PRACTICE EXERCISES

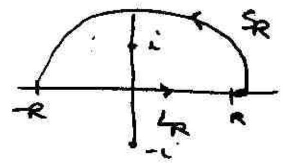
EX. PAG 257, N. 1

$$\int_0^\infty \frac{dx}{1+x^2} \left[= \int_0^\infty \frac{d}{dx}(\operatorname{arctg} x) = \lim_{R\to\infty} \int_0^R \frac{d}{dx}(\operatorname{arctg} x)\, dx = \lim_{R\to\infty} \operatorname{arctg} R = \frac{\pi}{2} \right]$$

Using the Residues:

Define $f(z) = \frac{1}{1+z^2}$ this function has simple poles at $z = \pm i$ $\quad \operatorname{Res}_{z=\pm i} f(z) = \frac{\pm 1}{2i}$

Consider the following contour C_R:



$R > 1$

$$\int_{C_R} f(z)\,dz \underset{\text{Residue thm}}{=} 2\pi i \operatorname{Res}_{z=i} f(z) = 2\pi i \left(\frac{1}{2i}\right) = \pi$$

$$\|$$

$$\int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{S_R} \frac{1}{1+z^2}\,dz$$

Moreover $\left| \int_{S_R} \frac{1}{1+z^2}\,dz \right| \le \frac{1}{R^2-1}\,\pi R \xrightarrow{R\to\infty} 0$

$$\Rightarrow \quad \pi = \lim_{R\to+\infty} \int_{-R}^{R} \frac{1}{1+x^2}\,dx = \text{P.V.} \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx \overset{f \text{ is even}}{=} 2\int_0^\infty \frac{1}{1+x^2}\,dx$$

$$\Rightarrow \quad \boxed{\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}}$$

EX. PAG. 257, N. 2

$$\int_0^\infty \frac{dx}{(1+x^2)^2}$$

Consider C_R as above (ex 1)

two poles at $z = \pm i$ w/ multiplicity 2

Define $f(z) = \frac{1}{(1+z^2)^2} = \frac{1}{(z-i)^2(z+i)^2}$

We are only interested in the residue at $z = i$

$$\operatorname{Res}_{z=i} f(z) = \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]_{z=i} = -2(z+i)^{-3}\Big|_{z=i} = -2\,(2i)^{-3} = \frac{-2}{8i^3} = \frac{1}{4i}$$

$$\Longrightarrow \int_{C_R} f(z)\,dz = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$$

$$\| $$

$$\int_{-R}^{R} f(x)\,dx + \int_{S_R} f(z)\,dz \qquad \text{where } \left|\int_{S_R} f(z)\,dz\right| \overset{R>1}{\le} \frac{1}{(R^2-1)^2}\pi R \xrightarrow{R\to\infty} 0$$

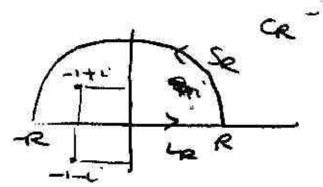
$$\Longrightarrow \frac{\pi}{2} = \lim_{R\to+\infty}\int_{-R}^{R}\frac{1}{(1+x^2)^2}\,dx = \text{P.V.}\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^2}\,dx = 2\int_{0}^{\infty}\frac{1}{(1+x^2)^2}\,dx$$

$$\Longrightarrow \boxed{\int_{0}^{\infty}\frac{1}{(1+x^2)^2}\,dx = \frac{\pi}{4}}$$

EX. PAG. 257, N.6

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2}$$

consider $f(z) = \dfrac{1}{z^2+2z+2}$



$R>2$

poles at $z = -1 \pm \sqrt{1-2} = -1 \pm i$

Simple poles

Consider only $z=-1+i$: $\underset{z=-1+i}{\text{Res}} f(z) = \dfrac{1}{((-1+i)-(-1-i))} = \dfrac{1}{2i}$

$\Longrightarrow$ by Residue's theorem: $\displaystyle\int_{C_R} f(z)\,dz = 2\pi i \frac{1}{2i} = \pi$

$$\|$$

$$\int_{-R}^{R} f(x)\,dx - \int_{S_R} f(z)\,dz$$

and $\displaystyle\left|\int_{S_R} f(z)\,dz\right| \underset{\substack{\uparrow \\ R>>0}}{\le} \frac{\pi R}{R^2-2R-2} \xrightarrow{R\to\infty} 0$

$$\Longrightarrow \boxed{\text{P.V.}\int_{-\infty}^{\infty}\frac{1}{x^2+2x+2}\,dz = \pi}$$

Does this integral exist in the ordinary sense?

