HOMEWORK ASSIGNMENT #2

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EX. 9 : PROVE THAT: $|z_1| = |z_2| \iff \exists\, c_1, c_2 \in \mathbb{C}: \; z_1 = c_1 c_2 \;\&\; z_2 = c_1 \bar{c_2}$

proof: ($\Leftarrow$) obviously $|z_1| = |c_1 \cdot c_2| = |c_1| \cdot |c_2| = |c_1| \cdot |\bar{c_2}| = |c_1 \cdot \bar{c_2}| = |z_2|$ $\square$

($\Rightarrow$) Using the hint; let $z_1 = \alpha e^{i\theta_1}$ & $z_2 = \alpha e^{i\theta_2}$

(where $\alpha = |z_1| = |z_2| \in \mathbb{R}_{\geq 0}$ and $\theta_1 = \operatorname{Arg} z_1$, $\theta_2 = \operatorname{Arg} z_2$)

Define: $c_1 = \sqrt{\alpha}\, e^{i\left(\frac{\theta_1 + \theta_2}{2}\right)}$ & $c_2 = \sqrt{\alpha}\, e^{i\left(\frac{\theta_1 - \theta_2}{2}\right)}$ $\square$

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EX. 22. (a) $\cos n\theta + i \sin n\theta \overset{\text{DEMOIVRE'S Formula}}{=} (\cos\theta + i\sin\theta)^n \overset{\text{BINOMIAL FORM.}}{=} \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\,(i\sin\theta)^k$

define $m = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$

observe that $(i)^k = \begin{cases} (-1)^{\frac{k}{2}} & \text{if } k \text{ even} \\ (-1)^{\frac{k-1}{2}} i & \text{if } k \text{ odd} \end{cases}$

$\Rightarrow \cos n\theta = \operatorname{Re}\left(\sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\, (i\sin\theta)^k \right) = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} \cos^{n-k}\theta\; i^k \sin^k\theta =$

$$\underset{\substack{\uparrow \\ \text{call } k=2j \\ \text{and } m \text{ as} \\ \text{above}}}{=} \sum_{j=0}^{m} \binom{n}{2j} \cos^{n-2j}\theta\, (-1)^j \sin^{2j}\theta \quad \square$$

(b) $T_n(x) = \cos(n\cos^{-1}x) \overset{\downarrow}{=} \sum_{j=0}^{m} \binom{n}{2j} \cos^{n-2j}(\cos^{-1}x)\, (-1)^j \sin^{2j}(\cos^{-1}x) =$

$$= \sum_{j=0}^{m} \binom{n}{2j} x^{n-2j} (1-x^2)^j$$ $\Rightarrow$ it's a polynomial of degree n

$\uparrow$

I use $\sin^2\theta = 1 - \cos^2\theta$

· How many solutions of $T_m(x)=0$ are in $-1\le x\le 1$?

$$T_m(x) = \cos\left(m\cos^{-1}x\right) = 0 \iff m\cos^{-1}x = \frac{2k+1}{2}\pi \quad \exists k\in\mathbb{Z}$$

$$\iff \cos^{-1}x = \frac{2k+1}{2m}\pi \quad \exists k\in\mathbb{Z} \iff x_k = \cos\left(\frac{2k+1}{2m}\pi\right) \quad \exists k\in\mathbb{Z}$$

- These $x_k\in[-1,1]$ $\quad\forall k\in\mathbb{Z}$
- These x_k are not all distinct (b/c of the periodicity of $\cos\theta$)

$\rightarrow$ condition to be distinct: $\frac{2k+1}{2m}\pi \in [0,\pi] \iff 0\le \frac{2k+1}{2m}\le 1$

$$\iff 0\le 2k+1\le 2m \iff -1\le 2k\le 2m-1$$

$$\iff -\tfrac{1}{2}\le k\le n-\tfrac{1}{2} \underset{(\text{since } k\in\mathbb{Z})}{\iff} k=0,\dots,m-1$$

