c^2 \iff \overline{f(z)} = \frac{c^2}{f(z)}$

$(f(z) \neq 0$ on $D)$ $\qquad \uparrow$ $f(z)\neq 0$ on D

$\Rightarrow$ both f and $\bar{f}$ are analytic in D (since $\bar{f} = \frac{c^2}{f}$ and $f \neq 0$)

$\underset{\text{(Ex 3 pag. 73)}}{\Rightarrow}$ $f(z) \equiv$ constant on D.

(other way to solve it)

$|f(z)| = c$ and suppose $c \neq 0 \Rightarrow u^2 + v^2 = c^2$

$\Rightarrow \begin{array}{l} 2uu_x + 2vv_x = 0 \\ 2uu_y + 2vv_y = 0 \end{array} \longrightarrow 0 = (uu_x + vv_x)^2 + (uu_y + vv_y)^2 =$

$= u^2u_x^2 + v^2\underbrace{v_x^2}_{=u_y^2} + \cancel{2uvu_x\underbrace{v_x}_{=-u_y}} + u^2u_y^2 + v^2\underbrace{v_y^2}_{=u_x^2} + \cancel{2uvu_y\underbrace{v_y}_{=-v_x}}$

$= (u^2+v^2)(u_x^2+u_y^2) \underset{u^2+v^2\neq 0}{\Rightarrow} u_x^2 + u_y^2 = 0$

$\Rightarrow u_x = u_y = 0$ & (by C-R eq's) $v_x = v_y = 0$

$\Rightarrow u \equiv$ const & $v \equiv$ const $\Rightarrow f =$ const.

Bonus Exercise

$f(z) = u(x,y) + i\,v(x,y)$

Consider $z_0 = (x_0, y_0)$ st: $\quad \cdot\ f'(z_0) \neq 0$

$$\cdot \begin{cases} u(x_0;y_0) = c_1 \in \mathbb{C} \\ v(x_0,y_0) = c_2 \in \mathbb{C} \end{cases}$$

show that the level curves $u(x,y) = c_1$ & $v(x,y) = c_2$ are orthogonal at z_0.

1st method

- Observe that $\nabla u(x_0,y_0) \neq 0$. In fact, if $\nabla u(x_0,y_0) = 0 \overset{\text{CR eq's}}{\Longrightarrow} \nabla v(x_0,y_0) = 0$
  $\Longrightarrow f'(z_0) = 0$ $\ast$ contradiction.

- suppose $u_y(x_0,y_0) \neq 0 \quad \Longmapsto$ (CR eqn's) $v_x(x_0,y_0) \neq 0$
  (the case $u_y(x_0,y_0) = 0 \Rightarrow u_x(x_0,y_0) \neq 0 \Rightarrow v_y(x_0,y_0) \neq 0$ can be analyzed in a similar way)
  $\hookrightarrow$ (otherwise one can consider a new function $g = if$ and obtain $\hat{u}_y \neq 0 \dots$)

- Applying the implicit function theorem (I.F.T.), we can locally write these curves in a nbhd of z_0:

$$\gamma : (x_0-\varepsilon, x_0+\varepsilon) \longrightarrow \mathbb{C} \qquad \text{st} \qquad u(\gamma(x)) = c_1 \quad \forall x \in (x_0-\varepsilon, x_0+\varepsilon) \quad (1)$$
$$x \longmapsto (x, \hat{y}(x)) \qquad \qquad \& \ \gamma(x_0) = (x_0,y_0)$$

$$\tilde{\gamma} : (y_0-\varepsilon, y_0+\varepsilon) \longmapsto \mathbb{C}$$
$$y \longmapsto (\hat{x}(y), y) \qquad \text{st} \qquad v(\tilde{\gamma}(y)) = c_2 \quad \forall y \in (y_0-\varepsilon, y_0+\varepsilon) \quad (2)$$
$$\& \ \tilde{\gamma}(y_0) = (x_0, y_0)$$

differentiating (1) w.r.t. x:

$$0 = \frac{\partial u}{\partial x}(x_0,y_0) + \frac{\partial u}{\partial y}(x_0,y_0) \cdot \frac{\partial \hat{y}}{\partial x}(y_0) \quad \Longrightarrow \quad \frac{\partial \hat{y}}{\partial x}(x_0) = \hat{y}'(x_0) = \frac{-\frac{\partial u}{\partial x}(x_0,y_0)}{\underbrace{\frac{\partial u}{\partial y}(x_0,y_0)}_{\neq 0}}$$

similarly, differentiating (2):

$$\frac{\partial \hat{x}}{\partial y}(y_0) = \hat{x}'(y_0) = \frac{-\frac{\partial v}{\partial y}(x_0,y_0)}{\underbrace{\frac{\partial v}{\partial x}(x_0,y_0)}_{\neq 0}}$$

the tangent vector of γ at (x_0,y_0) is: $\quad \tau = (1, \hat{y}'(x_0))$
The tangent vector of $\tilde{\gamma}$ at (x_0,y_0) is: $\quad \tilde{\tau} = (\hat{x}'(y_0), 1)$

$\longrightarrow$

observe: $\tau \cdot \tau' = \hat{x}'(y_0) + \hat{y}'(x_0) = - \dfrac{u_y(x_0,y_0)\, v_y(x_0,y_0) + u_x(x_0,y_0)\, v_x(x_0,y_0)}{u_y(x_0,y_0) \cdot v_x(x_0,y_0)} =$

$= 0 \implies$ they are orthogonal.

Cauchy Riemann eq's

2nd method: The normal vectors to the curves at z_0, are respectively

$\nabla u(x_0,y_0)$ & $\nabla v(x_0,y_0)$ (they are both $\neq 0$ .. see above)

(This is a result of multivariable calculus)

$\longrightarrow$ using C-R equations, it's evident that $\nabla u(x_0,y_0) \cdot \nabla v(x_0,y_0) = 0$

$\implies$ the two curves are perpendicular at z_0.

$\square$

NOTE: the condition $f'(z_0) \neq 0$ is important (to apply I.F.T.)

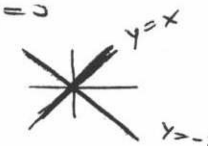
consider $f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + 2ixy$

and $z_0 = 0 = (0,0)$

$u(0,0) = 0 = v(0,0)$

$\longrightarrow$ the two level curves that intersect at 0 are: $\begin{cases} u(x,y)=0 \\ v(x,y)=0 \end{cases}$

$\iff \begin{cases} x^2-y^2=0 \\ 2xy=0 \end{cases} \iff \begin{cases} (x-y)(x+y)=0 \\ xy=0 \end{cases}$ $\longrightarrow$



$\longrightarrow$ coordinate axis

obviously they are not orthogonal at the origin!