YES, IT EXISTS !!
$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx = \int_{-\infty}^{\infty} \frac{1}{(x^2+2x+1)+1}\,dx =$$

$$= \int_{-\infty}^{\infty} \frac{1}{1+(x+1)^2}\,dx \underset{y=x+1}{\uparrow}= \int_{-\infty}^{\infty} \frac{dy}{1+y^2} = 2\int_0^{\infty} \frac{dy}{1+y^2} = 2\cdot\frac{\pi}{2} = \pi .$$

EXERCISE PAG. 257, N. 8

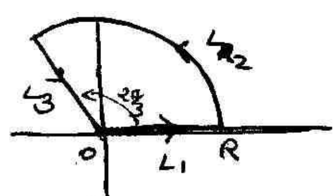
$$\int_0^{\infty} \frac{dx}{1+x^3} = ?$$

← NOTE: in class, Alfonso did the general case $\int_0^{\infty} \frac{1}{1+x^m}\,dx$

consider $f(z) = \dfrac{1}{1+z^3}$ → poles where $z^3+1=0 \iff z^3=-1=e^{i\pi}$

poles at $z_k = e^{i\left(\frac{\pi}{3}+\frac{2k\pi}{3}\right)}$ $k=0,1,2$

consider C_R:



$R>1$

there is only one pole inside C_R (for $R>1$); namely $z_0 = e^{i\frac{\pi}{3}}$

$$\operatorname*{Res}_{z=e^{i\frac{\pi}{3}}} f(z) = \frac{1}{3(e^{i\frac{\pi}{3}})^2} = \frac{1}{3\,e^{i\frac{2}{3}\pi}}$$

$$\Longrightarrow \int_{C_R} f(z)\,dz = \frac{2\pi i}{3\,e^{i\frac{2}{3}\pi}}$$

$$\| $$

$$\int_{L_1} + \int_{L_2} + \int_{L_3}$$

$$\int_{L_1} f(z)\,dz = \int_0^R \frac{1}{1+x^3}\,dx$$

$$\int_{L_3} f(z)\,dz = \int_0^R \frac{1}{1+(e^{i\frac{2\pi}{3}}x)^3}\,e^{i\frac{2}{3}\pi}\,dx$$

$$= e^{i\frac{2}{3}\pi}\int_0^R \frac{1}{1+x^3}\,dx$$

$$\left|\int_{L_2} f(z)\,dz\right| \le \frac{\pi R}{R^3-1} \xrightarrow{R\to\infty} 0$$

$$\Longrightarrow \frac{2\pi i}{e^{i\frac{2}{3}\pi}} = (1-e^{i\frac{2}{3}\pi})\int_0^{\infty} \frac{1}{1+x^3}\,dx$$

(5)

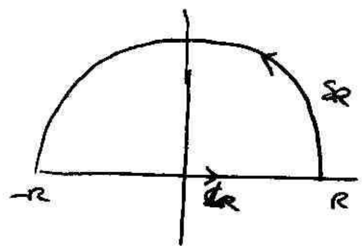
$$\Rightarrow \int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi i}{3e^{i\frac{2\pi}{3}}\left(1-e^{i\frac{2\pi}{3}}\right)} = \frac{2\pi i}{3e^{i\frac{2\pi}{3}}e^{i\frac{\pi}{3}}\left(e^{-i\frac{\pi}{3}}-e^{i\frac{\pi}{3}}\right)} =$$

$$= \frac{-2\pi}{3\underbrace{e^{i\pi}}_{=-1}\sin\frac{\pi}{3}} = \frac{\pi}{3\frac{\sqrt{3}}{2}} = \frac{2\pi}{3\sqrt{3}}$$

$\boxed{\text{EX. PAG 265, N. 1}}$

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} \qquad a>b>0$$

C_R:



consider $f(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)} = \dfrac{1}{(z-ia)(z+ia)(z-ib)(z+ib)}$ → Simple poles at $z=\pm ia$, $\pm ib$

$$\operatorname{Res}_{z=ia} e^{iz}f(z) = \frac{e^{-a}}{2ia(b^2-a^2)}$$

$$\operatorname{Res}_{z=ib} e^{iz}f(z) = \frac{e^{-b}}{2ib(a^2-b^2)}$$

on $|z|=R$, $R>a$: $\quad |f(z)| \le \dfrac{1}{(R^2-b^2)(R^2-a^2)} =: M_R \xrightarrow{R\to\infty} 0$

Hence I can apply Jordan's lemma (§74)