$\longrightarrow$ THERE ARE EXACTLY m solutions: $x_k = \cos\left(\frac{2k+1}{2m}\pi\right) = \cos\left(\frac{k}{m}\pi + \frac{\pi}{2m}\right)$
$(k=0,\dots m-1)$

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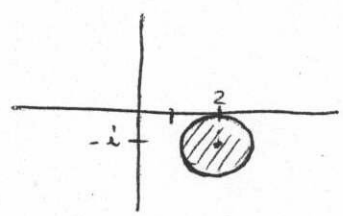
EX 1

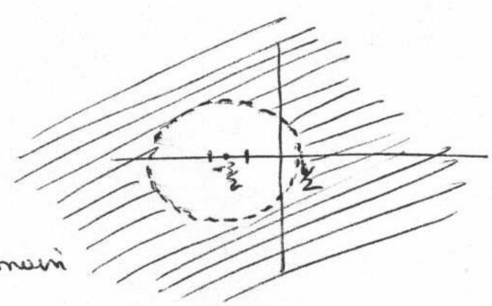
(a) $|z-2+i|\le 1$



circle of radius 1
& center $2-i$
(including the boundary)

IT'S connected, but closed $\Rightarrow$ no domain

(b) $|2z+3|>4 \iff |z+\tfrac{3}{2}|>2$



all points outside the closed disk of radius 2 and center $-\tfrac{3}{2}$

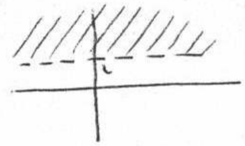
it's connected and open $\Rightarrow$ it's a domain

$\uparrow$

since it's the complement of a closed set.

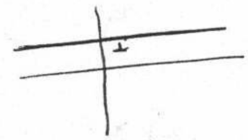
ⓒ $\operatorname{Im} z > 1$

IT'S connected
IT'S open $\Longrightarrow$ IT'S A Domain



ⓓ $\operatorname{Im} z = 1$

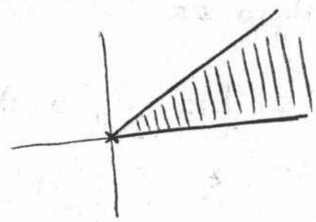
IT'S connected
IT'S closed $\Longrightarrow$ ~~It's not domain~~ IT'S NOT A domain



ⓔ $0 \leq \arg z \leq \frac{\pi}{4}$ $\quad z \neq 0$

IT'S connected
IT'S NEITHER OPEN NOR CLOSED

$\Rightarrow$ NO DOMAIN

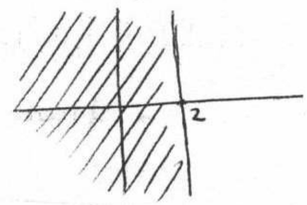


ⓕ $|z-4| \geq |z| \iff |(x+iy)-4|^2 \geq |x+iy|^2$

$\iff (x-4)^2 + y^2 \geq x^2 + y^2$

$\iff \cancel{x^2} - 8x + 16 \geq \cancel{x^2}$

$\iff x \leq 2 \iff \operatorname{Re} z \leq 2$



IT'S CONNECTED
IT'S CLOSED $\Rightarrow$ NO DOMAIN

$\boxed{\text{EX 2}}$ only SET in (e) is NEITHER open nor closed

$\boxed{\text{EX 3}}$ only SET in (a) is bounded

$\boxed{\text{EX 10}}$ prove that a finite set of points $z_1 \dots z_m$ cannot have an accumulation pt.

denote $\Delta = \min \{ |z_i - z_j| ,\ i \neq j \}$

obviously $\Delta > 0$ since there are only finitely many z_i's

$\forall z_i$ $(i = 1 \dots m)$ and $0 < \delta < \Delta$, the neighborhood $D_\delta(z_i) = \{ |z - z_i| < \delta \}$

is such that $D_\delta(z_i) \cap \{z_1 \dots z_m\} = \{z_i\}$ $\Longrightarrow$ z_i is not an accumulat. point

$\square$

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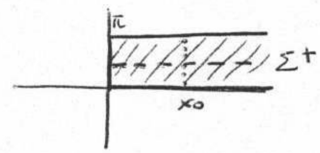
EX. 7

Semistrip $\Sigma^+ = \{x \geq 0 \ , 0 \leq y \leq \pi\}$



$w = e^z = e^x \cdot e^{iy} \quad \Rightarrow \quad |w| = e^x$

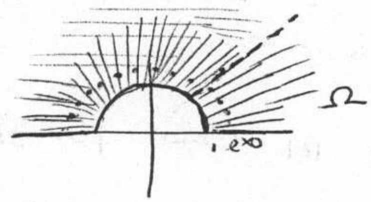
$\qquad\qquad\qquad\qquad\quad \text{Arg}\, w = y$

$\Rightarrow \ \forall z \in \Sigma^+ \qquad |w| \geq 1 \qquad (\text{since } x \geq 0)$

$\qquad\qquad 0 \leq \text{Arg}\, w \leq \pi$

Conversely $\forall \ \hat{w} \in \{|w| \geq 1 \ , \ 0 \leq \text{Arg}\, w \leq \pi\}$, $\exists \ z_0 \in \Sigma^+$ st $e^{z_0} = \hat{w}$

it suffices to consider $z_0 = \underbrace{\ln|\hat{w}|}_{\geq 0} + i \underbrace{\text{Arg}\, \hat{w}}_{\in [0,\pi]}$



$\Rightarrow$ the image is the following region

$$\Omega = \{|w| \geq 1 \ , \ 0 \leq \text{Arg}\, w \leq \pi\} =$$

$$= \{|w| \geq 1 \ , \ \text{Im}\, w \geq 0\}$$

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EX. 7

We know by hp: $\lim_{z \to z_0} f(z) = w_0 \iff \forall \varepsilon > 0 \ \exists \ \delta = \delta(\varepsilon);$ if $|z - z_0| < \delta$ $\Rightarrow |f(z) - w_0| < \varepsilon$

consider now $\varepsilon > 0$ fixed, and let's show that $\exists \ \hat{\delta} = \hat{\delta}(\varepsilon)$ st if $|z - z_0| < \hat{\delta}$

$\Rightarrow \ \big| |f(z)| - |w_0| \big| < \varepsilon$.

In fact: $\big| |f(z)| - |w_0| \big| \leq |f(z) - w_0| < \varepsilon$ if $|z - z_0| < \delta(\varepsilon)$ (by hp)

$\Rightarrow$ it's enough to take $\hat{\delta}(\varepsilon) = \delta(\varepsilon)$.

EX.9

Suppose $\lim_{z\to z_0} f(z) = 0$ (ie $\forall \varepsilon>0\ \exists\, \delta=\delta(\varepsilon)$: $|f(z)|<\varepsilon$ if $|z-z_0|<\delta$)

& $|g(z)| \le M \quad \forall\, z\in U$ (nbhd of z_0)

we want to show that $\lim_{z\to z_0} f(z)g(z) = 0$, ie

$\forall \varepsilon>0\ \exists\, \hat\delta=\hat\delta(\varepsilon)$ st if $|z-z_0|<\delta \implies |f(z)g(z)|<\varepsilon$

denote $\Delta = \operatorname{dist}(z_0, \partial U) = \inf\{|z-z_0| \mid z\in\partial U\} = \sup\{r>0 : \overline{D_r(z_0)} \subset U\}$

$$\implies |f(z)g(z)| \le |f(z)|\underbrace{|g(z)|}_{\substack{\wedge\!\!|\\ M \text{ if}\\ z\in U\\ \text{ie } \hat\delta<\Delta}} \le \underbrace{|f(z)|}_{\substack{\wedge\!\!|\\ \frac{\varepsilon}{M}\\ \text{if } \delta=\delta(\frac{\varepsilon}{M})}} \cdot M \le \varepsilon$$

$\Rightarrow$ it's enough to consider $\hat\delta(\varepsilon) = \min\{\Delta, \delta(\tfrac{\varepsilon}{M})\}$.