Consider $$\int_{C_R} f(z)e^{iz}\,dz = 2\pi i\left[\frac{e^{-a}}{2ia(b^2-a^2)} + \frac{e^{-b}}{2ib(a^2-b^2)}\right]$$

$$\|$$

$$\int_{-R}^{R} f(x)e^{ix}\,dx + \underbrace{\int_{S_R} f(x)}_{\to 0 \text{ as } R\to\infty}$$ by Jordan's Lemma

$$\Rightarrow \text{P.V.} \int_{-\infty}^{\infty} f(x)e^{ix}\,dx = \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

even function

$\Rightarrow$ Taking Real parts

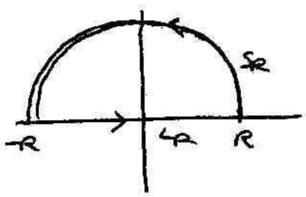
$$\boxed{\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{(a^2+x^2)(b^2+x^2)}\,dx = \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]}$$

$$\int_0^\infty \frac{x\sin 2x}{x^2+3}\,dx$$

Consider $f(z)=\dfrac{z}{z^2+3}$ $\Longrightarrow$ on $|z|=R$, $R>3$: $|f(z)|\le \dfrac{R}{R^2-3} =: M_R \xrightarrow{R\to\infty} 0$

hence we can apply Jordan's Lemma

Consider the following path C_R:



$R>>0$

$$\operatorname*{Res}_{z=i\sqrt3} f(z)e^{i2z} = \frac{i\sqrt3\, e^{-2\sqrt3}}{2i\sqrt3} = \frac{e^{-2\sqrt3}}{2}$$

$\Longrightarrow$ for $R>>0$:

$$\int_{C_R} f(z)e^{i2z}\,dz = 2\pi i\,\frac{e^{-2\sqrt3}}{2} = i\pi e^{-2\sqrt3}$$

$$\parallel$$

$$\int_{-R}^{R} f(x)e^{i2x}\,dx + \underbrace{\int_{S_R} f(z)e^{i2z}\,dz}_{\downarrow\, R\to\infty \atop 0}$$

$$\Longrightarrow \quad \text{P.V.}\int_{-\infty}^{\infty}\frac{x e^{i2x}}{x^2+3} = i\pi e^{-2\sqrt3}$$

$\Longrightarrow$ taking the imaginary part: $\quad \text{P.V.}\displaystyle\int_{-\infty}^{\infty}\frac{x\sin 2x}{x^2+3}\,dx = \pi e^{-2\sqrt3}$

$\parallel$ ← even function

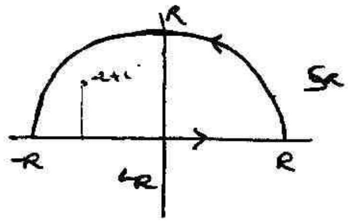
$$2\int_0^\infty \frac{x\sin 2x}{x^2+3}\,dx$$

$$\Longrightarrow \quad \boxed{\int_0^\infty \frac{x\sin 2x}{x^2+3}\,dx = \frac{\pi}{2}e^{-2\sqrt3}}$$

$\boxed{\text{EX. PAG. 266, N. 10}}$

$$P.V. \int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}\, dx$$

consider $f(z) = \dfrac{z+1}{z^2+4z+5}$ poles at $z = -2 \pm \sqrt{4-5} = -2 \pm i$



$\Big\} C_R$

on S_R $|f(z)| \leq \dfrac{R+1}{R^2-4R-5} \xrightarrow{R\to\infty} 0$

$R \gg 0$ we can apply Jordan's lemma

$$\operatorname*{Res}_{z=-2+i} \left[f(z)e^{iz}\right] = \frac{[-2+i]+1}{(-2+i)-(-2-i)}\, e^{i(-2+i)} = \frac{-1+i}{2i}\, e^{-2i} e^{-1}$$

$$\Longrightarrow \int_{C_R} f(z) e^{iz} dz = \frac{\pi}{e} e^{-2i}(-1+i)$$

$$\|$$

$$\int_{-R}^{R} f(x) e^{ix} dx + \underbrace{\int_{S_R} f(z) e^{iz} dz}_{\downarrow R\to\infty \; 0}$$

$$\Longrightarrow \quad P.V. \int_{-\infty}^{\infty} f(x)\, e^{ix} dx = \frac{\pi}{e}(-1+i)\, e^{-2i}$$

taking the real part:

$$P.V. \int_{-\infty}^{\infty} f(x) \cos x \, dx = \frac{\pi}{e}\left[-\cos 2 - \sin(-2)\right] = \frac{\pi}{e}\left[\sin 2 - \cos 2\right]$$