ADDITIONAL QUESTIONS

(1) Prove that for any non constant polynomial $p(z) = a_m z^m + \dots + a_0$ one has $p(z) \to +\infty$ as $z \to +\infty$

proof by induction on $\deg p \ge 1$ (since it's non constant)

- $\deg p = 1$: $\quad p(z) = a_1 z + a_0 \qquad a_1 \ne 0$

$$\Rightarrow |p(z)| = |a_1 z + a_0| \ge \underset{\ne 0}{|a_1|}\cdot|z| - |a_0| \xrightarrow{|z|\to+\infty} +\infty$$

- suppose is true for $\deg p \le m-1$ and let's show it for $\deg p = m$

$p(z) = a_m z^m + \dots + a_0 \qquad a_m \ne 0$

$$|p(z)| = |a_m z^m + \dots + a_0| \ge |a_m z^m + \dots + a_1 z| - |a_0| =$$

$$= \underset{\substack{\downarrow\\ +\infty}}{|z|}\ \underbrace{|a_m z^{m-1} + \dots + a_1|}_{\substack{\deg = m-1\\ \text{hence the claim holds}\\ \text{for this poly}\\ \downarrow\\ +\infty}} - |a_0| \xrightarrow{|z|\to\infty} +\infty$$

② $f(z) = e^{1/z}$

ⓐ what is the image under f of the deleted nbhd $0<|z|<\delta$ for $\delta>0$?

ⓑ does $\lim_{z\to 0} f(z)$ exist ?

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we will show that $\forall \alpha \in \mathbb{C}\setminus\{0\}$, $\exists \{z_n\} \subset \mathbb{C}$ st $z_n \to 0$ and $f(z_n) = \alpha$

THIS WILL IMPLY: ⓑ THE LIMIT $\lim_{z\to 0} f(z)$ DOESN'T EXIST (I can find different accum. points)

ⓐ $\forall \delta>0$ $f(\{0<|z-z_0|<\delta\}) = \mathbb{C}\setminus\{0\}$
(obviously $f(z)\neq 0$ $\forall z\in\mathbb{C}$)

LET'S SHOW our claim:

$\alpha\in\mathbb{C}\setminus\{0\} \to$ define $\omega_n = \log|\alpha| + 2\pi i(\operatorname{Arg}\alpha + n)$ $n=0,1,\ldots$

$\uparrow$ is the "real" log

in particular $|\omega_n| = [\log|\alpha|]^2 + 4\pi^2(\operatorname{Arg}\alpha+n)^2 \xrightarrow{n\to\infty} +\infty$ and $e^{\omega_n} = \alpha$

Define $z_n = \dfrac{1}{\omega_n} \in \mathbb{C}$ (since $\omega_n \neq 0$ $\forall n$)

and $|z_n| \xrightarrow{n\to\infty} 0$

$\{z_n\}_{n\geq 0} \subset \mathbb{C}$ is the sequence we were seeking. In fact $f(z_n) = e^{\frac{1}{z_n}} = e^{\omega_n} = \alpha$ □

$\cancel{\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} e^{\frac{1}{z_n}} = \lim_{n\to\infty} e^{\omega_n} = \lim_{n\to\infty} \alpha = \alpha}$ //

Hence, in any nbhd $0<|z|<\delta$, $\exists z_N$ st $f(z_N)=\alpha$

$\Rightarrow f(\{0<|z|<\delta\}) = \mathbb{C}\setminus\{0\}$

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another way to see that the limit doesn't exist.

$\lim_{\substack{x\to 0^+\\ x\in\mathbb{R}}} f(x) = \lim e^{\frac{1}{x}} = +\infty$ $\qquad$ $\lim_{\substack{x\to 0^-\\ x\in\mathbb{R}}} f(x) = \lim e^{\frac{1}{x}} = 0$